# On the Outage Probability of Large DS-CDMA Systems Using Higher Order Modulation

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***Abstract*— The outage behavior of a single cell DS-CDMA system using higher order modulation schemes is investigated for the uplink in presence of flat Rayleigh fading channels. For this purpose, the results of the asymptotic analysis for large CDMA systems with random spreading codes are used to obtain the signal-to-interference-and-noise ratio after the linear minimum mean-squared-error receiver. This information is used to determine the outage probabilities for different rates and both coding at capacity and modulation and coding at cut-off rate. The influence of the system load on the outage is determined analytically and compared to simulation results. Furthermore, an expression is presented that allows to easily determine the additional required signal-to-noise ratio for modulation and coding at cut-off rate from the results for coding at capacity. Finally, the investigations are extended to determine the maximum spectral efficiency for given outage constraints and modulation schemes. The optimum loads corresponding to the maximum efficiencies are presented and shown to vary for the different modulation schemes.**

## I. Introduction

Most third generation mobile communications systems are based on code division multiple access (CDMA). Currently, extensions to increase peak data rate and spectral efficiency are subject of research mainly for the downlink. Especially advanced receiver concepts such as multiuser detectors or interference cancellers have been studied. A promising trade-off between performance and complexity is given by the linear minimum mean-squared-error (LMMSE) receiver, which has been proposed in many different implementation variants.

Recently, also the spectral efficiency of the uplink of CDMA systems has been investigated. The reason is that due to interactive applications the uplink might become a bottleneck in the future. To increase the throughput, higher order modulation is an interesting alternative. In the 3GPP specification this study item is named "Enhanced Uplink". In the downlink, higher modulation of up to 64-QAM is already considered for the advanced modulation and coding schemes of the 3GPP UTRA High Speed Downlink Packet Access.

For the widely deployed matched filter (MF) and LMMSE receivers, the system spectral efficiency can be determined, for example, by using the results of the asymptotic analysis introduced in [1]. Based on this analysis, the uplink spectral efficiency was investigated in several publications, e.g. in [2] and [3] for higher order modulation and coding at the cut-off rate and flat Rayleigh fading channels in the ergodic limit. For a complete consideration of the performance of a communications system also the outage behavior has to be studied. So far, analysis of the outage probability has only received little attention in this context, see [4] as an example focusing on diversity issues with coding at capacity. In this contribution, outage will be investigated in detail for the LMMSE receiver with both coding at capacity and modulation and coding at cut-off rate. The analytical evaluation shows that the LMMSE has its strongest interference suppression capability for system load 1. Furthermore, an expression is presented to determine the signal-to-noise ratio (SNR) necessary to obtain a certain outage probability at given rate and load. Because the evaluation of the asymptotic analysis is quite complex, a simple expression is derived providing the information about the additional SNR, which is necessary to obtain the same rate and outage constraint with modulation and coding at cut-off rate in comparison to coding at capacity. Based on the asymptotic analysis for coding at capacity, it allows for fast system design without the need of a complete analysis for each modulation scheme.

Another important aspect is the maximum system efficiency that can be achieved under a given outage constraint. For coding in the ergodic sense, this spectral efficiency has been considered in [5]. In this paper, these results are extended to the outage efficiency, which strongly depends on the given outage constraint. Similar to the ergodic efficiency [5], the system load maximizing the outage efficiency depends on the modulation scheme, which is important for the choice of system parameters.

The structure of this paper is as follows. First, in section II the system model is introduced. Furthermore, the idea and underlying assumptions of the theory of the asymptotic analysis are presented. In section III the evaluation of the asymptotic analysis is shown in short for fading channels and coding at capacity in the ergodic limit. The outage probability for coding at capacity and an analytical treatment of the impact of the system load on the outage is given in section IV. Finally, the investigation is extended to modulation and coding at the cut-off rate in section V. In this section, also the outage capacity is shown, as well as an approximation of the SNR loss for modulation and coding at the cut-off rate in comparison to coding at capacity, which is valid for a given outage, rate and load. Conclusions are contained in section VI.

