EXIT Chart-based Design of LDPC Codes for Inter-Symbol Interference Channels

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Abstract— **The subject of this paper is the design of low-density parity-check (LDPC) codes concatenated with a Gray mapped quaternary phase shift keying (QPSK) modulator for transmission over inter-symbol (ISI) channels. In particular, we will present several results considering important partial response channels (PRCs). Using an extrinsic information transfer (EXIT) chart-based analysis, we give insights into the behavior of an iterative detector/decoder for LDPC codes transmitted over ISI channels. In particular, we will discuss if there is a real need for LDPC codes specifically optimized for ISI channels. Using the proposed analysis method, we design LDPC codes tailored for ISI channels. We show that, if the channel impulse response is short, an *ad hoc* optimized LDPC code is very similar to that optimized for an additive white Gaussian noise (AWGN) channel, reducing significantly the need for code optimization. On the other hand, if the impulse response of the ISI channel is long, *ad hoc* optimized LDPC codes achieve sensible gain with respect to LDPC codes optimized for AWGN channel.**

I. INTRODUCTION

THE recent rediscovery [1], [2] of a suboptimal, yet effective, decoding algorithm, denoted as *sum-product* [3], for a wide class of linear block codes, the Low-Density Parity-Check (LDPC) codes [1], has allowed the construction of communication systems for *memoryless channels* characterized by performance very close to the theoretical limits predicted by Shannon [4]. The successful use of LDPC codes for transmission over memoryless channels has motivated the use of LDPC codes, and, more generally, graph-based codes [5], for transmission over channels with memory as well as over channels characterized by stochastic parameters, such as phase-uncertain channels or fading channels [6]. Two possible approaches for the design of LDPC codes can be devised.

- A *graph-based* approach, which describes the transmission system as a graph representing the codeword symbols and the constraints, due to both code and channel, to which the symbols are subject. The decoding algorithm is typically a generalization of the standard belief propagation decoding algorithm for LDPC codes [7].
- A *concatenated* approach, which uses the LDPC code as an outer code concatenated with an inner modulator tailored for the specific channel. In this approach, the decoder comprises an inner soft-input soft-output (SISO) block relative to the modulator and exchanging extrinsic information [8] with an outer standard LDPC decoder.

In [9], [10], the authors choose the first approach and, by means of a *density evolution* analysis [11], they design LDPC codes optimized for partial response channels (PRCs). The second approach can be seen as a generalization, to modulation formats with memory, of bit interleaved coded modulation with iterative decoding (BICM-ID) schemes [12], [13]. In particular, in [14] the authors investigate the performance of a binary code concatenated through an interleaver with a high spectral efficiency modulator, allowing exchange of information between the binary decoder and the SISO demodulator. In [15], the authors use extrinsic information transfer (EXIT) charts to analyze the performance of classes of LDPC codes and design a simple, yet effective, LDPC code for a multiple input multiple output (MIMO) channel. A concatenated approach to the design of LDPC codes can also be found in in [16], where the transmission of a binary code mapped over a binary phase shift keying (BPSK) through a PRC is considered. In particular, in [16], the authors perform EXIT chart-based analysis for a decoder in which a SISO block, featuring a BCJR algorithm [17] for the PRC channel, iteratively exchanges extrinsic information with a SISO decoder associated to the binary code. In [18], the authors derive an optimization algorithm for LDPC codes based on a Gaussian approximation of the messages exchanged in the graph.

In this paper, LDPC codes mapped over a memoryless quaternary phase shift keying (QPSK) constellation and transmitted over an ISI channel are investigated. Following a *concatenated* approach, we use the EXIT chart-based analysis in [15] to investigate the impact of the impulse response of the channel on the convergence behavior of the receiver. In particular, we show that (i) LDPC codes optimized for ISI channels with short impulse response exhibit limited performance improvement with respect to LDPC codes optimized for the additive white Gaussian noise (AWGN) channel; (ii) when ISI channel impulse response is sufficiently long (i.e., interference of at least $3 \div 4$ samples) LDPC code optimized for ISI channel achieve significant performance improvement with respect to standard LDPC codes optimized for AWGN channel. Moreover, based on the proposed analysis technique, we perform optimization of the LDPC code parameters, i.e., the code's *degree distributions* [19], for both short and "long" impulse response ISI channels as well as AWGN channel without ISI. We then show, by simulating the overall com-

