The Misspecified and Semiparametric lower bounds and their application to inference problems with Complex Elliptically Symmetric (CES) distributed data

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Outline – Part I: The Misspecified CRB

1. Motivations and history
2. Formal description of a misspecified problem
3. The Misspecified Cramér-Rao Bound (MCRB)
4. The Mismatched Maximum Likelihood (MML) estimator
5. Simple examples: variance and power estimation
6. Radar applications:
   - Direction of Arrivals (DOAs) estimation
   - Scatter matrix estimation for Complex Elliptically Simmetric (CES) distributed data

Outline – Part II: The Semiparametric CRB

1. Why semiparametric models?
2. CRB for parametric models with finite-dimensional nuisance parameters:
   - ❑ Classical approach,
   - ❑ "Hilbert-space-based" approach.
3. Extension to semiparametric models.
4. Semiparametric interpretation of Real and Complex ES Distributions.
5. Examples.

Why study lower bounds (LBs)?

- **Lower Bounds (LBs)** provide a benchmark of comparison for the performance of any **unbiased** estimator.
- If the performance of a certain estimator achieves a relevant LB, then it is established that no other *unbiased* estimators can do better.
- A LB is said to be **tight** if it reasonably predicts the performance of the Maximum Likelihood (ML) estimator.
- The **Cramér-Rao Bound (CRB)** is an *asymptotically tight* bound for any *unbiased* estimator.

Model misspecification (1/2)

- **Classical "matched" assumption**: the true data model and the model assumed to derive the estimation algorithm are the same, i.e. <u>the model is *correctly specified*</u>.

- All the results on the ML estimator and the CRB rely on this implicit assumption.

- However, much evidence from engineering practice shows that this assumption is often violated.

- **Model misspecification**: the assumed data model (i.e. the data pdf) differs from the true model.

Model misspecification (2/2)

- There are two main reasons for model misspecification:
  1. An **imperfect knowledge** of the true data model that leads to a wrong specification of the data pdf.
  2. The true data model is known but it is **too involved** to pursue the optimal "matched" estimator.
- One may be forced (scenario 1) or may prefer (scenario 2) to derive an estimator by assuming a **simpler** but **misspecified** data model.
- This suboptimal procedure may lead to some degradation in the overall system performance.

Misspecified Lower Bounds (MLBs)

- Assessing the impact of **model misspecification** on the estimation performance is crucial to guarantee the reliability of the (mismatched) estimator.
- **Misspecified LBs** allow the assumed and true models to differ, yet establishing performance limits on estimation error covariance.
- **Misspecified LBs** indicate how the model mis-specification affects estimation performance.

Some history and recent applications (1/3)

- Properties of the **Mismatched ML estimator**: Huber [1] (1967), Akaike [2] (1972) and White [3] (1982).

- Generalization to the **Bayesian** framework: Berk [5] (1966), Bunke and Milhaud [6] (1998), Richmond and Basu [7] (2016).

- **Cramér-Rao inequality** under **model misspecification**: Vuong [4] (1986), Richmond and Horowitz [8] (2015), S. Fortunati, F. Gini, M. S. Greco [11] (2016).

- A tutorial introduction to the **Misspecified CRB** has been proposed in:

S. Fortunati, F. Gini, M. S. Greco and C. D. Richmond, "Performance Bounds for Parameter Estimation under Misspecified Models: Fundamental Findings and Applications," *IEEE Signal Processing Magazine*, vol. 34, no. 6, pp. 142-157, Nov. 2017.

Some history and recent applications (2/3)

- Recent applications of **misspecified LBs**:

1. Direction of Arrivals estimation in sensor arrays [8] and MIMO radars [9].
2. Covariance matrix estimation in non-Gaussian data ([10], [11], [12] and [23]).
3. Radar-communication systems coexistence [7].
4. Waveform parameter estimation in the presence of uncertainty in the propagation model [13].
5. Time of Arrivals (ToA) estimation problem for UWB signals in the presence of interferences [14].

Some history and recent applications (3/3)

6. Sparse Bayesian estimation [27].

7. Spectral estimation [28].

8. Estimation of hybrid sinusoidal FM-PPS signals [29].

Description of a misspecified problem (1/3)

- Let $\mathbf{x}_m \in \mathbb{C}^N$ be an N-dimensional measurement vector.

- Let $p_X(\mathbf{x}_m) \in \mathcal{P}$ its true probability density function (pdf) belonging to a possibly non-parametric model $\mathcal{P}$.

- To characterize the statistical behavior of $\mathbf{x}_m$, we adopt a parametric pdf, say $f_X(\mathbf{x}_m | \boldsymbol{\theta})$ with $\boldsymbol{\theta} \in \Theta \subset \mathbb{R}^d$.

- The assumed pdf $f_X(\mathbf{x}_m | \boldsymbol{\theta})$ is implicitly assumed to belong to a *parametric* model:

$$\mathcal{F} = \left\{ f_X \middle| f_X(\mathbf{x}_m | \boldsymbol{\theta}), \boldsymbol{\theta} \in \Theta \subset \mathbb{R}^d \right\}$$

Description of a misspecified problem (2/3)

- The classical "matched" estimation theory requires the existence of a parameter vector $\bar{\boldsymbol{\theta}} \in \Theta$ such that:

$$p_X(\mathbf{x}_m) = f_X(\mathbf{x}_m | \bar{\boldsymbol{\theta}}) \text{ or, equivalently, } p_X(\mathbf{x}_m) \in \mathcal{F}$$

- If this assumption is violated, the model is **misspecified**.

Model misspecification:

$f_X(\mathbf{x}_m | \boldsymbol{\theta})$ differs from $p_X(\mathbf{x}_m)$ for every $\boldsymbol{\theta} \in \Theta$.

$$p_X(\mathbf{x}_m) \neq f_X(\mathbf{x}_m | \boldsymbol{\theta}), \forall \boldsymbol{\theta} \in \Theta \quad \longrightarrow \quad \mathcal{P} \not\subset \mathcal{F}$$

Description of a misspecified problem (3/3)

- Suppose to collect M *independent, identically distributed* (i.i.d.) N-dimensional measurement vectors:

$$\mathbf{x} = \{\mathbf{x}_m\}_{m=1}^{M}, \quad \mathbf{x}_m \sim p_X(\mathbf{x}_m), \; m = 1,\dots,M$$

- Due to the independence assumption, the true joint pdf of the dataset **x** is:

$$p_X(\mathbf{x}) = \prod_{m=1}^{M} p_X(\mathbf{x}_m)$$

- The assumed joint pdf of the dataset **x** is:

$$f_X(\mathbf{x}|\boldsymbol{\theta}) = \prod_{m=1}^{M} f_X(\mathbf{x}_m|\boldsymbol{\theta})$$

Two fundamental questions

- This misspecified scenario raises two main questions:

  1. Is it still possible to derive LBs on the error covariance of any mismatched estimator of the parameter vector $\boldsymbol{\theta}$?

  2. How will the classical statistical properties of an estimator, e.g. *unbiasedness*, *consistency* and *efficiency*, change in this misspecified model framework?

- The **Misspecified Cramér-Rao Bound (MCRB)** provides answers to these questions.