## II. System Model and Asymptotic Analysis

Throughout this work, the uplink of a synchronous, single cell DS-CDMA system with $K$ active users and spreading factor $N$ is assumed. Furthermore, only AWGN and flat Rayleigh fading channel are considered as both were shown to provide best and worst performance in terms of spectral efficiency in the ergodic limit [5]. Then, the chip matched filter output of the received signal is

$$\mathbf{r} = \sum_{k=1}^{K} x_k h_k \mathbf{c}_k + \mathbf{n}, \tag{1}$$

where $\mathbf{c}_k$ denotes the spreading sequence of user $k$, $x_k$ is the complex modulated data symbol of user $k$, $h_k$ describes the channel coefficient of user $k$ and $\mathbf{n}$ is the additive white Gaussian noise (AWGN) vector with variance $\sigma^2/2$ per component. The model of such a system is shown in Fig. 1, with the source bits being coded and modulated to the symbols $x_k$, which are spread and then transmitted.

The signature sequences are modelled as random sequences with i.i.d. components. With the notation $\mathbf{c}_k = \frac{1}{\sqrt{N}}(c_{k,1}, \ldots, c_{k,N})^T$ for the normalized signatures of users $k = 1, \ldots, K$ we assume that $E[c_{k,i}] = 0$, $E[|c_{k,i}|^2] = 1$ and $E[|c_{k,i}|^4] < \infty$. The detection is done using a linear filter $\mathbf{w}$, such that the data estimate of desired user 1 is $\hat{x}_1 = \mathbf{w}^H \mathbf{r}$. The conventional receiver for CDMA systems is the MF receiver, usually implemented as Rake receiver. However, it is optimal only in presence of white interference, which is, in general, not true. The LMMSE receiver deploys knowledge of the signatures and propagation channels of all active users to suppress the multiple access interference (MAI), thus maximizing the signal-to-interference-and-noise ratio (SINR) of the symbols after the linear filter. For the desired user, its filter coefficients can be easily derived from standard LMMSE equations as e.g. in [6]. They are given by

$$\mathbf{w}_{mmse} = (\mathbf{C}\mathbf{D}\mathbf{C}^H + \sigma^2 \mathbf{I}_K)^{-1} \mathbf{c}_1 h_1, \tag{2}$$

where $\mathbf{C} = [\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_K]$ is the spreading code matrix, and $\mathbf{I}_K$ denotes the $K$-dimensional identity matrix. Matrix

$$\mathbf{D} = diag(E[h_1 h_1^H], \ldots, E[h_K h_K^H]) \tag{3}$$

contains the fading powers of all users. Then, based on the asymptotic analysis, for large CDMA systems, i.e. $K \to \infty$, with a fixed system load $\alpha = K/N$ and $P_k$ being the transmit power of user $k$, the SINR $\beta_1$ of the symbol estimates of

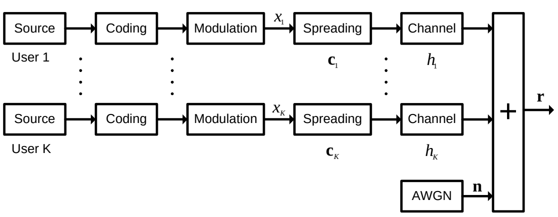


Fig. 1. DS-CDMA uplink: System model of transmitter and channel

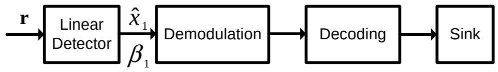


Fig. 2. DS-CDMA uplink: System model of receiver

desired user 1 after the LMMSE receiver can be determined as the solution of the implicit equation

$$\beta_1 = \frac{P_1}{\sigma^2 + \alpha E[I(P_k, P_1, \beta_1)]}, \tag{4}$$

where the interference term is given by $I(P_k, P_1, \beta_1) = P_k P_1/(P_1 + P_k \beta_1)$ [1]. Both interference and noise approach asymptotically, i.e. for large systems, to a Gaussian distribution as proven e.g. in [7]. The SINR $\beta_1$ describes the quality of the symbol estimates $\hat{x}_1$ before demodulation and decoding as shown in Fig. 2. Thus, by means of information theory, the SINR can be considered as the effective SNR of a single user transmission without spreading over an AWGN channel. A similar SINR expression for the MF receiver can be found as well, but in this work it is concentrated on the LMMSE receiver only.

