Recovery Guarantees for Slow Time Phase Coded Waveforms in MIMO Radar

Nithin Sugavanam, Emre Ertin
Department of ECE
The Ohio State University, USA

Laura Anitori and Wim van Rossum
Radar Technology, TNO
The Hague, The Netherlands

***Abstract*—Motivated by the widespread adoption of Multi-input Multi-output (MIMO) antenna systems, radar systems which can transmit independent waveforms on multiple antennas coupled with independently sampled receive arrays have been suggested for improving detection, spatial resolution, and clutter suppression capabilities. Improved spatial diversity enabled by simultaneous transmission of multiple transmit antennas can only be realized if incoherent waveforms are utilized in transmit. Recently, slow time coding across the pulses has been suggested to obtain quasi-orthogonal transmit waveforms. The slow time coding could take the form of uncorrelated uniformly distributed random phase streams or randomly staggered phase ramps. In this paper, we show that measurements obtained with a MIMO radar employing with slow time phase codes are equivalent to structured projections of the measurements of a SIMO radar with an extended receive array matching to the virtual co-array of the MIMO-radar. Based on this modeling strategy we obtain recovery guarantees for the undersampled system as a function of the number of antennas, pulse repetition frequency and Doppler bandwidth assuming a sparse scene of reflectors. Our results establish the limits of antenna diversity of MIMO radar systems with slow time phase coded waveforms.**

I. Introduction

Motivated by the widespread adoption of Multi-input Multi-output (MIMO) antenna systems, radar systems which can transmit independent waveforms on multiple antennas coupled with independently sampled receive arrays have been suggested for improving detection, parameter estimation and clutter suppression capabilities [1], [2], [3] . Although many traditional multi-antenna radar concepts such as phased-array receive beamforming, STAP, and interferometry can be seen as special cases of MIMO radar, the distinct advantage of a multi-antenna radar system with independent transmit waveforms is the increased number of degrees of freedom leading to improved resolution [4], [3], [5], detection [6], and parameter estimation [7].

Increased degrees of freedom afforded by transmitting multiple waveforms can only be realized if quasi-orthogonal waveforms are utilized at each transmitter. Multi-access schemes commonly used in wireless communication: Time Division Multiple Access (TDMA), Frequency Division Multiple Access (FDMA) and Code Division Multiple Access (CDMA) have been suggested to synthesize quasi-orthogonal waveforms for MIMO Radar on a pulse-to-pulse basis. However, each of these concepts has practical limitations when applied to radar system design. TDMA results in reduced average transmitted power and lower pulse repetition frequency as the different transmitters have to be interleaved in time. Application of FDMA leads to decorrelation of the clutter returns and loss of coherency. Similarly, CDMA leads to an increase in the clutter subspace dimension [8]. As Pulse-Doppler systems process multiple pulses across a coherent processing interval to perform target detection, an alternative strategy for synthesizing quasi-orthogonal waveforms is to modulate the transmit waveform across the slow time (pulse-to-pulse) using phase codes [8]. Since slow-time coding spreads energy in the Doppler domain, this strategy is also dubbed as Doppler Division Multiple Access (DDMA). In particular, Rabideau [9] suggested modulation with phase ramps across slow time, effectively shifting the support of the transmitted waveform in Doppler domain. For a sufficiently high PRF rate, Doppler frequency offsets applied to each transmitter can be chosen to avoid overlap between the support of different transmit waveform returns. However, uniformly spaced Doppler frequency offsets result in ambiguities in Doppler and introduction of new blind speeds when stationary clutter is present. Therefore, perturbing the order and spacing of Doppler offsets – frequency dithering– or applying phase modulations across antennas –phase dithering– was proposed [9], [10] to remove angle-Doppler ambiguities. Recently, random slow time phase modulation across both antennas and pulses (Slow-Time Code Division Multiple Access (ST-CDMA)) was studied in [11]. Both dithered DDMA and random ST-CDMA waveform designs spread sidelobe energy across the Doppler-Angle domain and avoid perfect ambiguities at a cost of high computational complexity recovery.