Regular models

- As for the classical CRB, in order to guarantee the existence of the MCRB, some regularity conditions need to be imposed.

- Specifically, the assumed parametric model $\mathcal{F}$ has to be **regular** with respect to (wrt) the true one $\mathcal{P}$.

- Among the rather technical assumptions that $\mathcal{F}$ has to satisfy to be regular wrt $\mathcal{P}$, the most important is:

  - Existence and uniqueness of the pseudo-true parameter vector $\boldsymbol{\theta}_0$

The pseudo-true parameter vector

- If $\mathcal{F}$ is regular wrt $\mathcal{P}$, then there exist a unique interior point $\boldsymbol{\theta}_0$ of Θ, such that:

$$\boldsymbol{\theta}_0 \triangleq \arg\min_{\boldsymbol{\theta}\in\Theta}\left\{-E_p\left\{\ln f_X\left(\mathbf{x}_m\,|\,\boldsymbol{\theta}\right)\right\}\right\} = \arg\min_{\boldsymbol{\theta}\in\Theta}\left\{D\left(p_X \,\|\, f_X\right)\right\}$$

where $E_p\{\cdot\}$ is the expectation operator wrt the true pdf $p_X(\mathbf{x}_m)$ and

$$D\left(p_X \,\|\, f_X\right) \triangleq \int \ln\left(\frac{p_X(\mathbf{x}_m)}{f_X(\mathbf{x}_m\,|\,\boldsymbol{\theta})}\right) p_X(\mathbf{x}_m)\,d\mathbf{x}_m$$

is the **Kullback-Leibler divergence (KLD)** between the true pdf and the assumed pdf.

Generalization of the FIM

- The vector $\boldsymbol{\theta}_0$ is the point the minimizes the KLD between the true pdf $p_X(\mathbf{x}_m)$ and the assumed pdf $f_X(\mathbf{x}_m|\boldsymbol{\theta})$.

- Let $\mathbf{A}_{\boldsymbol{\theta}_0}$ be the matrix whose entries are given by:

$$\left[\mathbf{A}_{\boldsymbol{\theta}_0}\right]_{ij} \triangleq E_p\left\{ \frac{\partial^2}{\partial\theta_i\partial\theta_j} \ln f_X\left(\mathbf{x}_m|\boldsymbol{\theta}\right)\Big|_{\boldsymbol{\theta}=\boldsymbol{\theta}_0} \right\}$$

If $\mathcal{F}$ is regular wrt $\mathcal{P}$, then the matrix $\mathbf{A}_{\boldsymbol{\theta}_0}$ is non singular

- In the matrix $\mathbf{A}_{\boldsymbol{\theta}_0}$ we can recognize a sort of generalization of the Fisher Information Matrix (FIM).

Another generalization of the FIM

- The second generalization of the FIM is given by:

$$\left[\mathbf{B}_{\boldsymbol{\theta}_0}\right]_{ij} \triangleq E_p\left\{ \left.\frac{\partial \ln f_X\left(\mathbf{x}_m|\boldsymbol{\theta}\right)}{\partial \theta_i}\right|_{\boldsymbol{\theta}=\boldsymbol{\theta}_0} \cdot \left.\frac{\partial \ln f_X\left(\mathbf{x}_m|\boldsymbol{\theta}\right)}{\partial \theta_j}\right|_{\boldsymbol{\theta}=\boldsymbol{\theta}_0} \right\}.$$

- If the model is **misspecified**, i.e. it does not exists a parameter vector $\overline{\boldsymbol{\theta}} \in \Theta$ such that $p_X(\mathbf{x}_m) = f_X(\mathbf{x}_m|\overline{\boldsymbol{\theta}})$, then: $\mathbf{B}_{\boldsymbol{\theta}_0} \neq -\mathbf{A}_{\boldsymbol{\theta}_0}$

The MS-unbiasedness property

- The Misspecified (MS)-unbiasedness property generalizes the classical notion of unbiased estimators.

- In the misspecified model framework, unbiasedness is defined wrt the pseudo-true parameter vector $\boldsymbol{\theta}_0$.

- Let $\mathbf{x} = \{\mathbf{x}_m\}_{m=1}^{M}$ be the available dataset and let $\hat{\boldsymbol{\theta}}(\mathbf{x})$ be an estimator derived by assuming the misspecified model $\mathcal{F}$.

$\hat{\boldsymbol{\theta}}(\mathbf{x})$ is an **MS-unbiased** estimator iff:

$$E_p\left\{\hat{\boldsymbol{\theta}}(\mathbf{x})\right\} = \int \hat{\boldsymbol{\theta}}(\mathbf{x}) p_X(\mathbf{x}) d\mathbf{x} = \boldsymbol{\theta}_0$$

The Misspecified Cramér-Rao Bound (1/2)

- Let $\mathcal{F}$ be a misspecified parametric model that is regular wrt a true model $\mathcal{P}$.

Theorem 1 (Vuong 1986): Let $\hat{\boldsymbol{\theta}}(\mathbf{x})$ be an MS-unbiased estimator derived under the misspecified model $\mathcal{F}$ from a dataset $\mathbf{x}=\{\mathbf{x}_m\}_{m=1}^{M}$. Then, for every possible $p_X(\mathbf{x}_m)\in\mathcal{P}$:

$$\mathbf{C}_p\left(\hat{\boldsymbol{\theta}}(\mathbf{x}),\boldsymbol{\theta}_0\right)\geq\frac{1}{M}\mathbf{A}_{\boldsymbol{\theta}_0}^{-1}\mathbf{B}_{\boldsymbol{\theta}_0}\mathbf{A}_{\boldsymbol{\theta}_0}^{-1}\triangleq\mathrm{MCRB}\left(\boldsymbol{\theta}_0\right)$$

where

$$\mathbf{C}_p\left(\hat{\boldsymbol{\theta}}(\mathbf{x}),\boldsymbol{\theta}_0\right)\triangleq E_p\left\{\left(\hat{\boldsymbol{\theta}}(\mathbf{x})-\boldsymbol{\theta}_0\right)\left(\hat{\boldsymbol{\theta}}(\mathbf{x})-\boldsymbol{\theta}_0\right)^T\right\}$$

is the error covariance matrix of the mismatched estimator.

[4] Q. H. Vuong, “Cramér-Rao bounds for misspecified models”, *Working paper 652, Division of the Humanities and Social Sciences* , Caltech, October 1986.

The Misspecified Cramér-Rao Bound (2/2)

- The MCRB is a local lower bound (LB) on the error variance of any MS-unbiased estimator of the pseudo-true parameter vector $\boldsymbol{\theta}_0$.

- MCRB for constrained estimation problem [20]:

  S. Fortunati, F. Gini, M. S. Greco, "The Constrained Misspecified Cramér-Rao Bound," *IEEE Signal Process. Letters*, Vol. 23, No. 5, pp. 718-721, May 2016.

- Extension to complex parameters [21]:

  S. Fortunati, "Misspecified Cramér-Rao Bounds for Complex Unconstrained and Constrained Parameters," *EUSIPCO 2017*, Kos, Greece, 28 Aug. 2017–2 Sept. 2017.