## III. Spectral Efficiency and Ergodic Behavior

An important performance criteria for every communications system is the spectral efficiency $\eta$. For multiuser systems with $K$ equal users, this spectral efficiency is commonly defined as $\eta = KR_b/W$, where $R_b$ is the information bit rate of the individual users and $W$ is the effective system bandwidth. From the assumption of ideal pulse-shaping with respect to the Nyquist criteria we obtain $N = W/R_s$, with $R_s$ being the coded symbol rate. For equal users with SINR $\beta_k = \beta_1$ after the linear detector (see Fig. 2) and rate $R = R_b/R_s$ comprising both modulation and coding, the efficiency is $\eta = RK/N$. For code rate $R_c$ and discrete modulation with a modulation size of $log_2 M$ coded bits per symbol, the rate is $R = R_c log_2 M$, such that $\eta = R_c log_2 M \alpha$. The instantaneous received SNR depends on the fading power $z_k = |h_k|^2$, which has a pdf of $p_z(z_k)$ for all users, such that

$$\frac{P_k}{\sigma^2} = z_k \frac{E_S}{N_0} = z_k R \frac{E_b}{N_0}. \tag{5}$$

Merging these definitions with the results from the asymptotic analysis (4) leads to the following implicit expression for the SINR after the LMMSE receiver

$$\beta_1 = \frac{z_1 R \frac{E_b}{N_0}}{1 + \eta \frac{E_b}{N_0} \int_0^\infty \frac{z_k p_z(z_k) dz_k}{1 + z_k \frac{\beta_1}{z_1}}}. \tag{6}$$

For flat Rayleigh fading $p_z(z) = e^{-z}$ holds. After introducing a fading power normalization to the SINR after the linear filter, i.e. $\beta_n = \beta_1/z_1$, the maximum (ergodic) rate of desired user 1 with coding at capacity is given by

$$R(\beta_n) = \int_0^\infty log_2(1 + \beta_n z_1) p_z(z_1) dz_1. \tag{7}$$

This equation and (6) with $R = R(\beta_n)$ allow to solve the implicit expression for the normalized SINR $\beta_n$ of the desired user. However, a closed-form solution can only be found for the AWGN channel. For practical implementation modulation and coding at the cut-off rate is also of interest, because it can be implemented with reasonable complexity. In comparison to coding at capacity, the spectral efficiency is reduced and also the optimal system load can be changed. Details on the definition, on the ergodic behavior for modulation and coding at the cut-off rate and simulation results for both flat and multipath fading channels can be found in [2] and [5].

## IV. Outage Probability for Gaussian Codes

For a complete analysis of a communications system, the outage probability has to be investigated in addition to the ergodic behavior as well. Outage probability is important for slow fading scenarios, where the channel coherence time is longer or in the same range as a codeword or data block. In this case, outage occurs when due to fading the instantaneous SINR $\beta_1$ of desired user 1 after the linear filter is smaller than a certain threshold $\bar{\beta}$, necessary for reliable transmission at rate $R$. Coding at capacity demands $\bar{\beta} = 2^R - 1$. In general, for flat Rayleigh fading we obtain the outage probability

$$P_{out} = P(\beta_1 < \bar{\beta}) = P\left(z_1 < \frac{\bar{\beta}}{\beta_n}\right), \tag{8}$$

and finally

$$P_{out} = \int_0^{\frac{\bar{\beta}}{\beta_n}} p_z(z_1)dz_1 = 1 - e^{-\frac{\bar{\beta}}{\beta_n}}. \tag{9}$$

This relation is shown in Fig. 3 for the LMMSE receiver for various loads and coding at capacity. The outage probability in Fig. 3 is increased significantly for loads larger than $\alpha = 1$. To understand this behavior the first derivative of the outage probability is computed. For coding at capacity this leads to

$$\frac{dP_{out}}{d\frac{E_b}{N_0}} = e^{-(2^R-1)/\beta_n} \frac{2^R - 1}{-\beta_n^2} \frac{d\beta_n}{d\frac{E_b}{N_0}}. \tag{10}$$