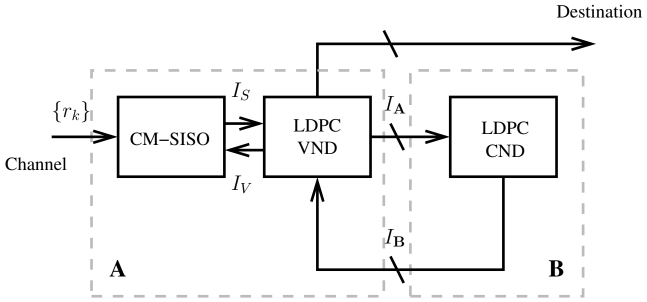

Fig. 1. Schematic representation of the receiver.

munication system, that the performance of LDPC codes optimized for short impulse response ISI channel and for AWGN channel without ISI, both transmitted over the short impulse response ISI channel, is very similar. Furthermore, we show that LDPC codes optimized for "long", i.e. ISI channel with 5 samples impulse response, outperform LDPC codes optimized for memoryless AWGN channel by more than 1 dB, when used over the channel they are optimized for.

This paper is structured as follows. In Section II, EXIT chart-based analysis of LDPC codes is introduced. In Section III, the need for optimized LDPC codes for ISI channels is discussed. In Section IV, two LDPC codes optimized for short and "long" impulse response ISI channel respectively is presented, and their performance is analyzed through simulations. In Section V, we discuss on the implications of the obtained results. Section VI concludes the paper.

II. SYSTEM ANALYSIS USING EXIT CHARTS

The considered communication system is characterized as follows. At the transmitter side, an LDPC code encodes bits coming from an information source. The resulting length-N codeword $\mathbf{x}$ is subdivided into a sequence of couples of bits which are encoded into QPSK symbols using a Gray mapped QPSK modulator. The considered channel is an ISI channel. At the receiver a SISO module, implementing the BCJR algorithm for the finite state machine (FSM) modeling the considered channel, iteratively exchanges soft information with the variable node detector (VND) and the check node detector (CND) associated with the LDPC code [15]—details of this decoding algorithm can be found in [15], [20]. In Fig. 1, the receiver is shown, decomposed into two main blocks labeled block **A** and block **B**, respectively. Block **A** comprises the SISO module for the coded modulator and the channel (CM-SISO) and the VND. Block **B** is formed by the CND, which is connected directly to the VND. In [15], it is shown how to perform EXIT chart-based analysis of the generic system represented in Fig. 1. More precisely, for ease of description, the labels of the EXIT curves associated to the subblocks are explicitly indicated. In particular, the EXIT curve relative to block **A**, denoted as $I_{\mathbf{A}}$, is obtained by composing the EXIT curves of the CM-SISO, denoted as I_S, and the VND, denoted as I_V. The EXIT curve of block **B**, denoted as $I_{\mathbf{B}}$, is simply the EXIT curve of CND. The decoding operation can be interpreted as a recursive update of mutual information (MI) at the output of each of blocks **A** and **B** corresponding to the respective EXIT curve. EXIT charts can be effectively used to predict convergence of the decoding process, by verifying that the following sequence $\{x_n\}$, which describes the recursive MI update, converges to 1 as n goes to infinity:

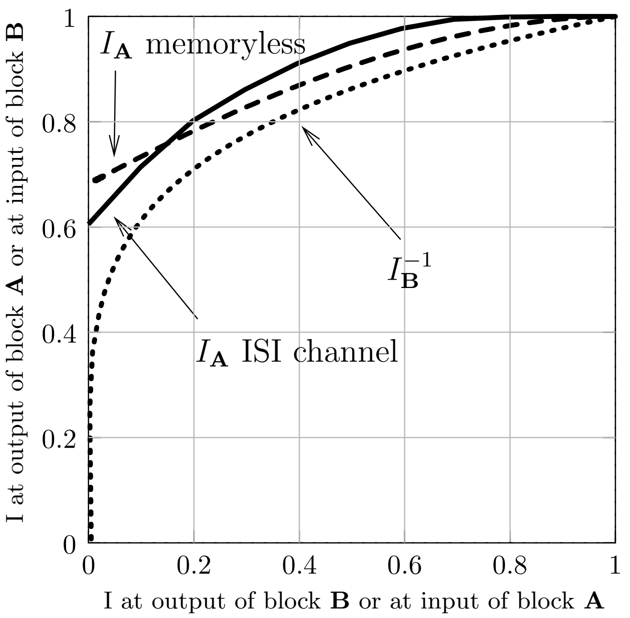

Fig. 2. Example of EXIT charts for blocks **A** and **B**.