MCRB vs CRB

- If the model is **correctly specified**, i.e. if there exists a parameter vector $\bar{\boldsymbol{\theta}} \in \Theta$, such that $p_X(\mathbf{x}_m) = f_X(\mathbf{x}_m | \bar{\boldsymbol{\theta}})$, then:

$$\boldsymbol{\theta}_0 = \bar{\boldsymbol{\theta}} \text{ and } \mathbf{B}_{\boldsymbol{\theta}_0} = -\mathbf{A}_{\boldsymbol{\theta}_0} = -\mathbf{A}_{\bar{\boldsymbol{\theta}}} \quad \Rightarrow \quad \mathrm{MCRB} = \mathrm{CRB}$$

$$E_p\left\{\left(\hat{\boldsymbol{\theta}}(\mathbf{x}) - \bar{\boldsymbol{\theta}}\right)\left(\hat{\boldsymbol{\theta}}(\mathbf{x}) - \bar{\boldsymbol{\theta}}\right)^T\right\} \geq -\frac{1}{M}\mathbf{A}_{\bar{\boldsymbol{\theta}}}^{-1} = \mathrm{MCRB}\left(\bar{\boldsymbol{\theta}}\right) = \mathrm{CRB}\left(\bar{\boldsymbol{\theta}}\right)$$

The misspecified framework is consistent with the classical matched theory!

The Mismatched ML estimator (MML) (1/3)

- As before, suppose to collect M *independent, identically distributed* (i.i.d.) N-dimensional measurement vectors:

$$\mathbf{x} = \{\mathbf{x}_m\}_{m=1}^{M}, \quad \mathbf{x}_m \sim p_X(\mathbf{x}_m), m = 1, \ldots, M$$

- The log-likelihood function for the dataset **x** under a misspecified model $\mathcal{F}$ is given by:

$$L_M(\boldsymbol{\theta}) \triangleq \frac{1}{M} \sum_{m=1}^{M} \ln f_X(\mathbf{x}_m | \boldsymbol{\theta}), \quad \mathbf{x}_m \sim p_X(\mathbf{x}_m)$$

- The **MML estimator** is the point that maximizes $L_M(\boldsymbol{\theta})$:

$$\hat{\boldsymbol{\theta}}_{MML}(\mathbf{x}) \triangleq \underset{\boldsymbol{\theta} \in \Theta}{\arg\max}\, L_M(\boldsymbol{\theta}) = \underset{\boldsymbol{\theta} \in \Theta}{\arg\max} \sum_{m=1}^{M} \ln f_X(\mathbf{x}_m | \boldsymbol{\theta})$$

The Mismatched ML estimator (MML) (2/3)

Theorem 2 (Huber 1967, White 1982): Under suitable regularity conditions, it can be shown that:

$$\hat{\boldsymbol{\theta}}_{MML}(\mathbf{x}) \underset{M\to\infty}{\overset{a.s.}{\rightarrow}} \boldsymbol{\theta}_0.$$

Moreover:

$$\sqrt{M}\left(\hat{\boldsymbol{\theta}}_{MML}(\mathbf{x})-\boldsymbol{\theta}_0\right) \underset{M\to\infty}{\overset{d}{\sim}} \mathcal{N}\left(\mathbf{0},\mathbf{A}_{\boldsymbol{\theta}_0}^{-1}\mathbf{B}_{\boldsymbol{\theta}_0}\mathbf{A}_{\boldsymbol{\theta}_0}^{-1}\right).$$

Huber "sandwich" covariance matrix = MCRB

- $\underset{M\to\infty}{\overset{a.s.}{\rightarrow}}$ indicates the *almost sure* (*a.s.*) convergence.
- $\underset{M\to\infty}{\overset{d}{\sim}}$ indicates the convergence *in distribution*.

The Mismatched ML estimator (MML) (3/3)

- The MML estimator is asymptotically MS-unbiased and its asymptotic error covariance is equal to the MCRB, i.e. it is an *asymptotically efficient* estimator wrt the MCRB.

- Consistent with the *classical* "matched" ML estimator:

If the model $\mathcal{F}$ is correctly specified, i.e. if there exists

$$\bar{\boldsymbol{\theta}} \in \Theta \text{ such that: } p_X(\mathbf{x}_m) = f_X(\mathbf{x}_m | \bar{\boldsymbol{\theta}})$$

then:

$$\hat{\boldsymbol{\theta}}_{ML}(\mathbf{x}) \underset{M \to \infty}{\overset{a.s.}{\rightarrow}} \boldsymbol{\theta}_0 = \bar{\boldsymbol{\theta}}$$

CRB

$$\sqrt{M}\left(\hat{\boldsymbol{\theta}}_{ML}(\mathbf{x}) - \bar{\boldsymbol{\theta}}\right) \underset{M \to \infty}{\overset{d}{\sim}} \mathcal{N}\left(\mathbf{0}, -\mathbf{A}_{\bar{\boldsymbol{\theta}}}^{-1}\right)$$

Consistent estimate of the MCRB

- Let us define the following data-dependent matrices:

$$[\mathbf{A}_M(\boldsymbol{\theta})]_{ij} \triangleq M^{-1}\sum_{m=1}^{M}\frac{\partial^2 \ln f_X(\mathbf{x}_m|\boldsymbol{\theta})}{\partial\theta_i\partial\theta_j}$$

$$[\mathbf{B}_M(\boldsymbol{\theta})]_{ij} \triangleq M^{-1}\sum_{m=1}^{M}\frac{\partial \ln f_X(\mathbf{x}_m|\boldsymbol{\theta})}{\partial\theta_i}\cdot\frac{\partial \ln f_X(\mathbf{x}_m|\boldsymbol{\theta})}{\partial\theta_j}$$

$$\mathbf{C}_M(\boldsymbol{\theta}) \triangleq [\mathbf{A}_M(\boldsymbol{\theta})]^{-1}\mathbf{B}_M(\boldsymbol{\theta})[\mathbf{A}_M(\boldsymbol{\theta})]^{-1}$$

- It can be shown (see [3, Theo 3.2]) that:

$$\left[\mathbf{C}_M(\hat{\boldsymbol{\theta}}_{MML})\right]_{i,j} \underset{M\to\infty}{\overset{a.s.}{\to}} \left[\mathbf{C}_{\boldsymbol{\theta}_0}\right]_{i,j} = \left[\mathrm{MCRB}(\boldsymbol{\theta}_0)\right]_{i,j} \quad \forall i,j = 1,\ldots,\dim(\boldsymbol{\theta}_0)$$

A test for model misspecification

- It is possible to infer from the collected dataset whether or not the assumed model $\mathcal{F}$ is correctly specified.
- Recall that, if $\mathcal{F}$ is correctly specified, then $\mathbf{B}_{\boldsymbol{\theta}} = -\mathbf{A}_{\boldsymbol{\theta}}$.
- Since $\mathbf{A}_{\boldsymbol{\theta}}$ and $\mathbf{B}_{\boldsymbol{\theta}}$ are unobservable, we can exploit their sample estimate to implement a composite hypothesis testing:

$$\begin{cases} H_0: & \mathbf{A}_M(\hat{\boldsymbol{\theta}}_{MML}) + \mathbf{B}_M(\hat{\boldsymbol{\theta}}_{MML}) = \mathbf{0} \\ H_1: & \mathbf{A}_M(\hat{\boldsymbol{\theta}}_{MML}) + \mathbf{B}_M(\hat{\boldsymbol{\theta}}_{MML}) \neq \mathbf{0} \end{cases}$$

- A Wald-type test can be derived to discriminate between the two hypotheses: *correct specification* (H_0) *vs model misspecification* (H_1) (see [3, Sec. 4]).

