A METHOD FOR DESIGNING THE DOUBLE-DENSITY DUAL-TREE DISCRETE WAVELET TRANSFORM

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ABSTRACT

In this paper, we design the second FIR filter bank of a dual-tree double-density discrete wavelet transform (DWT), given the first filter bank. We show that a good approximation of the highpass filters is given by the Hilbert transform of a transfer function involving a half-delay and express the corresponding H_∞ approximation problem in semidefinite programming (SDP) form. The obtained filters are then refined by nonlinear optimization, such that they satisfy perfect reconstruction constraints and the resulting pairs of wavelets have the analyticity property. We show two examples of design that are significantly better than those obtained with a previous method.

1. INTRODUCTION

In certain signal processing applications, like denoising, overcomplete transforms can offer a better tradeoff between performance and complexity, compared to critically sampled transforms. A distinguished member of the family of overcomplete discrete wavelet transforms (DWT) is the double density (DD) DWT [1], based on the filter bank shown in Figure 1. The input signal is split in three channels, each decimated by a factor of two. The signal on the first channel is processed by an identical filter bank etc. The DD-DWT is expansive with a factor of two, compared to the critically sampled DWT.

A dual tree (DT) [2, 3] is formed by two wavelet transforms processing the same input signal and satisfying a certain relationship: one of the wavelets is an approximate Hilbert transform of the other. The DT-DWT has several appealing properties, such as nearly shift invariance and directional selectivity in higher dimensions. Designed initially for the critically sampled DWT, the dual tree concept can be extended to other types of DWTs. The conditions for two DD-DWTs to form a dual tree are as follows [1].

Let us consider two filter banks with the structure from Figure 1, one (the primal) defined by filters $H_0(z)$, $H_1(z)$ and $H_2(z)$, the other (the dual) defined by filters $G_0(z)$, $G_1(z)$ and $G_2(z)$. Let $\psi_{h,i}(t)$ and $\psi_{g,i}(t)$, $i = 1, 2$, be

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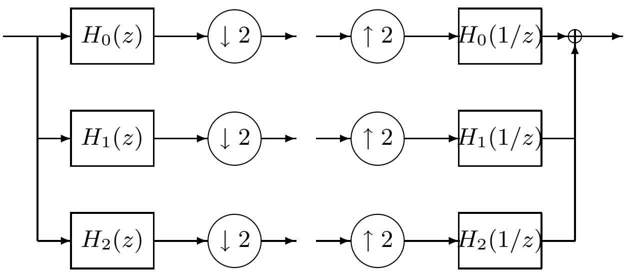

Figure 1. Filter bank used for the implementation of the DD-DWT.

the wavelets generated by the filters $H_0(z)$, $H_i(z)$ and $G_0(z)$, $G_i(z)$, respectively. The two DD-DWTs form a dual-tree if $\psi_{g,i}(t)$ is approximately equal to the Hilbert transform of $\psi_{h,i}(t)$. Equivalently, the complex wavelet $\psi_i(t) = \psi_{h,i}(t) + j\psi_{g,i}(t)$ is approximately analytic and so its spectrum $\Psi_i(\Omega)$, $\Omega \in \mathbb{R}$, is approximately zero for negative frequencies. A measure for this property is

$$E_{2,i} = \frac{\int_{-\infty}^{0} |\Psi_i(\Omega)|^2 d\Omega}{\int_{0}^{\infty} |\Psi_i(\Omega)|^2 d\Omega}. \quad (1)$$

An algorithm for the design of dual-tree DD-DWT was presented in [1], using allpass systems that approximate a half-sample delay. Here, we take a different approach. We assume that the primal DD-DWT is given: the FIR filters $H_0(z)$, $H_1(z)$, $H_2(z)$ are known. We present semidefinite programming (SDP) problems which allow us to compute FIR filters $G_0(z)$, $G_1(z)$, $G_2(z)$ that form a relatively good dual filter bank. These filters are ∞-norm approximations of $z^{-1/2}H_0(z)$ and of the Hilbert transforms of $z^{1/2}H_1(z)$, $z^{1/2}H_2(z)$, respectively. However, these filters do not respect exactly the perfect reconstruction (PR) conditions required for the DD filter bank, namely

$$\begin{array}{l} \sum_{i=0}^{2} G_i(z)G_i(z^{-1}) = 2 \\ \sum_{i=0}^{2} G_i(z)G_i(-z^{-1}) = 0 \end{array} \quad (2)$$

Although these conditions are not convex, we propose to refine the filters $G_0(z)$, $G_1(z)$, $G_2(z)$ by optimizing the analyticity criteria (1) subject to (2). The good initializations provided by the SDP problems mentioned above is crucial in the success of this approach. Moreover, we present further ways to improve the filters.