$$\begin{cases} x_0 &= I_{\mathbf{A}}(0) \\ x_{2n+1} &= I_{\mathbf{B}}(x_{2n}) \\ x_{2n} &= I_{\mathbf{A}}(x_{2n-1}). \end{cases} \tag{1}$$

If this is the case, the decoding process converges to MI$=1$, thus enabling virtually error free recovery of the transmitted bits.

III. LDPC CODES FOR ISI CHANNELS

In this section, the need for optimized LDPC codes for ISI channels is discussed. In Fig. 2, an example of EXIT curves $I_{\mathbf{A}}$ and $I_{\mathbf{B}}$ is shown. The EXIT curve of block **A** corresponds to a CM-SISO module associated with the concatenation of a memoryless QPSK Gray mapper and an ISI channel. The EXIT curve of block **B** does not depend on the particular CM-SISO block, but only on the check nodes' degree distribution of the used LDPC codes' family.

A possible optimization strategy would consist in modifying these curves, by manipulating the LDPC codes' degree distributions, in order to guarantee convergence in particular conditions, such as, for example, lowest signal to noise ratio (SNR), minimum number of iteration, etc.

In the previous section it has been stated that $I_{\mathbf{A}}$ can be expressed as a function of the CM-SISO EXIT curve $I_S(\cdot)$. In fact $I_{\mathbf{A}} = I_{\mathbf{A}}\big(I_{\mathbf{B}}, I_S(I_V(I_{\mathbf{B}}))\big)$, where $I_V(\cdot)$ and $I_{\mathbf{A}}(\cdot,\cdot)$ depend on the variable nodes' degree distribution. Experience suggests that each one of these functions is monotonically non-decreasing (recall that the underlying SISO algorithms compute *a posteriori* probabilities (APPs) relative to the

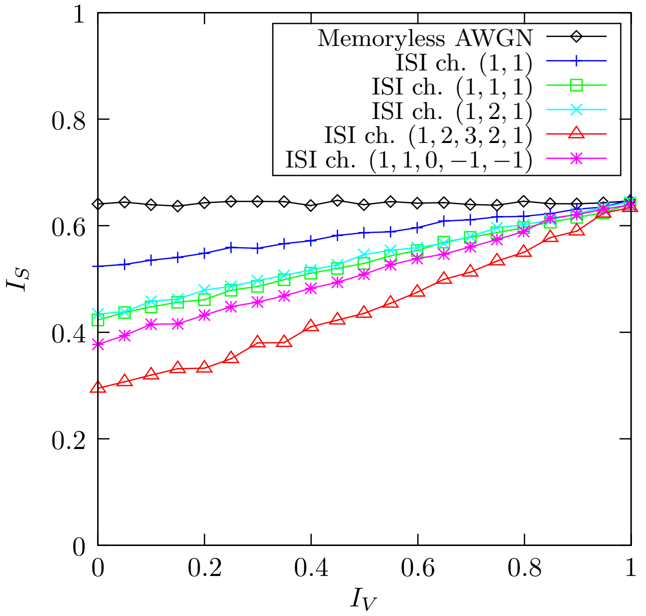

Fig. 3. EXIT charts SISO modules relative to several ISI channels and a memoryless AWGN channel, at SNR= 2 dB. The used modulation format is QPSK with Gray mapping.