Example 1: Variance estimation (1/6)

- **Problem**: we want to estimate the variance of a Gaussian data set in the presence of misspecified mean value, e.g. we erroneously assume that the mean value is zero.

- True data set:

$$\mathbf{x} = \{x_m\}_{m=1}^{M}, \quad x_m \sim p_X(x_m) \equiv \mathcal{N}(\bar{\mu}, \bar{\sigma}^2), \quad \bar{\mu} \neq 0$$

- Assumed data model:

$$\mathcal{F} = \left\{ f_X \middle| f_X(x_m|\theta) \equiv \mathcal{N}(0,\theta) \; \forall \theta \in \mathbb{R}^+ \right\}$$

- Note that, as long as $\bar{\mu} \neq 0$,

$$p_X(x_m) \notin \mathcal{F}$$

Example 1: Variance estimation (2/6)

- Is the misspecified model $\mathcal{F}$ regular wrt $p(x_m)$?

- We have to check if:

1. there exists the pseudo-true parameter θ_0;
2. the matrix $\mathbf{A}_{\boldsymbol{\theta}_0}$ is non singular.

- The KLD can be expressed as:

$$D\left(p_X \| f_X\right) = \frac{\bar{\mu}^2}{2\theta} + \frac{1}{2}\left(\frac{\bar{\sigma}^2}{\theta} - 1 - \ln\frac{\bar{\sigma}^2}{\theta}\right)$$

- Its minimum point, i.e. the pseudo-true parameter, *exists* and is *unique*:

$$\theta_0 = \bar{\sigma}^2 + \bar{\mu}^2, \quad \bar{\mu} \neq 0$$

Example 1: Variance estimation (3/6)

- The scalar A_{θ_0} can be evaluated as:

$$A_{\theta_0} \triangleq E_p\left\{ \left.\frac{\partial^2 \ln f_X(x_m \mid \theta)}{\partial \theta^2}\right|_{\theta=\theta_0} \right\} = \frac{1}{2\theta_0^2} - \frac{1}{\theta_0^3} E_p\left\{x_m^2\right\} = -\frac{1}{2\theta_0^2}$$

that is always different from zero, since $\bar{\mu} \neq 0, \bar{\sigma}^2 \in \mathbb{R}^+$.

Since θ_0 exists and is unique and the scalar $A_{\theta_0} \neq 0$, then $\mathcal{F}$ is regular wrt $p\left(x_m\right)$.

Example 1: Variance estimation (4/6)

- The scalar B_{θ_0} can be evaluated as:

$$B_{\theta_0} \triangleq E_p\left\{ \left(\frac{\partial \ln f_X(x_m|\theta)}{\partial \theta} \right)^2 \Bigg|_{\theta=\theta_0} \right\} = \frac{2\bar{\sigma}^4 + 4\bar{\sigma}^2\bar{\mu}^2}{4\theta_0^4}$$

- By collecting the previous results, the MCRB is given by:

$$\mathrm{MCRB}(\theta_0) = \frac{2\bar{\sigma}^4}{M} + \frac{4\bar{\sigma}^2\bar{\mu}^2}{M} \geq \mathrm{CRB}(\bar{\sigma}^2) = \frac{2\bar{\sigma}^4}{M}$$

- The MCRB is always greater than the CRB, as expected.

- The MCRB equates the CRB when there is no model misspecification, i.e. when $\bar{\mu} = 0$.

Example 1: Variance estimation (5/6)

- Regarding the MML estimator, we have that:

$$\hat{\theta}_{MML}(\mathbf{x}) = \frac{1}{M}\sum_{m=1}^{M} x_m^2 \underset{M\to\infty}{\overset{a.s.}{\longrightarrow}} \theta_0 = \bar{\sigma}^2 + \bar{\mu}^2 \neq \bar{\sigma}^2$$

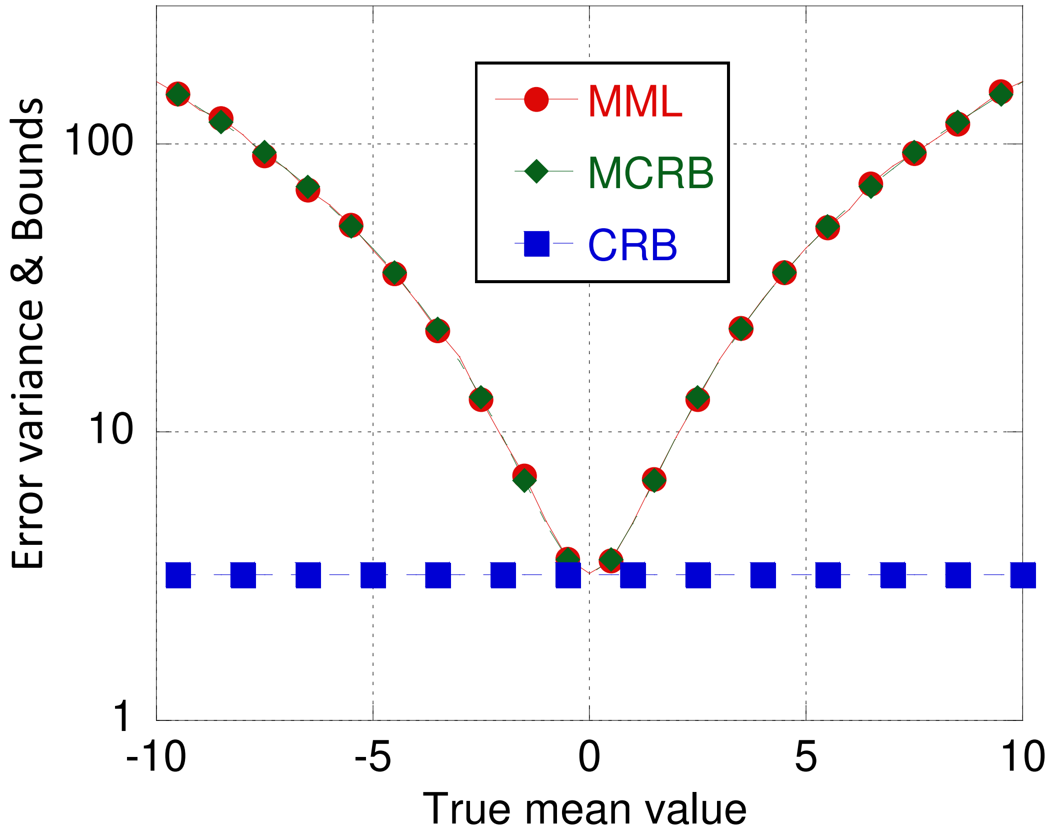
- The MML estimator is not consistent, since it converges to a value different from the true variance.