Detection of targets in Doppler-Angle domain using ST-CDMA waveforms can be posed as a sparse signal recovery problem where a signal vector $\mathbf{x}$ denoting the amplitudes of the targets across the Doppler-angle domain is to be recovered from under-determined set of linear measurements encoded by the sensing matrix $\boldsymbol{A}(\overline{\phi})$ corrupted by additive noise $\mathbf{n}$, where $\overline{\phi}$ represents the vector of phase code symbols.

$$\mathbf{y} = \boldsymbol{A}(\overline{\phi})\mathbf{x} + \mathbf{n},$$

In this case, the recovery problem is underdetermined as the number of measurements scale with the size of the receive array whereas the number of unknowns scale with the size of the virtual array- a product of the transmit and receive arrays. If $\mathbf{x}$ is sparse in that there are relatively few entries in $\mathbf{x}$ which are non-zero, then $\mathbf{x}$ can be recovered from an underdetermined set of measurements [12], [13]. The sparse recovery problem is stable in the presence of noise and approximately sparse signals, where the reconstruction error scales proportionally with the approximation error (a difference between the signal and its closest sparse approximation) and measurement noise [14]. Recovery from an incomplete set of measurements under sparse target scene assumption has been successfully applied to a variety of radar problems [15].

In this paper, we show that measurements obtained with a MIMO radar employing slow time phase codes can be expressed as:

$$\mathbf{y} = \mathbf{\Phi}(\overline{\phi})\mathbf{\Psi}\mathbf{x} + \mathbf{n},$$

where $\mathbf{\Psi}$ is the square sensing matrix encoding a 2D FFT operation corresponding to the fully sampled system with antenna locations given by the virtual co-array of the MIMO system and $\mathbf{\Phi}(\overline{\phi})$ are linear projections to the reduced dimension measurement space. Next, analyzing the coherence structure of the rows of $\mathbf{\Phi}$ and columns of $\mathbf{\Psi}$, we obtain recovery guarantees for the undersampled system as a function of the number of antennas, pulse repetition frequency and Doppler bandwidth assuming a sparse scene of reflectors. These results establish the limits of antenna diversity of MIMO radar systems with slow time coded waveforms.

II. System Model

We consider the setup with N_T transmitters and N_R receivers. Each transmitter employs a common waveform denoted by $s(t)$ with a random phase shift for each pulse. The total number of pulses per transmitter is set as N_p. The transmitted waveform from all transmitters is given by

$$T_x(t) = \sum_{p=1}^{N_p}\sum_{i=1}^{N_T} s(t-pT)\exp\left(j\phi(p,i)\right), \tag{1}$$

where $\phi(p,i) \sim Unif(0,2\pi)$. We consider N_R receivers with M samples per pulse per receiver. The received waveform due to K point scattering centers is given by

$$Rx(t,r,p) = \sum_{k=1}^{K} a_k \sum_{i=1}^{N_T} \exp(j\phi(p,i) + j\frac{2\pi}{\lambda}(i+r-2)\theta_k)\times \exp\left(j\frac{4T\pi}{\lambda}pv_k\right) s_p(t-\tau_k), \tag{2}$$

where $t = [t_1,\cdots,t_M]$, $r = 0,1,\cdots,N_r-1$, $p = 0,1,\cdots,N_p-1$, a_k is the scattering coefficient, θ_k is the angle of arrival with respect to the center of the array geometry, v_k is the Doppler velocity of the target, $s_p(t)$ is the common transmitted pulse. We propose to solve the delay-Doppler-angle estimation problem. This problem can be solved approximately in the range domain by using the matched filter for the transmitted waveform, which is common to all the transmitters. The angle-Doppler velocity estimation problem is subsequently solved for each range bin. The received signal at a particular range bin after pulse-compression is given by

$$y(r,p) = \sum_{k=1}^{K} a_k \sum_{i=1}^{N_T} \exp(j\phi(p,i) + j\frac{2\pi}{\lambda}(i+r-2)\theta_k)\times \exp\left(j\frac{4T\pi}{\lambda}pv_k\right). \tag{3}$$

The angle of arrival domain $(-1,1)$ is discretized with a resolution $\Delta\theta = 2/\left(N_T N_R\right)$ and the Doppler velocity domain $\Delta_v = \lambda/\left(2TN_p\right)$, where λ is the wavelength of the transmitted signal by assuming a narrow-band signal is transmitted and T is the pulse repetition interval. The angular and Doppler velocity bins are denoted by