transmitted bits). Fig. 3 depicts the EXIT curve for several possible CM-SISO modules: a memoryless QPSK demapper for transmission over an AWGN channel and BCJR algorithms for QPSK transmitted over the ISI channels characterized by the following impulse responses: $(1,1)$, $(1,1,1)$, $(1,2,1)$, $(1,2,3,2,1)$ and $(1,1,0,-1,-1)$. In the following, the actual signal to noise ratio (SNR) will be measured at the receiver, i.e., even if the notation for impulse responses does not account for it, actual impulse responses have unit energy. In all cases, Gray mapping is considered and the SNR is set to 2 dB. One can observe that, while the memoryless QPSK Gray demapper is characterized by a flat EXIT curve, the CM-SISO EXIT curves associated with the ISI channels are increasing functions of the input MI. The composition of the CM-SISO EXIT curve for a ISI channel and the VND EXIT curve, i.e., $I_{\mathbf{A}}$, is then expected to differ from the composition of the CM-SISO EXIT curve for a memoryless demapper and the VND EXIT curve. Thus, the optimal LDPC code for an ISI channel should differ significantly from the optimal LDPC code for memoryless Gray QPSK (which is not different from that of memoryless BPSK). From a practical viewpoint, as we will readily see, the difference between the composition of the CM-SISO EXIT curve for a memoryless demapper and the VND EXIT curve and the composition of the CM-SISO EXIT curve and the VND EXIT curve is very small in the case of short length impulse response, such as, e.g., a $(1,1)$ ISI channel, and becomes significant for longer impulse responses, such as, e.g., a $(1,2,3,2,1)$ ISI channel.

By looking at Fig. 3, one can observe that all the curves "touch" when the input MI is equal to one. This is not surprising, since the fact that the input MI is equal to one corresponds to *complete knowledge of all codeword bits but the one for which the APP is being computed*. In other words, we are transmitting, through a linear modulation, a single bit over an ISI channel, which, in case of single bit transmission simplifies to an AWGN channel. Therefore, the BCJR EXIT curves for QPSK (or BPSK) transmitted over any possible ISI channels converge to the same point $I_S(1) = I_{\text{AWGN}}$, where I_{AWGN} is the BPSK input capacity of an AWGN channel with noise sample variance equal to that of the considered ISI channel.

IV. LDPC CODES OPTIMIZED FOR ISI CHANNELS

Following the approach in [20], we perform optimization of degree distributions for three different LDPC-coded concatenated schemes: (i) an LDPC-coded QPSK transmitted over a channel with impulse response $(1,1)$, (ii) an LDPC-coded QPSK transmitted over a channel with impulse response $(1,2,3,2,1)$ and (iii) an LDPC-coded QPSK transmitted over a memoryless AWGN channel. The code optimized for AWGN channel will be used as a reference code, since LDPC codes currently in use are usually optimized for a memoryless (AWGN) channel. The optimization consists in a semi-random walk in the parametric space of LDPC code degree distributions, in order to obtain the degree distributions which guarantee the lowest possible convergence threshold (i.e., the lowest SNR for which convergence is guaranteed). The degree distributions are characterized by degrees in the ensemble $\{2,\ldots,12\}$, in order to enable the construction of moderate length LDPC codes with small number of short cycles [1]. The optimized codes have rate 0.5. Based on the obtained optimized degree distributions, we construct LDPC codes with codeword length $N = 12000$. In Fig. 4, the bit error rate (BER) performance of the obtained codes is shown. The LDPC code optimized for AWGN channel is transmitted both over the $(1,1)$ ISI channel and the $(1,2,3,2,1)$ ISI channel. For both the considered ISI channels the performance of the relative *ad hoc* optimized LDPC code is also shown. The maximum number of iterations is set to 100. If a codeword is found earlier, the decoding process stops. The BER curves exhibit a change in convexity, characteristic of the spring of an error floor. This can be attributed to the presence of very short cycles in the code's graph [1]. This issue arises since actual codes are extracted at random from given degree distributions for *finite* codeword length—this contradicts the implicit assumption, in the EXIT chart-based analysis, of infinite codeword length.

Considering the BER performance in Fig. 4, one can observe that the LDPC code optimized for the $(1,1)$ ISI channel leads to a performance very close (i.e., about 0.1 dB) to that of the LDPC code optimized for the AWGN channel and Gray QPSK modulation. This is in apparent contrast with the conjecture stated in Section III. However, this lack of optimization gain can be interpreted by considering that, even if the EXIT curves in Fig. 3 differ from each other, the EXIT curve for the considered ISI channel is "almost flat," leading to a very small difference in the block $\mathbf{A}$ EXIT curves for the $(1,1)$ ISI channel and the AWGN channel. Therefore, this leads to a small difference between optimized codes. On the other hand, considering the BER curves relative to