- However, the MML is MS-unbiased, since:

$$E_p\left\{\hat{\theta}_{MML}(\mathbf{x})\right\} = E_p\left\{\frac{1}{M}\sum_{m=1}^{M} x_m^2\right\} = \bar{\sigma}^2 + \bar{\mu}^2 = \theta_0$$

- Hence, the error variance of this MML estimator is lower bounded by the MCRB.

Example 1: Variance estimation (6/6)



$\bar{\sigma}^2 = 4$

$M = 10$

- MCRB = CRB when $\bar{\mu} = 0$: matched case.
- The MML estimator is efficient wrt the MCRB.

Example 2: Power estimation (1/6)

- **Problem**: we want to estimate the statistical power of a multivariate Gaussian data set under misspecification of the correlation structure.

- True dataset:

$$\mathbf{x} = \{\mathbf{x}_m\}_{m=1}^{M} \qquad p_X(\mathbf{x}_m) \equiv \mathcal{N}\left(\mathbf{0}, \bar{\sigma}^2 \mathbf{\Sigma}\right) \in \mathcal{P} \qquad [\mathbf{\Sigma}]_{ij} = \rho^{|i-j|}$$

- Assumed data model:

$$\mathcal{F} = \left\{ f_X \mid f_X(\mathbf{x}_m \mid \theta) \equiv \mathcal{N}\left(\mathbf{0}, \theta \mathbf{I}_N\right) \ \forall \theta \in \mathbb{R}^+ \right\}$$

- Note that, as long as $\rho \neq 0$,

$$p_X\left(\mathbf{x}_m\right) \notin \mathcal{F}$$

Example 2: Power estimation (2/6)

- Is the misspecified model $\mathcal{F}$ regular wrt $p(\mathbf{x}_m)$?

- As before, we have to check if:

1. there exists the pseudo-true parameter θ_0;
2. the scalar $A_{\boldsymbol{\theta}_0}$ is non singular.

- The KLD can be expressed as:

$$D\left(p_X \| f_X\right) = \frac{1}{2}\left[\operatorname{tr}(\theta^{-1}\bar{\sigma}^2\mathbf{\Sigma}) - N + \ln\theta - \ln\det(\bar{\sigma}^2\mathbf{\Sigma})\right]$$

- Its minimum point, i.e. the pseudo-true parameter, *exists* and is *unique*: $\theta_0 = \bar{\sigma}^2$

Example 2: Power estimation (3/6)

- The scalar A_{θ_0} can be evaluated as:

$$A_{\theta_0} \triangleq E_p\left\{ \frac{\partial^2 \ln f_X(\mathbf{x}_m|\theta)}{\partial \theta^2}\bigg|_{\theta=\theta_0} \right\} = \frac{N}{2\theta_0^2} - \frac{1}{\theta_0^3} E_p\left\{\mathbf{x}_m^T \mathbf{x}_m\right\} = -\frac{N}{2\bar{\sigma}^4}$$

that is always different from zero, since $\bar{\sigma}^2 \in \mathbb{R}^+$.

Since θ_0 exists and is unique and the scalar $A_{\theta_0} \neq 0$, then $\mathcal{F}$ is regular wrt $p(\mathbf{x}_m)$.

Example 2: Power estimation (4/6)

- The scalar B_{θ_0} can be evaluated as:

$$B_{\theta_0} = \frac{N\theta_0^2 + E_p\left\{(\mathbf{x}_m^T\mathbf{x}_m)^2\right\} - 2N\theta_0 E_p\left\{\mathbf{x}_m^T\mathbf{x}_m\right\}}{4\theta_0^4} = \frac{\mathrm{tr}\left(\boldsymbol{\Sigma}^2\right)}{2\bar{\sigma}^4}$$

- By collecting the previous results, the MCRB is given by:

$$\mathrm{MCRB}(\theta_0) = \mathrm{MCRB}(\bar{\sigma}^2) = \frac{2\bar{\sigma}^4}{MN^2}\mathrm{tr}\left(\boldsymbol{\Sigma}^2\right) \geq \mathrm{CRB}(\bar{\sigma}^2) = \frac{2\bar{\sigma}^4}{MN}$$

- The MCRB is always greater than the CRB, as expected.

- MCRB = CRB when there is no model misspecification, i.e. when $\rho = 0$.

Example 2: Power estimation (5/6)

- Regarding the MML estimator, we have that:

$$\hat{\theta}_{MML}(\mathbf{x}) = \frac{1}{M}\sum_{m=1}^{M}\frac{\mathbf{x}_m^T\mathbf{x}_m}{N} \xrightarrow[M\to\infty]{a.s.} \theta_0 = \frac{E_p\{\mathbf{x}_m^T\mathbf{x}_m\}}{N} = \frac{\bar{\sigma}^2}{N}\mathrm{tr}(\mathbf{\Sigma}) = \bar{\sigma}^2$$

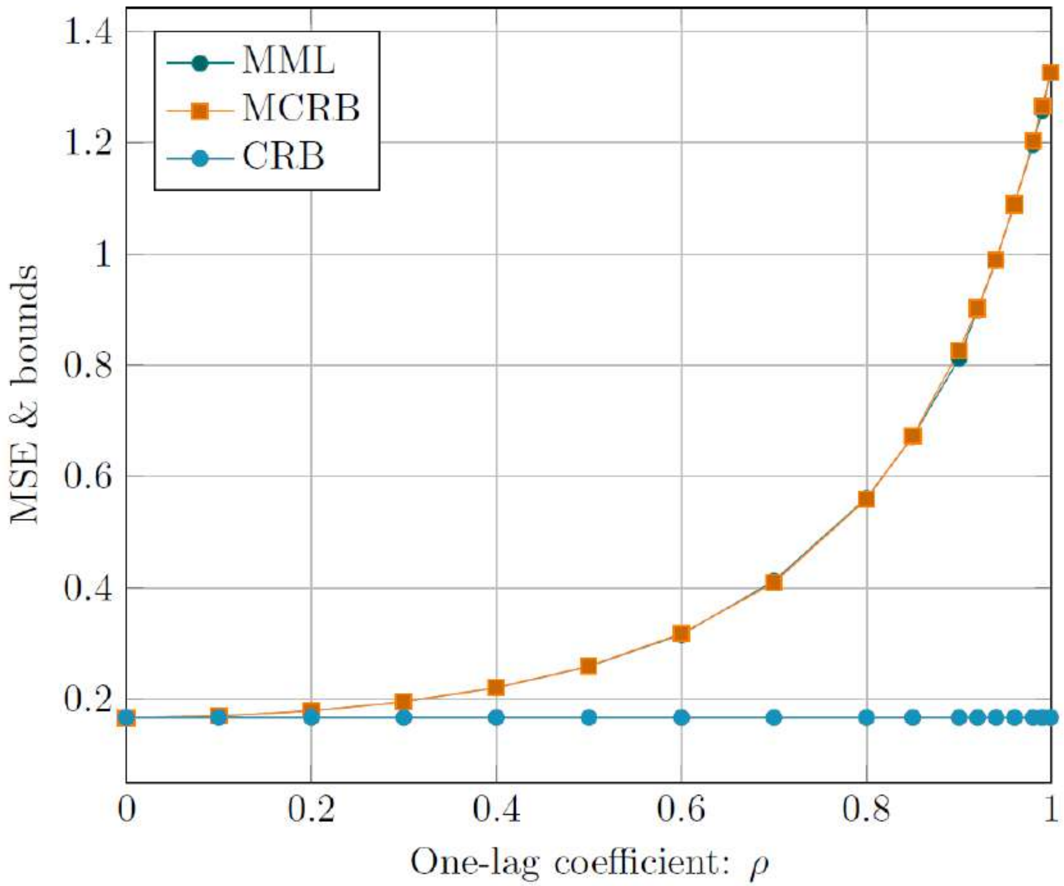
- The MML estimator is consistent, since it converges to the true value of the statistical power.