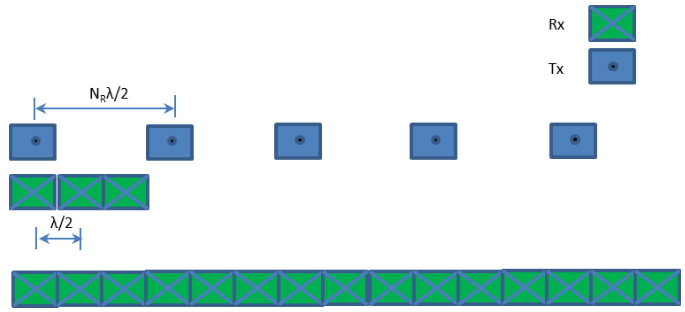

Fig. 1. Illustration of the MIMO system with $N_T = 5$ transmitters and $N_R = 3$ receivers. An equivalent array with receiver elements having the same aperture length of the MIMO system is also shown. We note that the system with receiver elements is fully sampled in the angle domain.

$$\boldsymbol{\theta} = \left[\theta_1,\cdots,\theta_{N_\theta}\right], N_\theta = N_T N_R,$$
$$\mathbf{v} = \left[v_1,\cdots,v_{N_P}\right],$$

where $\theta_1 = -1$, $\theta_{N_\theta} = 1$, $v_1 = -v_{max}$, and $v_{N_P} = v_{max}$ such that v_max is the maximum unambiguous Doppler velocity. The resulting signal due to K scattering centers in the discretized angle and Doppler domain for a given range bin is

$$y(r,p) = \sum_{v=v_1}^{v_{N_p}}\sum_{\theta=\theta_1}^{\theta_{N_\theta}} x_{(v,\theta)} \sum_{i=1}^{N_T}\exp(j\frac{2\pi}{\lambda}(id_T + rd_R)\theta) \times \exp\left(j(\phi(p,i) + \frac{4T\pi}{\lambda}pv)\right), \tag{4}$$

where $d_T = N_R\lambda/2$ and $d_R = \lambda/2$. Figure 1 shows a particular example of the MIMO system with $N_T = 5$ and $N_R = 3$. We observe that this system is partially sampled with only N_R measurements as opposed to the fully sampled receiver array with $N_T N_R$ receiver elements. Combining all the measurements into $\mathbf{y} \in \mathbb{C}^{N_R N_P}$ and the scattering coefficients $\mathbf{x} \in \mathbb{C}^{N_\theta N_P}$ we get

$$\mathbf{y} = \sum_{i=1}^{N_T} \boldsymbol{\alpha}_{R,T_i}(\boldsymbol{\theta}) \otimes \left(\mathbf{H}_i \boldsymbol{\alpha}_D\left(\mathbf{v}\right)\right) + \mathbf{n}, \tag{5}$$

where $\otimes$ represents the Kronecker product, $\boldsymbol{\alpha}_{R,T_i}(\boldsymbol{\theta}) \in \mathbb{C}^{N_R \times N_R N_T}$ represents the terms in the array factor for all the receivers and spatial frequencies and a fixed transmitter Tx_i, $\mathbf{H}_i \in \mathbb{C}^{N_p \times N_p}$ is given random phase added to each pulse for a particular transmitter, and $\boldsymbol{\alpha}_D\left(\mathbf{v}\right) \in \mathbb{C}^{N_p \times N_p}$ is the response in the Doppler velocity domain for each of the

Doppler velocity bins in $\mathbf{v}$. We give the details of the below

$$\boldsymbol{\alpha}_{R,T_i}(\boldsymbol{\theta}) = \begin{pmatrix} \exp\left(j\hat{\theta}_1\left[d_T i\right]\right) & \cdot & \exp\left(j\hat{\theta}_{N_\theta}\left[d_T i\right]\right) \\ \vdots & \cdot & \vdots \\ \exp\left(j\hat{\theta}_1\left[d_T i + d_R N_R\right]\right) & \cdot & \exp\left(j\hat{\theta}_{N\theta}\left[d_T i + d_R N_R\right]\right) \end{pmatrix},$$

$$\mathbf{H}_i = diag\left(\exp\left(j\phi\left(1,i\right)\right), \cdots, \exp\left(j\phi\left(N_p,i\right)\right)\right),$$