The analytical solution $d\beta_n/d(E_b/N_0)$ of the implicit equation (6) can be determined and is presented in equation (11). The

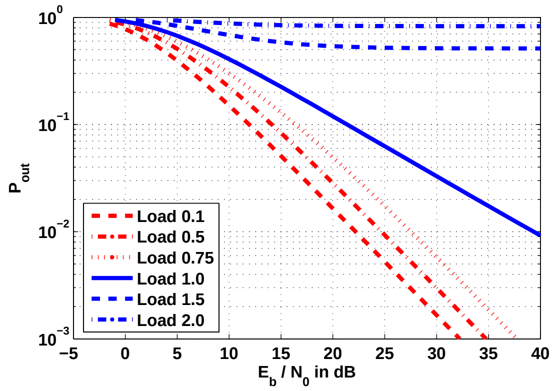


Fig. 3. LMMSE outage probability vs. SNR for coding at capacity ($R = 2$)

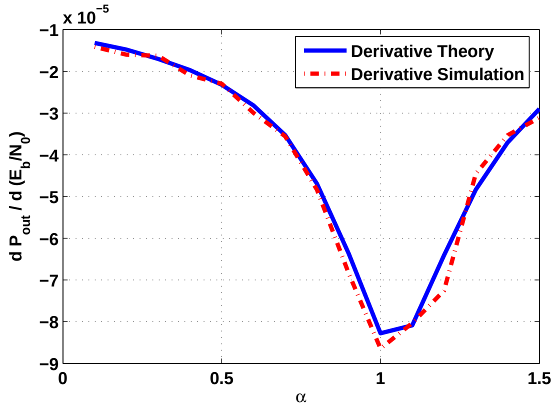


Fig. 4. LMMSE derivative $dP_{out}/d(E_b/N_0)$ vs. load for coding at capacity ($E_b/N_0 = 25dB$ and $R = 1.5$)

parameter $\Psi$ defines the transmit SNR and is given by $\Psi = R\frac{E_b}{N_0}$, such that

$$\frac{d\beta_n}{d\frac{E_b}{N_0}} = \frac{\beta_n^4 R}{\beta_n^4 + 2\beta_n^3\alpha\Psi + \left(\beta_n^2(\alpha-1) - \beta_n\right)\alpha\Psi^2 + B}. \tag{11}$$

Expression $B$ is shown in equation (12), with the incomplete Gamma function defined as $\Gamma(a,b) = \int_a^\infty t^{a-1}e^{-t}dt$, thus

$$\begin{aligned} B \quad = \quad & \alpha e^{\frac{1}{\beta_n}}\Psi\Gamma\left(0,\frac{1}{\beta_n}\right)\left(\Psi - 2\beta_n\left(\beta_n + (\alpha-1)\Psi\right)\right) \\ & + \left(\alpha e^{\frac{1}{\beta_n}}\Psi\Gamma\left(0,\frac{1}{\beta_n}\right)\right)^2. \end{aligned} \tag{12}$$

Equation (10) using (11) and (12) is visualized for different loads $\alpha$, $E_b/N_0 = 25dB$ and rate $R = 1.5$ for coding at capacity in Fig. 4, together with simulation results of a CDMA system with spreading factor $N = 64$. Interestingly, the steepest slope occurs for $\alpha = 1$, which means that the strongest interference suppression capability of the LMMSE receiver occurs at this load. However, the SINR after the LMMSE is lowest for $\alpha \rightarrow 0$, because the remaining interference level is larger for higher loads. This behavior is shown in Fig. 5, where both evaluation of the asymptotic analysis (6) and simulation results ($N = 64$) lead to similar results. It follows that the outage probability, which is proportional to $(-e^{-1/\beta_n})$ as described in equation (9), will increase with the load for a given SNR $E_b/N_0$ and rate $R$, because a higher load decreases the normalized SINR $\beta_n$.