- Moreover, the MML is MS-unbiased, since:

$$E_p\{\hat{\theta}_{MML}(\mathbf{x})\} = \frac{1}{M}\sum_{m=1}^{M}\frac{E_p\{\mathbf{x}_m^T\mathbf{x}_m\}}{N} = \bar{\sigma}^2 = \theta_0$$

- Hence, the error variance (and the MSE) of this MML estimator is lower bounded by the MCRB.

Example 2: Power estimation (6/6)



$\bar{\sigma}^2 = 4$

$M = 3N$

$N = 8$

- MCRB = CRB when $\rho = 0$: matched case.
- The MML estimator is efficient wrt the MCRB.

Applications: DOA estimation (1/8)

- **Problem**: we want to estimate the Directions of Arrival (DOAs) of plane-waves signals by means of an array of sensors.
- This is a core research within the SP community [15].
- The fundamental prerequisite for any DOA estimation algorithm is that the positions of the sensors in the array are known exactly.
- Many authors have investigated the impact on the DOA estimation performance of an imperfect knowledge of the sensor positions or of the misscalibration of the array itself (see e.g. [16] and [17]).

Applications: DOA estimation (2/8)

- Some authors have proposed *hybrid* or *modified* CRB with the aim to predict the lowest MSE of the DOA estimators in the presence of the position uncertainties ([18], [19]).
- All these classical approaches, although reasonable, are application-dependent and not general.
- The application of the general misspecified estimation framework to DOA estimation problems has been firstly proposed by Richmond and Horowitz in their seminal paper:

[8] Richmond, C.D.; Horowitz, L.L., "Parameter Bounds on Estimation Accuracy Under Model Misspecification," *IEEE Trans. on Signal Process.*, vol.63, no.9, pp.2263-2278, 2015

Applications: DOA estimation (3/8)

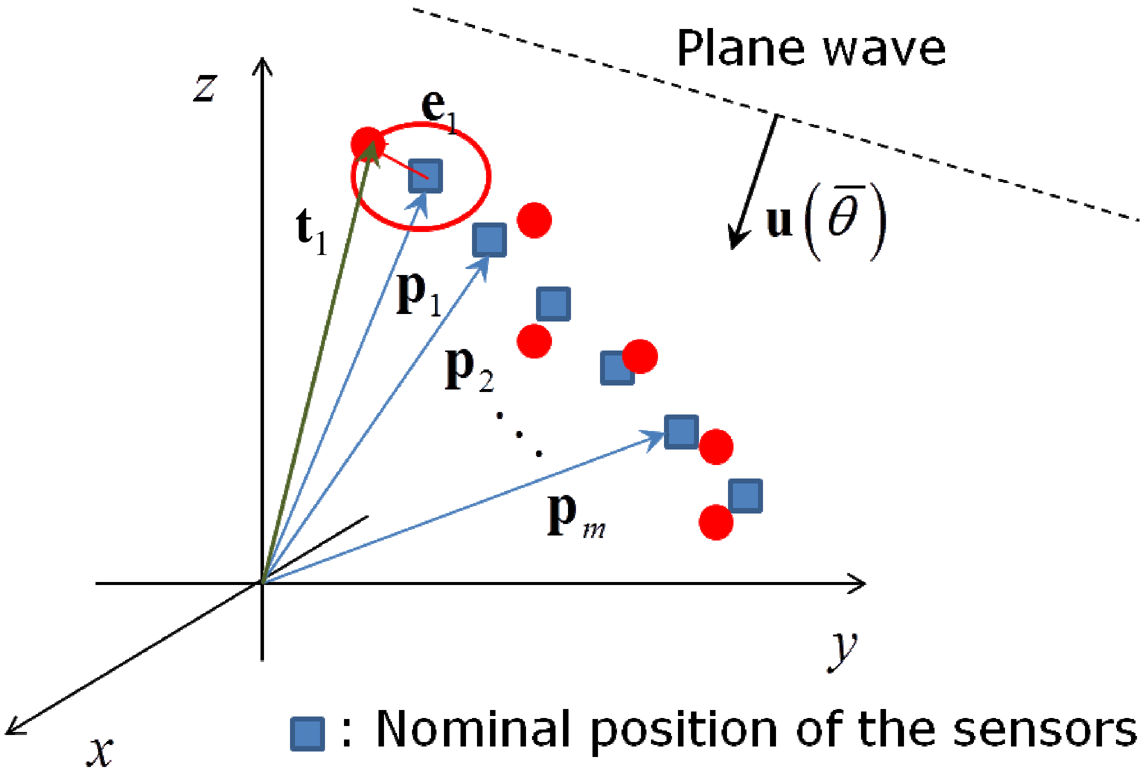
- Following [8], consider a ULA of M sensors and a single plane wave signal impinging on the array from a conic angle $\bar{\theta}$.

- Due to array misscalibration, the true position vector of the m^{th} sensor is known up to an error term modeled as a zero-mean Gaussian random vector $\mathbf{e}_m \sim \mathcal{N}(0, \sigma_e^2 \mathbf{I}_3)$.

- Define as $\mathbf{u}(\bar{\theta})$ the unit vector pointing at the direction of the impinging plane wave.

- Define $\mathbf{k}_{\bar{\theta}} \triangleq (2\pi/\lambda)\mathbf{u}(\bar{\theta})$ where λ is the wavelength.

Applications: DOA estimation (4/8)



■ : Nominal position of the sensors $\mathbf{p}_m$

● : True position of the sensors $\mathbf{t}_m = \mathbf{p}_m + \mathbf{e}_m$

- The true (perturbed) steering vector is given by:

$$[\mathbf{d}(\bar{\theta})]_m = \exp\left(j\mathbf{k}_{\bar{\theta}}^T (\mathbf{p}_m + \mathbf{e}_m) \right), \quad m = 1, \ldots, M$$

Applications: DOA estimation (5/8)

- The signal received at the *m*th sensor is:

$$x_m = \bar{s} \cdot [\mathbf{d}(\bar{\theta})]_m + [\mathbf{c}]_m, \quad m = 1, \ldots, M$$

- $\bar{s}$ is the deterministic unknown complex amplitude.

- $\mathbf{c} = \mathbf{n} + \mathbf{j}$ is the disturbance vector composed of a white Gaussian noise $\mathbf{n}$ and possibly also of a jammer $\mathbf{j}$.