The contents of the paper is as follows. In section 2, we derive the conditions satisfied by the dual highpass filters, interpret them as a Hilbert transform approximation problem, and express them in SDP form. In section 3, we give a complete description of our design algorithm. In section 4, we present two examples of design that are clearly better than their counterparts from [1].

2. HILBERT TRANSFORM VIA SDP

As a first step in the discussion of relations between the filters of a dual-tree structure, let us consider first orthogonal FIR two-channel filter banks; in Figure 1, we simply remove the third channel. It was suggested in [4] and proved in [5] that the only condition needed to build a dual tree is

$$G_0(z) = z^{-1/2} H_0(z). \tag{3}$$

(This is an ideal condition, impossible to attain with FIR filters.) In an orthogonal CQF (conjugate quadrature filter bank), the filter on the second channel is completely determined by the first channel filter, by

$$H_1(z) = z^{-N} H_0(-z^{-1}), \tag{4}$$

where N is the order of the filters. Since a similar condition holds for the dual tree, by using (3) we obtain

$$\begin{aligned} G_1(z) &= z^{-N} G_0(-z^{-1}) \\ &= z^{-N}(-z^{-1})^{-1/2} H_0(-z^{-1}) \\ &= (-1)^{-1/2} z^{1/2} H_1(z) \end{aligned} \tag{5}$$

In the typical case where the filters have real coefficients, we have to take a branch of the complex square root that has an odd (antisymmetric) phase response. So, the relation between the frequency responses is

$$G_1(\omega) = \begin{cases} -je^{j\omega/2} H_1(\omega), & \text{if } \omega > 0, \\ je^{j\omega/2} H_1(\omega), & \text{if } \omega < 0. \end{cases} \tag{6}$$

We conclude that $G_1(z)$ is the Hilbert transform of $z^{1/2}H_1(z)$. (We have not seen this remark and its trivial deduction in the literature.)

In the case of DD filter banks, the relation (4) is no longer true. However, it was proved in [1] that the ideal dual filters have the form

$$G_i(\omega) = H_i(\omega) e^{-j\theta_i(\omega)}, \tag{7}$$

where $\theta_i(\omega)$, $i = 0 : 2$, are 2π-periodic real functions defined by

$$\theta_0(\omega) = \omega/2, \;\; |\omega| < \pi, \tag{8}$$

and

$$\theta_i(\omega) = -\theta_0(\omega - \pi), \;\; i = 1, 2. \tag{9}$$

These relations are equivalent to (3) for the first channel filters and to (6) (and a similar relation linking $G_2(\omega)$ to $H_2(\omega)$) for the other two (highpass) channels.

In [6], we have shown how to solve the approximation problem

$$|G(\omega) - e^{-j\omega/2} H(\omega)| \le \gamma, \;\; \forall \omega \in [0, \omega_0], \tag{10}$$

where $H(z)$, $G(z)$ are FIR filters

$$H(z) = \sum_{k=0}^{N} h_k z^{-k}, \;\; G(z) = \sum_{k=0}^{N} g_k z^{-k}, \tag{11}$$

$H(z)$ is given and $G(z)$ has to be found. In (10), γ is a positive constant and $\omega_0 \in (0, \pi]$; we can take $\omega_0 = \pi$ and thus (10) becomes equivalent to $\|G(z) - z^{-1/2}H(z)\|_\infty \le \gamma$; in certain cases it is useful to take $\omega_0 < \pi$. It is clear that a solution of (10), especially in the case where γ is minimized, provides a good approximation to (3).

We consider here the approximation (6) of the highpass filters and the related problem

$$|G(\omega) + je^{j\omega/2} H(\omega)| \le \gamma, \;\; \forall \omega \in [\omega_0, \pi], \tag{12}$$

with $\omega_0 \ge 0$. Note that if the filters have real coefficients and (12) holds, then, by complex conjugation, the approximation of (6) is good also for $\omega \in [-\pi, -\omega_0]$. The approach is based on a Bounded Real Lemma for trigonometric polynomials, as in [6], but with some differences due to the presence of a function with complex coefficients in (12).