$$\boldsymbol{\alpha}_D(\mathbf{v}) = \begin{pmatrix} 1 & \cdots & 1 \\ \exp\left(\frac{4\pi v_1}{\lambda}\right) & \cdots & \exp\left(\frac{4\pi v_{N_P}}{\lambda}\right) \\ \vdots & \cdots & \vdots \\ \exp\left(\frac{4\pi(N_p-1)v_1}{\lambda}\right) & \cdots & \exp\left(\frac{4\pi(N_p-1)v_{N_P}}{\lambda}\right) \end{pmatrix}$$

$$\phi\left(p,i\right) = \begin{cases} Unif\left(0,2\pi\right), & \text{Random Slow-time CDMA} \\ 2\pi T f_i p, & \text{DDMA} \\ 2\pi T \hat{f}_i p + \phi_i, & \text{Dithered DDMA} \end{cases} \tag{6}$$

$\hat{\theta}_i = \frac{2\pi\theta_i}{\lambda}$, $f_i = i/\left(TN_T\right)$ is the frequency offset used in DDMA, $\hat{f}_i$ is the i^{th} element of the randomly permuted realization of $\mathbf{f} = [f_1, f_2, \cdots, f_{N_T}]$ used in Dithered DDMA, and ϕ_i is the randomly chosen phase for each transmitter in Dithered DDMA.

Next, we show that the measurements in (5) are a projected version of the measurement model of the fully sampled receiver array with $N_R N_T$ array elements and obtain the expression for the projection operator. The measurements from the fully sampled receiver array are

$$\mathbf{y}_{full} = \boldsymbol{\alpha}_R(\boldsymbol{\theta}) \otimes \boldsymbol{\alpha}_D(\mathbf{v}), \tag{7}$$

where $\boldsymbol{\alpha}_R(\boldsymbol{\theta}) \in \mathbb{C}^{N_T N_R \times N_T N_R}$ is the array factor term, $\mathbf{y}_{full} \in \mathbb{C}^{N_R N_T N_P}$ is the fully sampled measurements . The measurements in (7) can be expanded in terms of the parameters described in (6).

$$\mathbf{y}_{full} = \begin{pmatrix} \boldsymbol{\alpha}_{R_1,T_1}(\boldsymbol{\theta}) \otimes \boldsymbol{\alpha}_D\left(\mathbf{v}\right) \\ \boldsymbol{\alpha}_{R_1,T_2}(\boldsymbol{\theta}) \otimes \boldsymbol{\alpha}_D\left(\mathbf{v}\right) \\ \vdots \\ \boldsymbol{\alpha}_{R_1,T_{N_T}}(\boldsymbol{\theta}) \otimes \boldsymbol{\alpha}_D\left(\mathbf{v}\right) \\ \vdots \\ \boldsymbol{\alpha}_{R_{N_R},T_{N_T}}(\boldsymbol{\theta}) \otimes \boldsymbol{\alpha}_D\left(\mathbf{v}\right) \end{pmatrix} \mathbf{x}, \tag{8}$$

where $\boldsymbol{\alpha}_{R_1,T_i}(\boldsymbol{\theta}) \in \mathbb{C}^{1 \times N_R N_T}$ is the first row in $\boldsymbol{\alpha}_{R,T_i}(\boldsymbol{\theta})$. We note that the operator

$$\boldsymbol{\Psi} = \begin{pmatrix} \boldsymbol{\alpha}_{R_1,T_1}(\boldsymbol{\theta}) \otimes \boldsymbol{\alpha}_D\left(\mathbf{v}\right) \\ \boldsymbol{\alpha}_{R_1,T_2}(\boldsymbol{\theta}) \otimes \boldsymbol{\alpha}_D\left(\mathbf{v}\right) \\ \vdots \\ \boldsymbol{\alpha}_{R_1,T_{N_T}}(\boldsymbol{\theta}) \otimes \boldsymbol{\alpha}_D\left(\mathbf{v}\right) \\ \vdots \\ \boldsymbol{\alpha}_{R_{N_R},T_{N_T}}(\boldsymbol{\theta}) \otimes \boldsymbol{\alpha}_D\left(\mathbf{v}\right) \end{pmatrix}, \tag{9}$$

is the flattened version of the 2-D Fourier transform operator. The measurement model in (5) can be expressed as follows using (9) and using the fact that $\mathbf{H}_i$ is a diagonal matrix

$$\mathbf{y} = \boldsymbol{\Phi}\boldsymbol{\Psi}\mathbf{x} + \mathbf{n}, \tag{10}$$