- Given particular realizations of $\mathbf{e}_m$, the disturbance can be modeled as:

$$\mathbf{c} \sim \mathcal{N}\left(0, \sigma_n^2 \mathbf{I}_M + \sigma_j^2 \mathbf{d}(\theta_j)\mathbf{d}^H(\theta_j)\right)$$

Power and DOA of the jammer

Applications: DOA estimation (6/8)

- Since the particular realizations of $\mathbf{e}_m$ is generally unknown, one may decide to assume the <u>nominal steering vector</u> in the estimation algorithm:

$$[\mathbf{v}(\theta)]_m = \exp(j\mathbf{k}_\theta^T \mathbf{p}_m), \quad m = 1, \ldots, M$$

- The true (but unknown) data model is:

$$p_X(\mathbf{x}) = \mathcal{N}\left(\bar{s}\mathbf{d}(\bar{\theta}), \sigma_n^2 \mathbf{I}_M + \sigma_j^2 \mathbf{d}(\theta_j)\mathbf{d}^H(\theta_j)\right) \in \mathcal{P}$$

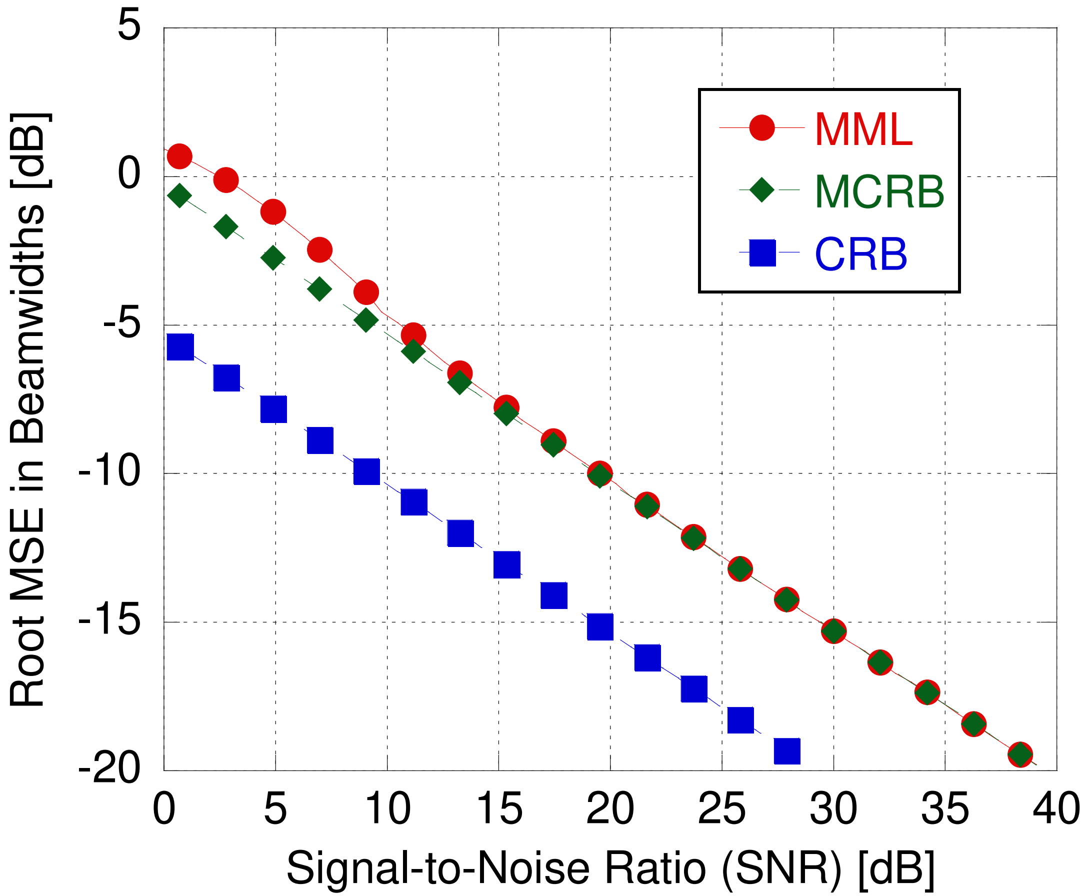
- The assumed parametric data model is:

$$\mathcal{F} = \left\{ f_X \,\middle|\, f_X(\mathbf{x} \mid s, \theta) \equiv \mathcal{N}\left(s\mathbf{v}(\theta), \sigma_n^2 \mathbf{I}_M + \sigma_j^2 \mathbf{v}(\theta_j)\mathbf{v}^H(\theta_j)\right) \right\}$$

where $s \in \mathbb{C}, \theta \in [0, 2\pi)$

Applications: DOA estimation (7/8)

- The MCRB can predict how large is the performance loss in the estimation of $\bar{\theta}$ due to this model mismatch.



Applications: DOA estimation (8/8)

- The MCRB accurately predicts performance of the Mismatched ML (MML) estimator.

- If the system goal is a 10-to-1 beamsplit ratio, i.e. -10dB RMSE in beamwidths, then this could be accomplished with an *SNR* of ~10dB when the model is perfectly known.

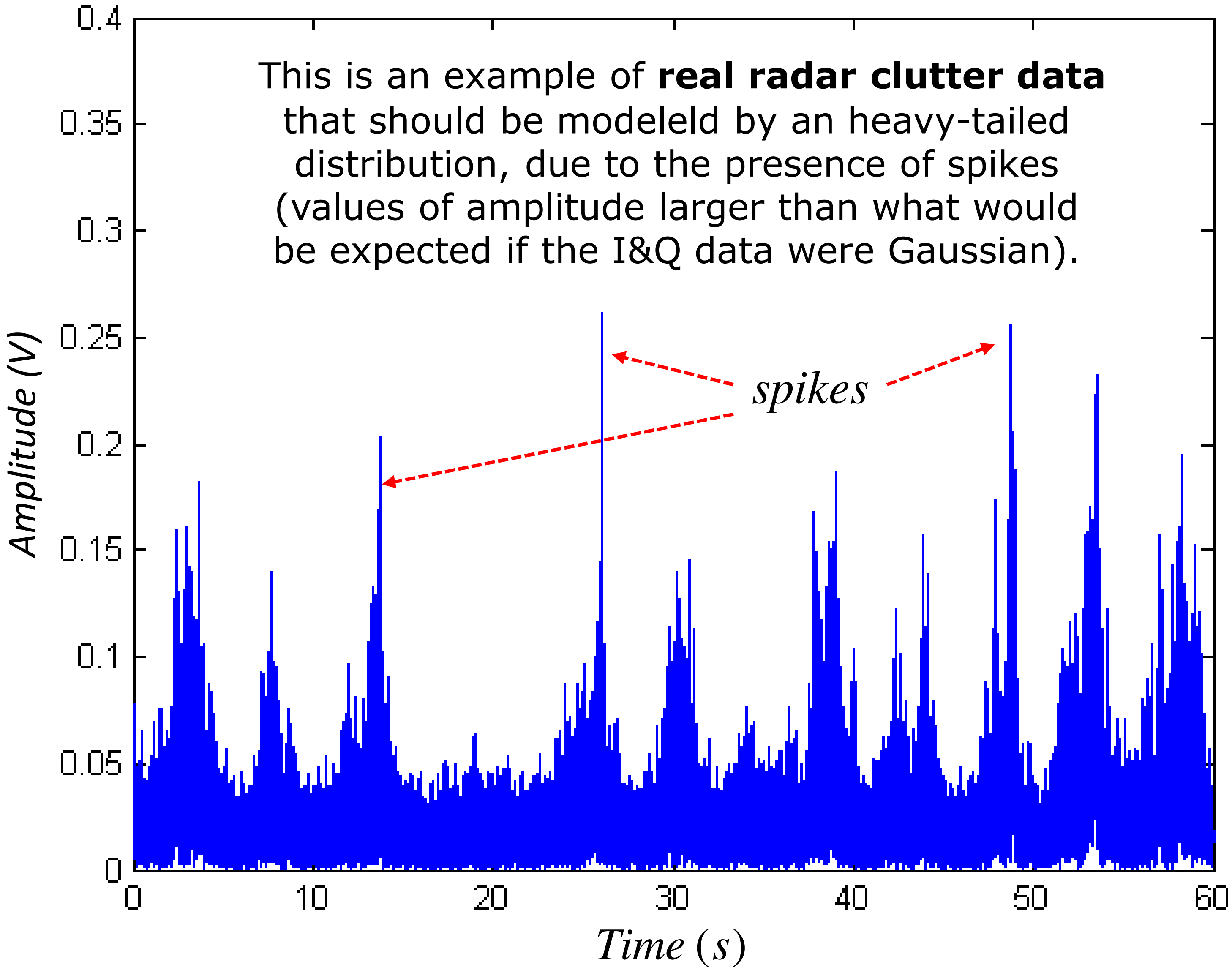
- However, not knowing precisely the true sensor positions requires an additional ~10dB of *SNR* to achieve the same goal [8].