To eliminate the fractional delay, we consider the following equivalent of (12):

$$|e^{-j\omega} G(2\omega) + jH(2\omega)| \le \gamma, \;\; \forall \omega \in [\omega_0/2, \pi/2]. \tag{13}$$

Let us denote

$$F(z) = z^{-1} G(z^2) + jH(z^2) \tag{14}$$

the polynomial appearing in (13) and

$$\boldsymbol{f} = [jh_0 \; g_0 \; jh_1 \; g_1 \; \ldots \; jh_N \; g_N]^T \tag{15}$$

the vector of its coefficients, of size $2(N+1)$.

To put (13) in LMI form, we need a result on trigonometric polynomials (with complex coefficients) that are positive on a given interval. Let

$$R(z) = \sum_{-M}^{M} r_k z^{-k}, \;\; r_{-k} = r_k^*, \tag{16}$$

be a symmetric trigonometric polynomial. This polynomial is nonnegative on the unit circle ($z = e^{j\omega}$), on the interval $[\omega_1, \omega_2]$, if and only if (see e.g. [7, Th. 1.15], [8]) there exist globally nonnegative (symmetric) trigonometric polynomials $A(z)$ and $B(z)$ of degrees M and $M-1$, respectively, such that

$$R(z) = A(z) + D(z)B(z), \tag{17}$$

where

$$D(z) = d_1 z^{-1} + d_0 + d_1^* z, \tag{18}$$

$$d_0 = -\frac{ab+1}{2},\ \ d_1 = \frac{1-ab}{4} + j\frac{a+b}{4}, \tag{19}$$

$$a = \tan(\omega_1/2),\ \ b = \tan(\omega_2/2). \tag{20}$$

In our case, since the interval is $[\omega_0/2, \pi/2]$, the relation (20) is replaced with

$$a = \tan(\omega_0/4),\ \ b = 1. \tag{21}$$

Since the proof follows the same lines as in [6], we give here only the final result, based on (17) and on the trace parameterization [9, 10, 11] of positive trigonometric polynomials.

Theorem. The inequality $|F(\omega)| \le \gamma$ holds for all $\omega \in [\omega_0/2, \pi/2]$ if and only if there exist positive semidefinite matrices $\boldsymbol{Q}_a \in \mathbb{C}^{2(N+1)\times 2(N+1)}$, $\boldsymbol{Q}_b \in \mathbb{R}^{(2N+1)\times(2N+1)}$ such that

$$\gamma^2\delta_k = \mathrm{tr}[\boldsymbol{\Theta}_k\boldsymbol{Q}_a] + \mathrm{tr}[d_0\boldsymbol{\Theta}_k + d_1\boldsymbol{\Theta}_{k-1} + d_1^*\boldsymbol{\Theta}_{k+1})]\boldsymbol{Q}_b \tag{22}$$

and

$$\begin{bmatrix} \boldsymbol{Q}_a & \boldsymbol{f} \\ \boldsymbol{f}^H & 1 \end{bmatrix} \succeq 0. \tag{23}$$

The matrices $\boldsymbol{\Theta}_k$ from (22) are elementary Toeplitz, with ones on diagonal k and zeros elsewhere, and tr$\boldsymbol{X}$ is the trace of the matrix $\boldsymbol{X}$.

We note that the best approximant $G(z)$ is obtained by minimizing γ in (13), i.e. by solving the problem

$$\begin{array}{cl} \min\limits_{\gamma^2,\boldsymbol{g},\boldsymbol{Q}_a,\boldsymbol{Q}_b} & \gamma^2 \\ \text{subject to} & (22), (23) \\ & \boldsymbol{Q}_a \succeq 0,\ \boldsymbol{Q}_b \succeq 0 \end{array} \tag{24}$$

This is an SDP problem. Moreover, we can impose some roots of $G(z)$ in predefined positions. Typical to our case is the presence of roots in $z = 1$; we can write $G(z) = \tilde{G}(z)(1 - z^{-1})^L$. The vector $\tilde{\boldsymbol{g}} \in \mathbb{R}^{N-L+1}$ is the new variable and it can be expressed as $\boldsymbol{f} = \boldsymbol{A}\tilde{\boldsymbol{g}} + \boldsymbol{b}$; so, the vector $\tilde{\boldsymbol{g}}$ appears affinely in (23), thus preserving the SDP character of the problem.