$$\boldsymbol{\Phi} = \begin{pmatrix} \mathbf{H}_1 \cdots \mathbf{H}_{N_T} & \mathbf{0} & \cdots \\ \mathbf{0} & \mathbf{H}_1 \cdots \mathbf{H}_{N_T} & \mathbf{0} \\ \vdots & \cdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{H}_1 \cdots \mathbf{H}_{N_T} \end{pmatrix}, \tag{11}$$

where $\boldsymbol{\Psi} \in \mathbb{C}^{N_R N_P \times N_R N_T N_P}$ is the projection operator relating to the fully sampled receiver and the slow-time modulated MIMO system. This also generalizes the measurements obtained from MIMO systems employing DDMA and dithereed DDMA waveforms. In the next section, we analyze these systems.

III. Recovery guarantees

To establish theoretical guarantees we make use of the incohernce in measurement operator and the underlying basis functions. We use results from [16], [12], [17] to establish the non-uniform recovery guarantees for the MIMO system employing random slow-time CDMA, dithered DDMA and DDMA waveforms. The normalize the matrix $\boldsymbol{\Phi}$ to have row norm as $\sqrt{N_T N_R N_P}$. Therefore, $\bar{\boldsymbol{\Phi}} = \boldsymbol{\Phi}\sqrt{N_R N_T}$.

We evaluate the coherence defined by $\mu(\bar{\boldsymbol{\Phi}}, \boldsymbol{\Psi}) = \max_{k,l}\left|\langle\bar{\Phi}_k, \Psi_l\rangle\right|$, $\bar{\Phi}_k$ is the k^{th} row or measurement and Ψ_l is the l^{th} column of the basis.

Lemma 1: For the measurement operator representing slow-time CDMA defined as $\bar{\boldsymbol{\Phi}} \in \mathbb{C}^{N_R N_P \times N_R N_T N_P}$ and the basis system $\boldsymbol{\Psi} \in \mathbb{C}^{N_R N_T N_P \times N_R N_T N_P}$, the mutual coherence is given by

$$\mu_{ST-CDMA}(\bar{\boldsymbol{\Phi}}, \boldsymbol{\Psi}) \leq \sqrt{\log\left(N_R N_P N_T \delta^{-1}\right)}, \tag{12}$$

with probability δ.

Proof: The inner-product between the k^{th} measurment coming from receiver r and pulse p and the l^{th} basis vector denoting the angle θ_u and Doppler v_m is given by

$$\langle\bar{\Phi}_k, \Psi_l\rangle = \frac{1}{\sqrt{N_T}}\sum_{i=1}^{N_T}\exp\left(-j\phi\left(p,i\right)\right)\exp\left(\frac{2\pi\theta_u}{\lambda}\left(d_T i\right)\right) \exp\left(\frac{2\pi\theta_u}{\lambda}\left(d_R r\right)\right)\exp\left(\frac{4\pi v_l T}{\lambda}\right). \tag{13}$$

Each term in the summation is bounded by $\frac{1}{\sqrt{N_T}}$ and $E\left(\exp\left(-j\phi\left(p,i\right)\right)\right) = 0$ by definition. Using Hoeffding's inequality we get a tail bound on the inner-product

$$P\left(\left|\langle\bar{\Phi}_k, \Psi_l\rangle\right| \geq t\right) \leq 2\exp\left(-2t^2\right), \tag{14}$$

$$P\left(\max_{k,l}\left|\langle\bar{\Phi}_k, \Psi_l\rangle\right| \geq t\right) \leq 2N_R N_T N_P \exp\left(-2t^2\right). \tag{15}$$

Using $t = \sqrt{\log\left(N_R N_T N_P \delta^{-1}\right)}$, the tail probability is bounded by δ ■