## V. Outage Probability for Coding at Cut-off Rate

As mentioned in section III, channel capacity, in contrast to cut-off rate that is introduced e.g. in [3], is a theoretical limit that practically cannot be implemented. Assuming a rate $R$, the SINR threshold for correct transmission at cut-off rate $R0 = R$ is $\bar{\beta}_{R0} > \bar{\beta}$. Such a stronger restriction on the SINR $\beta_n$ can only be met with higher SNR $E_b/N_0$. Thus, the question has to be answered, which additional SNR is necessary at a certain outage probability constraint to achieve

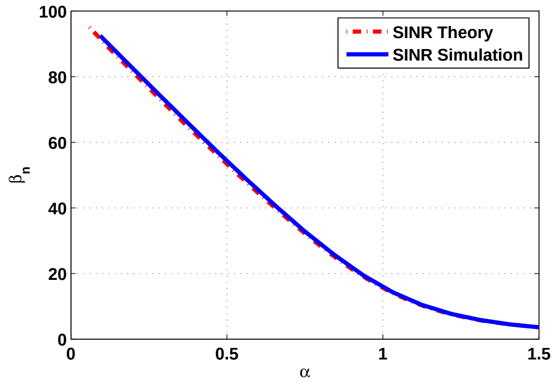


Fig. 5. Normalized SINR after LMMSE receiver vs. load for coding at capacity ($E_b/N_0 = 25dB$ and $R = 1.5$)

the same rate with modulation and coding at the cut-off rate in comparison to coding at capacity. It would be desirable to obtain this information based on the asymptotic analysis for coding at capacity, without the need for a new evaluation or simulation of the whole system for modulation and coding at the cut-off rate.

For this purpose, first the measure of the required SNR has to be introduced. It is defined as the $E_b/N_0$ value, which is necessary to transmit reliably with rate $R$ and load $\alpha$ at the outage probability level $P_{out}$. It is denoted by $(E_b/N_0)_{req}$, and can be derived from equation (6). For the interference integral in (6), it can be shown for the flat Rayleigh fading channel, that it is solved to

$$E[I(\beta_n)] = \int_0^\infty \frac{zp_z(z)dz}{1+z\beta_n} = \frac{\beta_n - e^{1/\beta_n}\Gamma(0,1/\beta_n)}{\beta_n^2}. \quad (13)$$

Now, the minimum required SNR for reliable transmission at rate $R$ can be defined to

$$\left(\frac{E_b}{N_0}\right)_{req} = \frac{1}{R/\beta_n - \alpha R E[I(\beta_n)]}. \quad (14)$$

It should be noted that the normalized SINR depends on the allowed outage probability level through equation (9), i.e. $\beta_n = -\bar{\beta}/ln(1-P_{out})$. Therefore, we need the previously introduced threshold $\bar{\beta}$, which is given by $\bar{\beta} = 2^R - 1$ for coding at capacity. Using the definition of the required SNR (14), we can further define the SNR loss, which occurs for modulation and coding at the cut-off rate in comparison to coding at capacity with the same rate $R$, to

$$SNR_{loss} = \left(\frac{E_b}{N_0}\right)_{req,R0} - \left(\frac{E_b}{N_0}\right)_{req}. \quad (15)$$

Using equations (14) and (13) we obtain for the SNR loss

$$SNR_{loss} = \frac{-\bar{\beta}_{R0}/\big(Rln(1-P_{out})\big)}{1-\alpha\Big(1-\frac{e^{1/\beta_{n,R0}}\Gamma(0,1/\beta_{n,R0})}{\beta_{n,R0}}\Big)} - \frac{-\bar{\beta}/\big(Rln(1-P_{out})\big)}{1-\alpha\Big(1-\frac{e^{1/\beta_n}\Gamma(0,1/\beta_n)}{\beta_n}\Big)}. \quad (16)$$

The value $\bar{\beta}_{R0}$ describes the SNR threshold that is necessary to achieve a rate $R$ with modulation and coding at the cut-off rate. It can be easily obtained from the inverse of the cut-off rate definition presented e.g. in [3]. Usually, the SNR $E_b/N_0$ is expressed in dB. Considering the SNR loss (16) in the dB scale as well leads to

$$SNR_{loss,dB} = 10log_{10}\frac{\bar{\beta}_{R0}}{\bar{\beta}} - 10log_{10}D, \quad (17)$$

where the term $D$ is given by

$$D = \frac{1-\alpha\Big(1-\frac{e^{1/\beta_{n,R0}}\Gamma(0,1/\beta_{n,R0})}{\beta_{n,R0}}\Big)}{1-\alpha\Big(1-\frac{e^{1/\beta_n}\Gamma(0,1/\beta_n)}{\beta_n}\Big)}. \quad (18)$$