3. DESIGN ALGORITHM

Design data. The design data are the FIR filters $H_0(z)$, $H_1(z)$, $H_2(z)$, all of order N, forming a valid DD filter bank. We have to design FIR filters $G_0(z)$, $G_1(z)$, $G_2(z)$ of order N, also forming a valid DD filter bank, i.e. satisfying relations (2), such that the analyticity criteria $E_{2,1}$ and $E_{2,2}$ defined by (1) are minimized; since we have to work with a single optimization criterion, we choose to minimize the function $E_{2,1} + \lambda E_{2,2}$, where $\lambda > 0$ is a constant; we take $\lambda = 1$ in the experiments, which gives equal weight to the analyticity criteria associated with the two wavelets. (An alternative would be to minimize $\max(E_{2,1}, E_{2,2})$, which tends to provide almost equal values of $E_{2,1}$ and $E_{2,2}$.) Moreover, the filter $G_0(z)$ must have K_0 roots in $z = -1$ and $G_1(z)$, $G_2(z)$ must have K_1, K_2 roots in $z = 1$, respectively.

Initialization. We compute initial approximations of the desired filters by solving the SDP problems described in the previous section. We find $G_0(z)$ with K_0 roots in $z = -1$ by minimizing γ in (12) (where $G(z) = G_0(z)$, $H(z) = H_0(z)$), as described in [6]. Then, we compute $G_1(z)$ with K_1 roots in $z = 1$ by solving (24), where we take $G(z) = G_1(z)$, $H(z) = H_1(z)$. Similarly, we compute $G_2(z)$.

Enforcing PR conditions. The obtained filters $G_0(z)$, $G_1(z)$, $G_2(z)$ satisfy only approximately the perfect reconstruction conditions (2); since the conditions are not convex, they cannot be added to the SDP problems. So, we must impose them explicitly. To this purpose, we solve the optimization problem

$$\begin{array}{cl} \min\limits_{\boldsymbol{g}_0,\boldsymbol{g}_1,\boldsymbol{g}_2} & E_{2,1} + \lambda E_{2,2} \\ \text{s.t.} & (2) \end{array} \tag{25}$$

This is a nonconvex problem. We have solved it using the Matlab function `fmincon`, initialized with the filters given by the initialization step. We have noticed that reliable results are obtained by replacing the equality constraints (2) with inequalities depending on a tolerance ϵ. For example, the first equality form (2) is implemented in (25) as

$$\left| \sum_{i=0}^{2} \sum_{k=0}^{N-\ell} g_{i,k} g_{i,k+\ell} - 2\delta_\ell \right| \le \epsilon, \ \ \text{for } \ell = 0 : N. \tag{26}$$

Iterative refinement. The solution of (25) may be satisfactory, but it can be further improved. We simply switch the roles of the primal and dual filters (consider $G_i(z)$ known and optimize $H_i(z)$) and solve (25) again. The optimization criterion decreases at each iteration and, usually in relatively few iterations, a local minimum is attained.

4. EXPERIMENTAL RESULTS

We report here two examples of design. The SDP problem corresponding to (24) has been solved using the library SeDuMi [12]. The optimization problem (25) has been solved with the large scale version of the Matlab function `fmincon`, with default parameters, excepting `TolCon` (the tolerance for satisfying the constraints), which has been set to 10^{-12}. The tolerance ϵ from (26) has been set to 10^{-11}, which means that the PR constraints are satisfied with very good precision. We report here the results obtained after 10 refinement iterations, although in the second example fewer iterations give approximately the same result.

Example 1. We consider first the filters from [1], Example 1, where $N = 9$, $K_0 = 4$, $K_1 = K_2 = 2$. The coefficients of the filters obtained by our method, starting from the filters $H_0(z)$, $H_1(z)$ and $H_2(z)$ from [1], are shown in Table 1. The values of the analyticity criteria (1) are $E_{2,1} = 5.19 \cdot 10^{-5}$, $E_{2,2} = 4.10 \cdot 10^{-5}$. For comparison, for the dual tree DD-DWT from [1], the values are $E_{2,1} = 1.16 \cdot 10^{-3}$, $E_{2,2} = 9.12 \cdot 10^{-4}$, i.e. about 20 times larger. The wavelets $\psi_{h,i}(t)$, $\psi_{g,i}(t)$, $i = 1, 2$, are shown in Figure 2; the wavelets generated by the dual tree are delayed with approximately half-sample with respect

Table 1. Coefficients of optimized filters for Example 1.