Lemma 2: For the measurement operator representing DDMA defined as $\bar{\boldsymbol{\Phi}} \in \mathbb{C}^{N_R N_P \times N_R N_T N_P}$ and the basis system $\mathbf{\Psi} \in \mathbb{C}^{N_R N_T N_P \times N_R N_T N_P}$, the mutual coherence is given by

$$\mu_{DDMA}(\bar{\mathbf{\Phi}}, \mathbf{\Psi}) = \sqrt{N_T}. \tag{16}$$

Proof: The inner-product between the k^{th} measurment coming from receiver r and pulse p and the l^{th} basis vector denoting the angle θ_u and Doppler v_m is given by

$$\begin{aligned} \langle \bar{\Phi}_k, \Psi_l \rangle = &\frac{1}{\sqrt{N_T}} \sum_{i=1}^{N_T} \exp\left(-j\phi\left(p,i\right)\right) \exp\left(\frac{2\pi\theta_u}{\lambda}\left(d_T i\right)\right) \\ &\exp\left(\frac{2\pi\theta_u}{\lambda}\left(d_R r\right)\right) \exp\left(\frac{4\pi v_l T}{\lambda}\right), \\ = &\frac{1}{\sqrt{N_T}} \sum_{i=1}^{N_T} \exp\left(-j\frac{2\pi i p}{N_T}\right) \exp\left(\frac{2\pi\theta_u}{\lambda}\left(d_T i\right)\right) \\ &\exp\left(\frac{2\pi\theta_u}{\lambda}\left(d_R r\right)\right) \exp\left(\frac{4\pi v_l T}{\lambda}\right). \end{aligned} \tag{17}$$

The angle $\theta_u = 2u/\left(N_T N_R\right)$ is used to obtain the expression as follows

$$\begin{aligned} \langle \bar{\Phi}_k, \Psi_l \rangle = &\frac{1}{\sqrt{N_T}} \sum_{i=1}^{N_T} \exp\left(-j\frac{2\pi i\left(p-u\right)}{N_T}\right) \\ &\exp\left(\frac{2\pi\theta_u}{\lambda}\left(d_R r\right)\right) \exp\left(\frac{4\pi v_l T}{\lambda}\right), \\ = &\begin{cases} \sqrt{N_T} \exp\left(\frac{2\pi\theta_u}{\lambda}\left(d_T i\right)\right) \exp\left(\frac{4\pi v_l T}{\lambda}\right), & \textbf{if } p = u \\ 0, & \textbf{otherwise}. \end{cases} \end{aligned} \tag{18}$$

Consequently, we get $\mu_{DDMA}(\bar{\mathbf{\Phi}}, \mathbf{\Psi}) = \sqrt{N_T}$. ■

Lemma 3: For the measurement operator representing dithered DDMA defined as $\bar{\mathbf{\Phi}} \in \mathbb{C}^{N_R N_P \times N_R N_T N_P}$ and the basis system $\mathbf{\Psi} \in \mathbb{C}^{N_R N_T N_P \times N_R N_T N_P}$, the mutual coherence is given by

$$\mu_{Dit-CDMA}(\bar{\mathbf{\Phi}}, \mathbf{\Psi}) \leq \sqrt{\log\left(N_R N_P N_T \delta^{-1}\right)}, \tag{19}$$

with probability δ.

The random phase-term added to each receiver leads to the same result obbtained in lemma 1.

We consider the matrix $\mathbf{U} = \bar{\mathbf{\Phi}}\mathbf{\Psi}$ to establish the isotropy and incoherence condition to obtain the recovery guarantees.

$$\mathbf{U} = \begin{bmatrix} \mathbf{u}_1^T \\ \vdots \\ \mathbf{u}_{N_R N_P}^T \end{bmatrix} = \mathbf{\Phi}\mathbf{\Psi} = \begin{bmatrix} \Phi_1^T \mathbf{\Psi} \\ \vdots \\ \Phi_{N_R N_P}^T \mathbf{\Psi} \end{bmatrix} \tag{20}$$

Lemma 4: For the matrix $\mathbf{U} \in \mathbb{N_R N_P} \times \mathbb{N_R N_T N_P}$ denoting the random slow-time CDMA scheme and dithered DDMA, we have

$$E\left(\mathbf{u_i}\mathbf{u_i}^*\right) = \mathbf{I}, \qquad \textbf{Isotropy property} \tag{21}$$

$$\mu\left(\mathbf{U}\right) \leq \log\left(N_R N_P N_T \delta^{-1}\right) \qquad \textbf{Incoherence property} \tag{22}$$

Proof: The vector u_i is expressed as follows

$$\begin{aligned} &\mathbf{u}_i = \mathbf{\Psi}^T \phi_1, \\ &E\left(\mathbf{u}_i \mathbf{u}_i^*\right) = \mathbf{\Psi}^{\boldsymbol{T}} E\left(\phi_1 \phi_1^*\right) \mathbf{\Psi}^{\boldsymbol{T}^*}. \end{aligned} \tag{23}$$