The value of the term $D$ depends on the load $\alpha$ and the normalized SINR $\beta_n$ and $\beta_{n,R0}$, which themselves depend on the rate $R$ and the outage probability $P_{out}$. It turns out, that for common intervals of the values $\alpha$ and $R$ it holds that

$$\lim_{P_{out}\ll 1} D = 1. \quad (19)$$

Because the outage probability is usually chosen quite small, the SNR loss in dB can be approximated by

$$SNR_{loss,dB} \approx 10log_{10}\frac{\bar{\beta}_{R0}}{\bar{\beta}}. \quad (20)$$

Equation (20) provides a very good rule-of-thumb to obtain the outage performance for modulation and coding at the cut-off rate from the asymptotic analysis evaluation for coding at capacity. Of course, this also holds vice versa, and allows for fast efficiency estimation during system design. The high accuracy of the SNR loss approximation in equation (20) is shown in Fig. 6 for outage probability $P_{out} = 0.01$ and system load $\alpha = 0.5$.

Another important aspect for system design is the maximum spectral efficiency for some outage probability constraint. From the definition in section III, it is determined by the highest rate and corresponding load product that fulfills (6) and the outage requirements given through $P_{out}$, $\bar{\beta}$ and $\bar{\beta}_{R0}$,

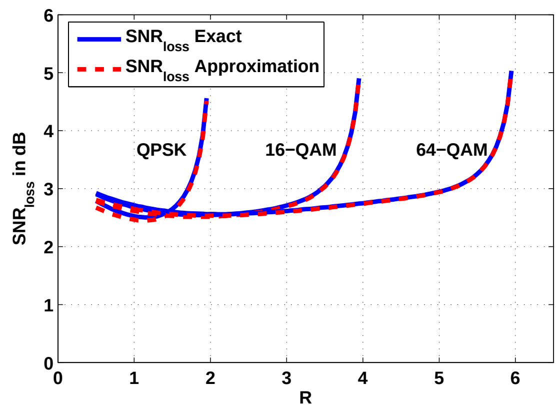


Fig. 6. LMMSE SNR loss between coding at cut-off rate and coding at capacity vs. rate ($P_{out} = 0.01$ and $\alpha = 0.5$)

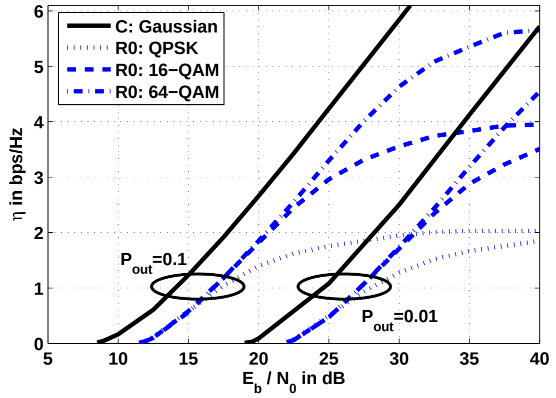


Fig. 7. LMMSE maximum outage spectral efficiency vs. SNR

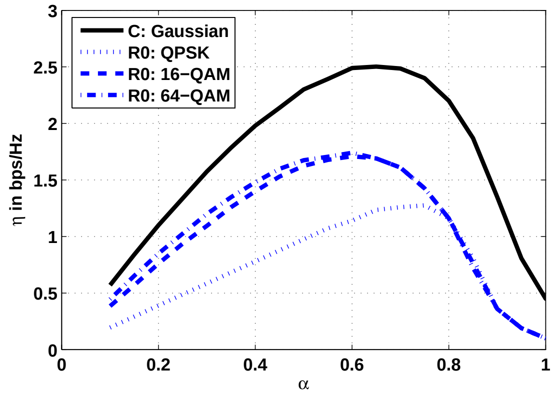


Fig. 8. LMMSE outage spectral efficiency vs. load ($P_{out} = 0.01$ and $E_b/N_0 = 30dB$)

respectively. It is shown in Fig. 7 for coding at capacity and for coding and QPSK, 16-QAM and 64-QAM modulation at the cut-off rate. Similar to the ergodic system efficiency shown in [5], by using the LMMSE receiver the outage efficiency is increased with higher order modulation schemes. However, this is only true for large SNR. In presence of low SNR, transmission with QPSK modulation results in the maximum efficiency. Nevertheless, the loss due to higher order modulation is small in this case.