$h_{0,k}$	$h_{1,k}$	$h_{2,k}$
0.07172370159505	0.00076076901999	0.00133098703411
0.36230036193850	0.00384289856633	0.00672326075998
0.66494703815050	-0.00043349459069	-0.00568339454318
0.47140309737195	-0.03281690886095	-0.08229185377013
-0.01799895170798	-0.06136454414349	-0.10103370970581
-0.15312080154057	-0.00923768881032	0.21406057417676
-0.01354874233901	0.13102206413624	0.39782529534987
0.02857352887159	0.33787985813471	-0.64575713709268
0.00198373546859	-0.56270662712939	0.21473455499402
-0.00204940547432	0.19305367367758	0.00009142279707
$g_{0,k}$	$g_{1,k}$	$g_{2,k}$
0.01976798190714	0.00095991650970	0.00203303024798
0.18796630962425	0.00912748189359	0.01933129639730
0.54463516675450	0.02370521240285	0.04519448638570
0.64365432452264	0.00518426608962	-0.03667185824797
0.21767214390247	-0.06791654815954	-0.24127168408985
-0.14706575138756	-0.12444007940910	-0.04034877318920
-0.08852593177102	-0.04960090813396	0.64638211422618
0.02465246521978	0.55291356612824	-0.44847344350248
0.01355742037414	-0.39838253651420	0.05627615891019
-0.00210056681188	0.04844962919279	-0.00245132713785

to those generated by the primal tree. The Fourier transforms of the complex wavelets $\psi_i(t) = \psi_{h,i}(t)+j\psi_{g,i}(t)$, $i = 1, 2$, are shown in Figures 3 and 4, in solid line. They have very small values for negative frequencies. The Fourier transforms of the complex wavelets designed in [1] are represented with dashed lines in the same figures. The improvement brought by our method is obvious.

It is also interesting to see the quality of the solution of the approximation problem (24). In Figure 5, we represent the error

$$E(\omega) = |G_1(\omega) + je^{j\omega/2}H_1(\omega)| \qquad (27)$$

for three pairs of filters. The dashed line corresponds to the filters from ([1]). The dash-dot line is obtained with the original $H_1(z)$ and the filter $G_1(z)$ obtained by solving (24); the error has the typical equiripple aspect of an ∞-norm solution. The solid line corresponds to the optimized filters from Table 1. It is difficult to draw a conclusion on the optimal shape of the error (27), but it is clear that the solution (24) is likely to be a good initialization of the optimization process, thus confirming the validity of our approach.

Example 2. The second example comes also from the filters from [1] (Example 2), with $N = 14$, $K_0 = 6$, $K_1 = K_2 = 3$. The coefficients of our optimized filters are given in Table 2. We show now only the Fourier transforms of the complex wavelets, in Figures 6 and 7. They are nearly analytic and much better than the original results from [1]. The values of the analyticity criteria are $E_{2,1} = 1.08 \cdot 10^{-5}$, $E_{2,2} = 1.05 \cdot 10^{-5}$. For the filters in [1], the corresponding values are $E_{2,1} = 7.36 \cdot 10^{-5}$, $E_{2,2} = 9.66 \cdot 10^{-5}$.

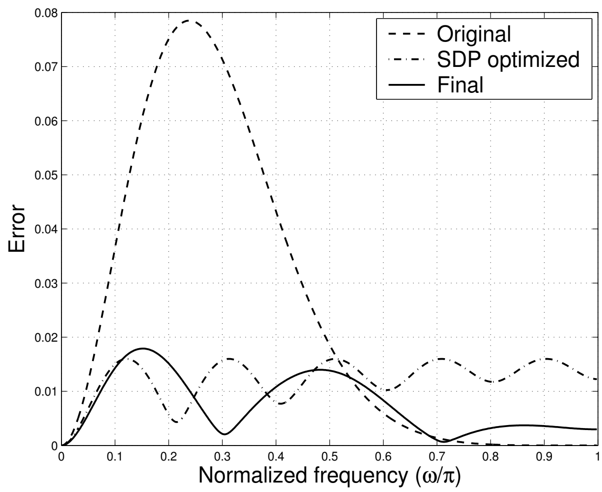

Figure 5. Values of the error function (27) for three pairs of filters generated in Example 1.

5. CONCLUSION

The contribution of this paper is twofold. Firstly, we show that the Hilbert transform FIR approximation problem (12) can be expressed as an SDP problem and solved reliably. Secondly, we give a complete algorithm to compute the second filter bank of a dual-tree double-density DWT. The algorithm comprises an initialization step based on SDP, followed by iterative refinement via nonlinear optimization. Design examples have shown the viability of this approach.