Since there are N_T complex exponential phase terms that are chosen at random and independently we can further reduce the number of terms to N_T. This leads to the following expression evaluated at some arbitrary pulse p and receiver r

$$\begin{aligned} E\left(\mathbf{u}_i \mathbf{u}_i^*\right) = &\left[\mathbf{\Psi}^{\boldsymbol{T}}_{\boldsymbol{T_1,r,p}} \cdots \mathbf{\Psi}^{\boldsymbol{T}}_{\boldsymbol{T_{N_T},r,p}}\right] E\left(\hat{\phi}_1 \hat{\phi}_1^*\right) \\ &\left[\mathbf{\Psi}^{\boldsymbol{T}}_{\boldsymbol{T_1,r,p}} \cdots \mathbf{\Psi}^{\boldsymbol{T}}_{\boldsymbol{T_{N_T},r,p}}\right]^*, \end{aligned} \tag{24}$$

where $\hat{\phi_1} \in \mathbb{C}^{N_T}$ is the set of phases used by all the transmitters in the first pulse, $\left[\mathbf{\Psi}^{\boldsymbol{T}}_{\boldsymbol{T_1,r,p}} \cdots \mathbf{\Psi}^{\boldsymbol{T}}_{\boldsymbol{T_{N_T},r,p}}\right] \in \mathbb{C}^{N_R N_P N_T \times N_T}$ is the selected subset from the 2-D FFT operator. By exploiting the independence of the random variables and utilizing the transmitter spacing of $d_T = N_R\lambda/2$ we obtain the covariance matrix as identity matrix.

The coherence for matrix $\mathbf{U}$ is given by $\mu\left(\mathbf{U}\right) \leq \mu(\bar{\mathbf{\Phi}}, \mathbf{\Psi})^2$, which was obtained in lemma 1, 2. ■

The DDMA system does not satisfy the isotropy property. The incoherence of the system $\mu_{DDMA}(\mathbf{U}) = N_T$ is also much higher than the Dithered DDMA and the random CDMA. The isotropy property plays an integral part in the existence of the left inverse of the measurement operator as the number of measurements increase. Therefore, the recovered parameters from DDMA system suffers from ambiguities as opposed to the dithered DDMA and random CDMA waveforms.

Using the result stated in Theorem 1.2 in [17], we obtain a condition on the number of measurements required to recover a fixed $S-$ sparse signal using the LASSO program.

Lemma 5: Given the measurment operator $\mathbf{U} \in \mathbb{C}^{N_R N_P \times N_R N_T N_P}$ and an arbitrary signal $\mathbf{x} \in \mathbb{C}^{N_R N_P N_T}$ and the noise variance σ_n, then the LASSO program with regularization $\lambda = 10\sigma_n\sqrt{\log\left(N_R N_T N_P\right)}$ successfully recovers the signal $\mathbf{x}$ with probability $1-6/\left(N_R N_T N_P\right)-6\exp\left(-\beta\right)$ such that the ℓ_2 reconstruction error is given by

$$\begin{aligned} &\left\|\hat{\mathbf{x}} - \mathbf{x}\right\|^2 \leq \min_{1\leq s\leq \bar{s}} C\left(1+\alpha\right) \times \\ &\left[\frac{\left\|\mathbf{x} - \mathbf{x_s}\right\|_{\ell_2}}{s} + \sigma\sqrt{\frac{s\log\left(N_R N_T N_P\right)}{m}}\right] \end{aligned} \tag{25}$$

if the number of measurements

$$m \geq C_\beta \log\left(N_R N_T N_P \delta^{-1}\right)^2 \bar{s},$$

and C is a universal constant, $\alpha = \sqrt{\frac{(1+\beta)s\log^5(N_R N_T N_P)}{m}}$, and $\mathbf{x_s}$ is best $s - sparse$ approximation.

IV. Conclusion

In this paper, we analyzed the performance MIMO pulse radar systems employing slow time coding across the pulses to synthesize quasi-orthogonal transmit waveforms across the coherent processing interval. We have shown that measurements obtained with an ST-CDMA MIMO can be seen as structured projections of the measurements of a SIMO radar with an extended receive array matching to the virtual co-array of the MIMO-radar. Based on this modeling strategy we obtained recovery guarantees for the undersampled system as a function of the number of antennas, pulse repetition frequency and Doppler bandwidth assuming a sparse scene of reflectors. In the extended version of the paper, we will present simulation results to analyze the phase transition of successful recovery as predicted by our theoretical results.