Fig. 7 does not provide any information about the load that maximizes the efficiency, which would be of interest for system design. This optimal load depends on the SNR, modulation scheme and outage probability. To find it, the efficiency has to be investigated separately for each parameter set. In Fig. 8 an example is provided for $E_b/N_0 = 30dB$ and outage probability $P_{out} = 0.01$. Obviously, in this case the optimum load for QPSK modulation is $\alpha = 0.75$, and for both 16-QAM and 64-QAM constellations $\alpha = 0.6$. The corresponding rates can be obtained from the same diagram by using equation $\eta = \alpha R$. The maximum efficiencies for all parameter sets form the outage efficiency shown in Fig. 7.

## VI. Conclusions

In this paper the outage probability of large DS-CDMA systems with random spreading codes was investigated. The results are based on the asymptotic analysis for the uplink, which was considered for the linear minimum mean-squared-error receiver and the flat Rayleigh fading channel. It could be shown that this receiver has its strongest interference suppression capability for load $\alpha = 1$. However, due to the remaining interference level, at a particular signal-to-noise ratio the outage probability level is lowest for small loads. Furthermore, it was derived that the results for the outage probability with coding at capacity can be easily extended to the relevant case of discrete modulation and coding at the cut-off rate. The outage efficiency, describing the maximum system spectral efficiency achievable at a certain outage probability constraint, behaves similar to the case of ergodic fading but at a higher signal-to-noise ratio. Thus, higher order modulation schemes should be used to increase the efficiency for sufficiently high signal-to-noise ratios. For system design, it also has to be considered that the corresponding optimum system load varies for different modulation schemes, which was exemplarily shown in the paper. Future work could be done to investigate the outage behavior for multi cell systems. Based on the results for the ergodic behavior, it can be assumed that for the ideal linear minimum mean-squared-error receiver the conclusions will not substantially differ from those of single cell systems. In contrast, for suboptimal implementations the impact of multiple cells might be significant.

## References

[1] D. N. C. Tse and S. V. Hanly, "Linear Multiuser Receivers: Effective Interference, Effective Bandwidth and User Capacity," *IEEE Transactions on Information Theory*, vol. 45, no. 2, pp. 641–657, March 1999.

[2] C.-H. Yih and E. Geraniotis, "Spectral Efficiency and Cufoff Rate of Random Spreading BICM-Based CDMA Signals With MMSE Multiuser Receivers," *IEEE Communications Letters*, vol. 8, no. 3, pp. 129–131, March 2004.

[3] O. Prätor and G. Fettweis, "On the Spectral Efficiency of Large CDMA Systems Using Higher Order Modulation Schemes," in *IEEE PIMRC 2003*, vol. 2, September 2003, pp. 1282–1286.

[4] O. Ertug, B. Sayrac, B. Baykal, and M. D. Yücel, "Total Outage Capacity of Randomly-Spread Coded-CDMA with Linear Multiuser Receivers over Multipath Fading Channels," in *Proceedings of the Eighth IEEE ISCC*, 2003.

[5] O. Prätor and G. Fettweis, "On the Spectral Efficiency of DS-CDMA Systems with Higher Order Modulation in Presence of Fading," in *IEEE WCNC 2005*, March 2005.

[6] U. Madhow and M. L. Honig, "MMSE Interference Suppression for Direct-Sequence Spread-Spectrum CDMA," *IEEE Transactions on Communications*, vol. 42, no. 12, pp. 3178–3188, December 1994.

[7] D. Guo, S. Verdú, and L. K. Rasmussen, "Asymptotic Normality of Linear Multiuser Receiver Outputs," *IEEE Transactions on Information Theory*, vol. 48, no. 12, pp. 3080–3095, December 2002.