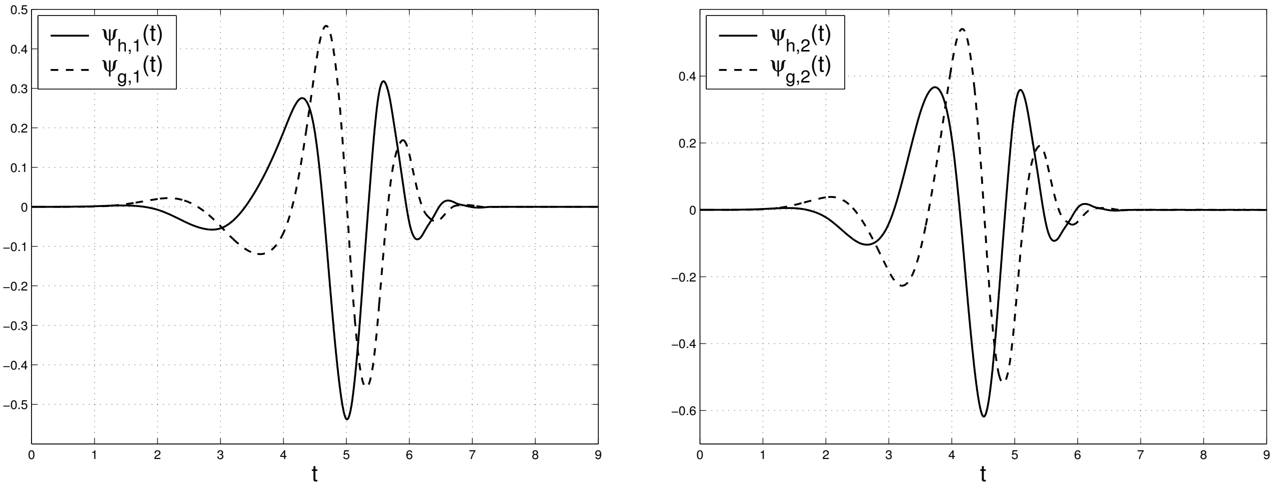

Figure 2. Wavelets generated by the filter banks designed in Example 1.

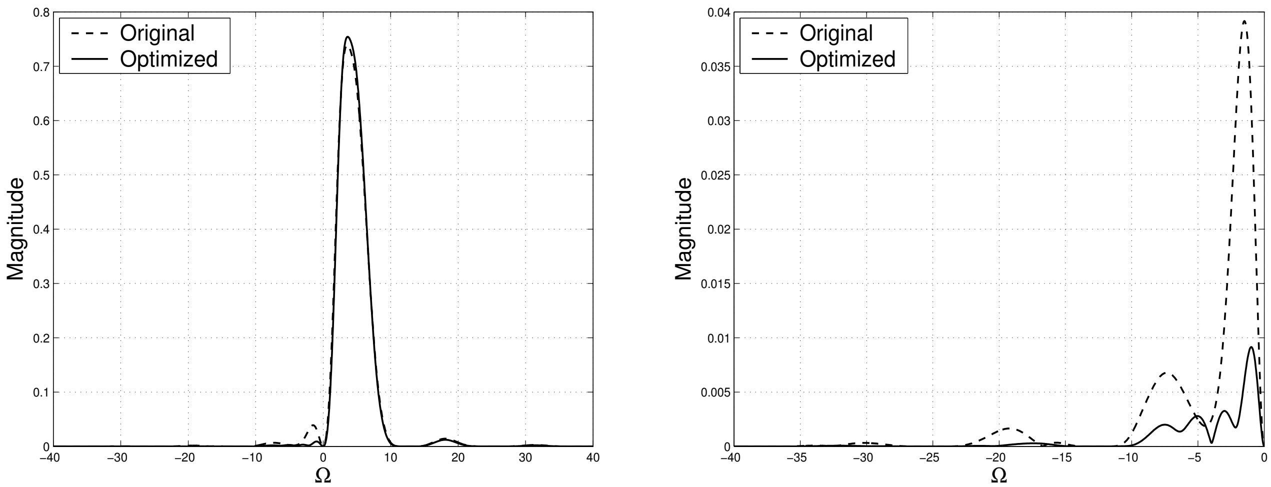

Figure 3. Left: Fourier transforms of the complex wavelets $\psi_1(t)$ obtained with our method (solid line) and with the method from [1] (dashed line), for Example 1. Right: detail.