REFERENCES

[1] E. Fishler, A. Haimovich, R. Blum, D. Chizhik, L. Cimini, and R. Valenzuela, "MIMO radar: An idea whose time has come," in *Radar Conference, 2004. Proceedings of the IEEE*. IEEE, 2004, pp. 71–78.

[2] E. Fishler, A. Haimovich, R. Blum, R. Cimini, D. Chizhik, and R. Valenzuela, "Performance of MIMO radar systems: Advantages of angular diversity," in *Signals, Systems and Computers, 2004. Conference Record of the Thirty-Eighth Asilomar Conference on*, vol. 1. IEEE, 2004, pp. 305–309.

[3] N. H. Lehmann, A. M. Haimovich, R. S. Blum, and L. Cimini, "High resolution capabilities of MIMO radar," in *Signals, Systems and Computers, 2006. ACSSC'06. Fortieth Asilomar Conference on*. IEEE, 2006, pp. 25–30.

[4] D. Bliss and K. Forsythe, "Multiple-input multiple-output (mimo) radar and imaging: degrees of freedom and resolution," in *Signals, Systems and Computers, 2004. Conference Record of the Thirty-Seventh Asilomar Conference on*, vol. 1. IEEE, 2003, pp. 54–59.

[5] G. San Antonio, D. R. Fuhrmann, and F. C. Robey, "MIMO radar ambiguity functions," *IEEE Journal of Selected Topics in Signal Processing*, vol. 1, no. 1, pp. 167–177, 2007.

[6] E. Fishler, A. Haimovich, R. S. Blum, L. J. Cimini, D. Chizhik, and R. A. Valenzuela, "Spatial diversity in radars—models and detection performance," *IEEE Transactions on Signal Processing*, vol. 54, no. 3, pp. 823–838, 2006.

[7] J. Li, P. Stoica, L. Xu, and W. Roberts, "On parameter identifiability of MIMO radar," *IEEE Signal Processing Letters*, vol. 14, no. 12, pp. 968–971, 2007.

[8] D. Bliss, K. Forsythe, S. Davis, G. Fawcett, D. Rabideau, L. Horowitz, and S. Kraut, "GMTI MIMO radar," in *Waveform Diversity and Design Conference, 2009 International*. IEEE, 2009, pp. 118–122.

[9] D. J. Rabideau, "Doppler-offset waveforms for MIMO radar," in *Radar Conference (RADAR), 2011 IEEE*. IEEE, 2011, pp. 965–970.

[10] ——, "MIMO radar waveforms and cancellation ratio," *IEEE Transactions on Aerospace and Electronic Systems*, vol. 48, no. 2, pp. 1167–1178, 2012.

[11] W. van Rossum and L. Anitori, "Doppler ambiguity resolution using random slow- time code division multiple access MIMO radar dopple ambguity resolution using random slow-time code division multiple access MIMO radar with sparse signal processing," in *Radar Conference, 2018 IEEE*. IEEE, 2018, pp. 441–446.

[12] E. Candes and J. Romberg, "Sparsity and incoherence in compressive sampling," *Inverse problems*, vol. 23, no. 3, p. 969, 2007.

[13] D. L. Donoho, "Compressed sensing," *IEEE Transactions on information theory*, vol. 52, no. 4, pp. 1289–1306, 2006.

[14] E. J. Candes, J. K. Romberg, and T. Tao, "Stable signal recovery from incomplete and inaccurate measurements," *Communications on pure and applied mathematics*, vol. 59, no. 8, pp. 1207–1223, 2006.

[15] L. C. Potter, E. Ertin, J. T. Parker, and M. Cetin, "Sparsity and compressed sensing in radar imaging," *Proceedings of the IEEE*, vol. 98, no. 6, pp. 1006–1020, 2010.

[16] J. Romberg, "Compressive sensing by random convolution," *SIAM J. Img. Sci.*, vol. 2, no. 4, pp. 1098–1128, Nov. 2009.

[17] E. J. Candes and Y. Plan, "A probabilistic and RIPless theory of compressed sensing," *IEEE Transactions on Information Theory*, vol. 57, no. 11, pp. 7235–7254, Nov. 2011.