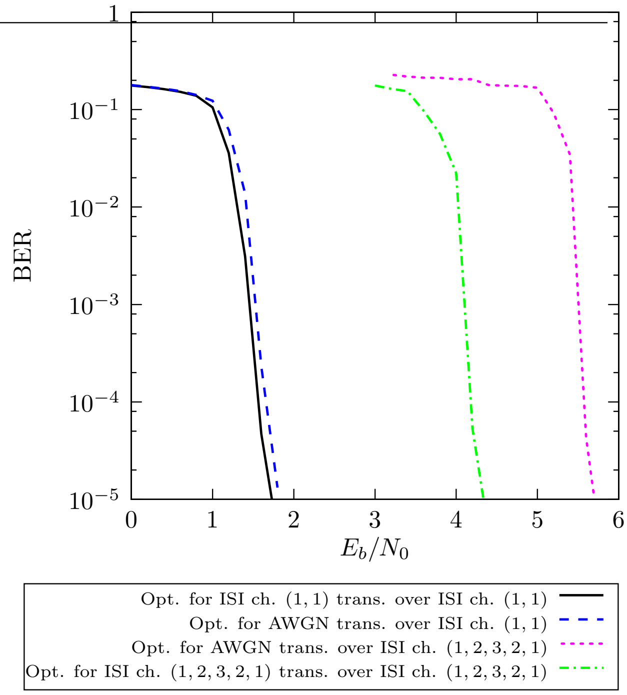

Fig. 4. BER performance for four systems operating on two different ISI channels: (i) $(1,1)$ ISI channel and (ii) $(1,2,3,2,1)$ ISI channel. For each channel both an *ad hoc* optimized LDPC code and an LDPC code optimized for AWGN channel are considered. The codeword length is 12000 LDPC codes.

the $(1,2,3,2,1)$ ISI channel one can readily notice that the performance difference is increased to 1.1 dB. This confirms our conjecture in Section III, since, the EXIT curve for the CM-SISO for the $(1,2,3,2,1)$ ISI channel exhibits greater slope than the $(1,1)$ ISI channel one.

It is possible to compare the codes simulated in Fig. 4 with other codes proposed in the literature. In particular, in [9], [10] an optimization algorithm is proposed, based on density evolution: the codes are designed in order to exhibit a BER lower than a given constant at a specified iteration. Since the underlying analysis model involves infinite dimensional quantities, this optimization algorithm is computationally very demanding. In [18], a simplified version of the analysis proposed in [10] is given, and used to optimize LDPC code degree distributions for a $1+D$ PRC, i.e., $(1,1)$ ISI channel: the theoretical threshold of 1.35 dB, obtained allowing degree-20 variable nodes, can be compared with the actual codes in Fig. 4, which show a BER equal to 10^{-3} at an SNR equal to approximately 1.4 dB.

V. DISCUSSION

The results presented in the previous sections can be easily extended to any ISI channel. Nevertheless, by looking at the EXIT curves of the BCJR for the ISI channel with impulse responses $(1,1,1)$ and $(1,2,1)$, shown in Fig. 3, one could conclude that the EXIT curve steepness depends almost only on the length of the impulse response. In particular, channels with short impulse response exhibit "almost constant" EXIT curves. This finding is the subject of our current research, since it would imply that a standard LDPC code, i.e., an LDPC code optimized for a memoryless channel, would be a good choice even if the channel were slowly time-varying, as long as the impulse response of the channel is short. On the other hand, it should be noted that long impulse response channels exhibit a steeper BCJR EXIT curve, making the optimization of LDPC codes important.

VI. CONCLUSIONS

In this paper, LDPC-coded QPSK transmitted over ISI channels have been investigated using EXIT chart-based analysis. The proposed analysis method has been used to design new LDPC codes tailored for the particular channels. The proposed approach has two main advantages with respect to previously known analysis and optimization techniques:

- *its simplicity*, which allows simple interpretation of the convergence behavior of the receiver,
- *its low computational burden*, which enables fast LDPC code design algorithms, according to the desired criterion (low convergence thresholds, *fast convergence*, *short codeword length* [21], etc.).

The BER performance of the obtained LDPC codes has been investigated, showing that, besides the excellent performance, LDPC codes designed for the $(1,1)$ ISI channel, i.e., $1+D$ PRC, exhibit very small difference from codes designed for AWGN channel. This has the important implication that, for transmission over ISI channels with short impulse response, standard LDPC codecs designed for memoryless channels are still a good choice. On the other hand, ISI channels with long impulse response benefit from an LDPC code optimization, since the overall statistical characteristics, summarized through the EXIT charts, of the CM-SISO output exhibit significant differences with respect to the memoryless AWGN channel CM-SISO output.

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