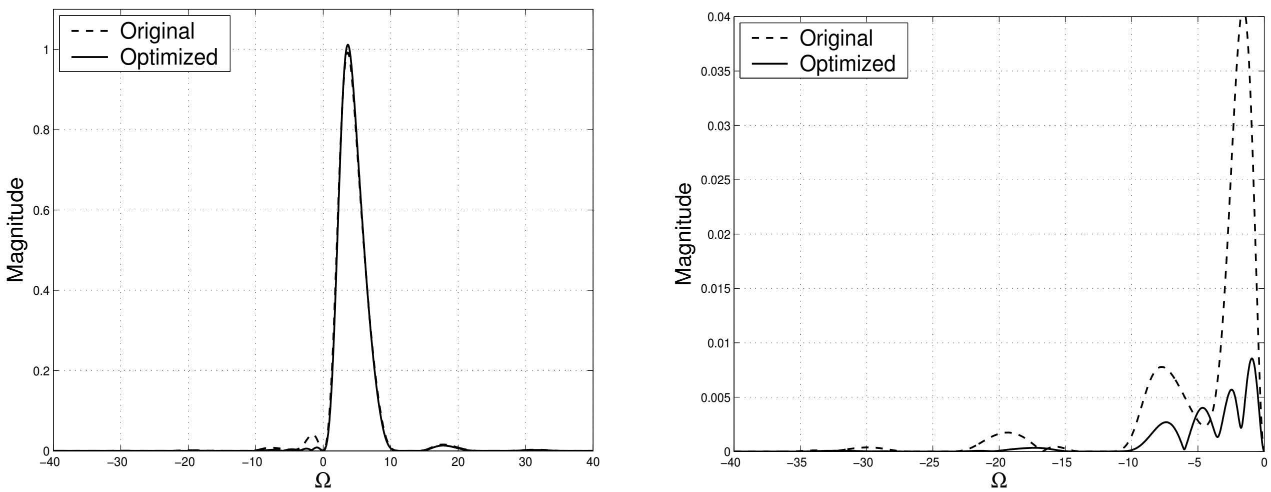

Figure 4. Left: Fourier transforms of the complex wavelets $\psi_2(t)$ obtained with our method (solid line) and with the method from [1] (dashed line), for Example 1. Right: detail.

Table 2. Coefficients of Hilbert pair filters for Example 2.

$h_{0,k}$	$h_{1,k}$	$h_{2,k}$
0.01111197186038	-0.00000230838534	-0.00001527139805
0.11039227653418	-0.00002293113414	-0.00015170339866
0.38902610591783	-0.00020620813676	-0.00086760994953
0.63818943401801	-0.00137828440398	-0.00418500279332
0.45228466042635	-0.00417576611217	-0.01014957601479
-0.01607065382050	-0.00413616604876	0.00205699162817
-0.18744972778104	0.00699749931054	0.05632918936109
-0.03718113309396	0.03053724188791	0.07384425497228
0.04879823115348	0.05076771374211	-0.07578365034196
0.01351037057998	-0.01059152902846	-0.23774281447438
-0.00697671262326	-0.21359528408043	-0.04190437042227
-0.00174350123350	-0.13307268309012	0.57504004593315
0.00029934749369	0.61620280114163	-0.40411847724883
0.00000998819012	-0.40324653791717	0.06822210248699
0.00001290472689	0.06592244225515	-0.00057410833988
$g_{0,k}$	$g_{1,k}$	$g_{2,k}$
0.00036230000097	0.00000036012417	0.00000110557247
0.03874605225675	0.00003851604271	0.00011828315911
0.22984657328096	0.00022020030508	0.00067055918977
0.54869989430634	-0.00034012612936	-0.00165228831777
0.60956733850079	-0.00464753468690	-0.01776242835558
0.21335037705652	-0.01234798520570	-0.03417024815518
-0.15862478971221	-0.01408589446686	0.01994310730092
-0.12472520864020	0.00729500767720	0.13591711596687
0.02875103256991	0.07737044882196	0.09107541313470
0.03585936254907	0.12472670075821	-0.20410731701029
-0.00293286477606	-0.13139025258379	-0.36257983634571
-0.00499159113906	-0.43610256214146	0.57105943976484
0.00014175042169	0.58483856853065	-0.20869330083268
0.00016789479096	-0.20384208784145	0.01137888290925
-0.00000455910566	0.00826664079553	-0.00119848798070

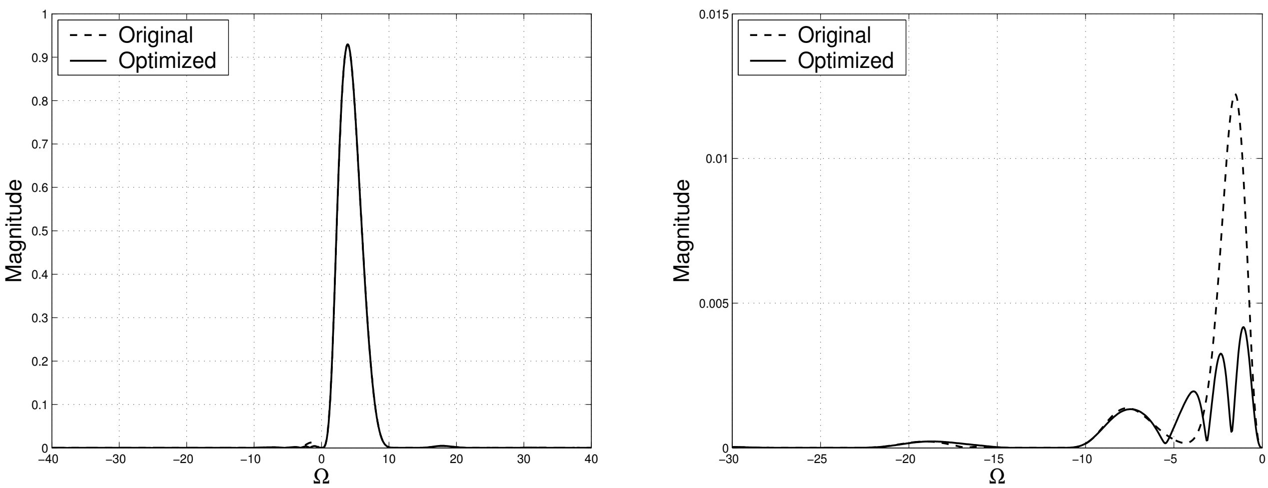

Figure 6. Left: Fourier transforms of the complex wavelets $\psi_1(t)$ obtained with our method (solid line) and with the method from [1] (dashed line), for Example 2. Right: detail.

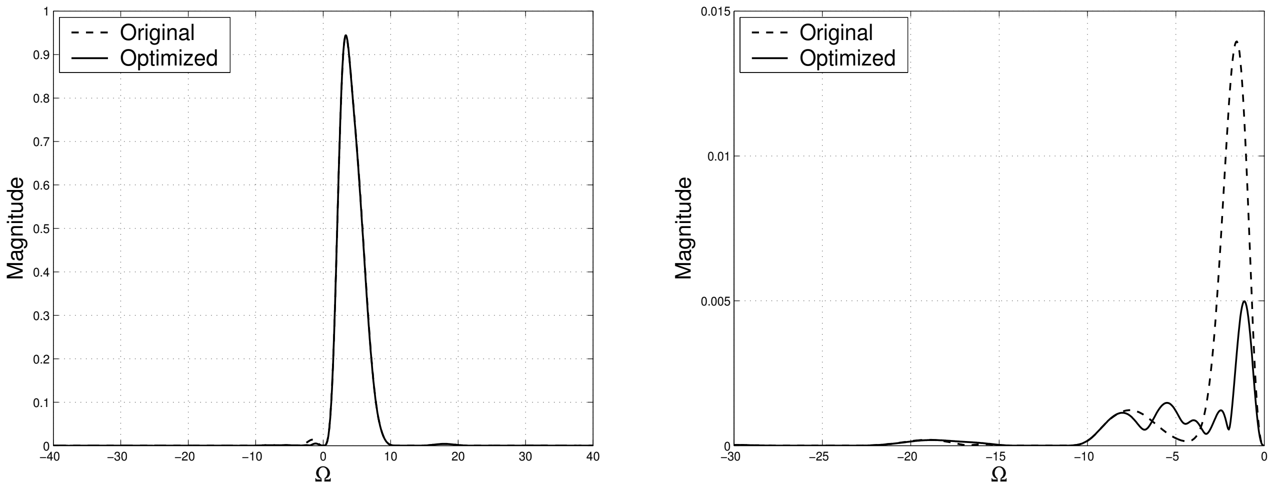

Figure 7. Left: Fourier transforms of the complex wavelets $\psi_2(t)$ obtained with our method (solid line) and with the method from [1] (dashed line), for Example 2. Right: detail.

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