[8] Richmond, C.D.; Horowitz, L.L., "Parameter Bounds on Estimation Accuracy Under Model Misspecification," *IEEE Trans. on Signal Process.*, vol.63, no.9, pp.2263-2278, 2015

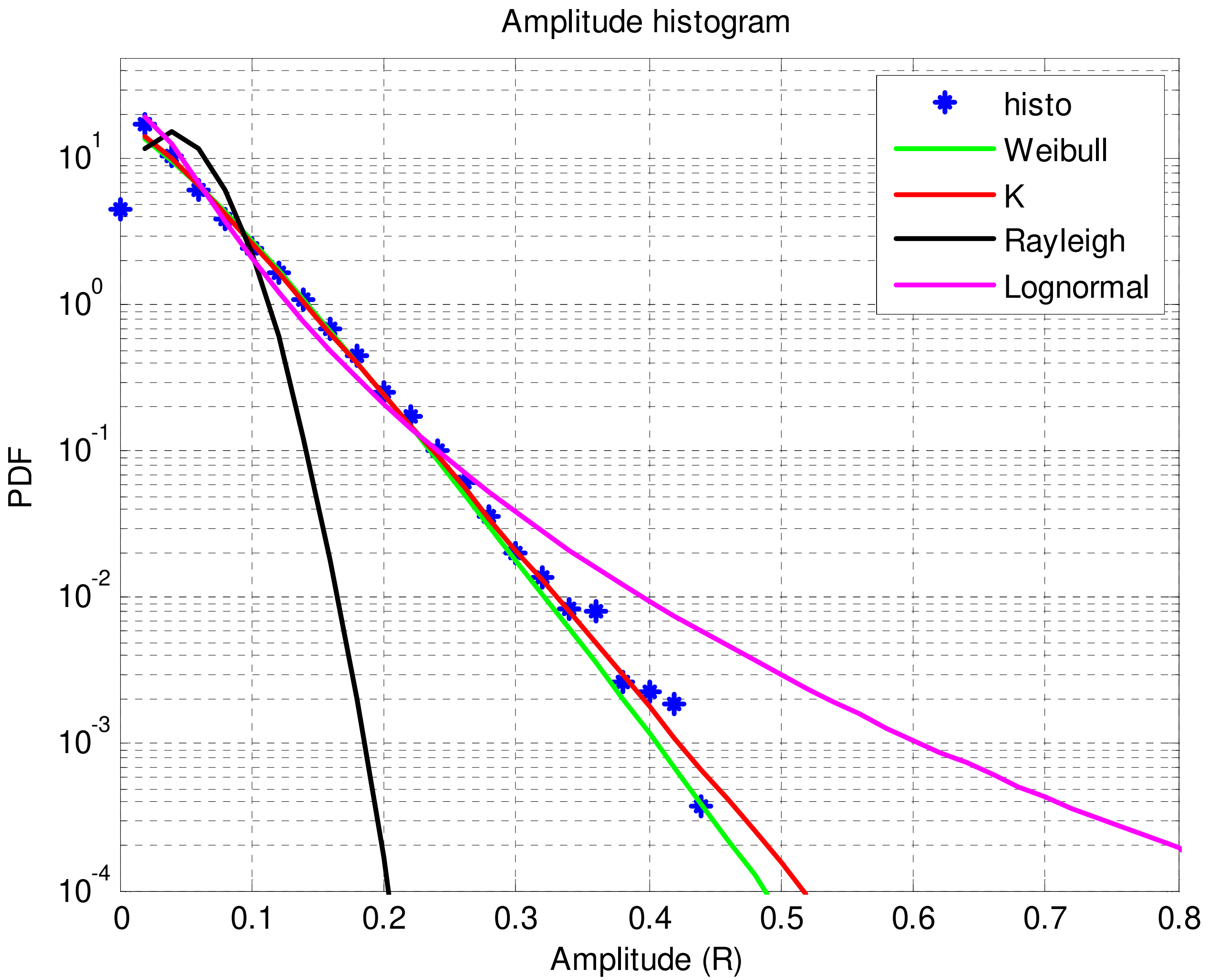
Applications: Scatter matrix estimation

- The **estimation of the correlation structure**, i.e. the scatter or covariance matrix, of a dataset is another common problem in many SP applications:
  - Adaptive detection in radar and sonar systems,
  - DOA estimation in array processing,
  - Principal Component Analysis (PCA),
  - Interference cancellation,
  - Portfolio optimization.
- Even if the data may come from disparate applications, they often share a non-Gaussian heavy-tailed statistical behavior (in radar and sonar applications, typically it is due to the high-resolution of the receiving sensor).

Real high-resolution sea clutter data



Heavy-tailed radar clutter



$$\text{Clutter amplitude:} \quad R = \left| X_I + jX_Q \right|$$

Scatter matrix estimation

- Estimating the scatter matrix of a set of non-Gaussian data is generally not a trivial task.

- The statistical characterization of non-Gaussian data requires additional parameters that generally have to be jointly estimated with the scatter matrix.

- The joint ML estimator of all unknown parameters often encounters computational difficulties and convergence (or even existence) issues.

- One could be led to use a simpler (mismatched) model.

A family of non-Gaussian distributions

- A popular family of non-Gaussian pdf's is the class of **Complex Elliptically Symmetric (CES)** distributions.

- Thanks to their flexibility, CES distributions represent a reliable data model in many areas such as radar, sonar, and communications [22].

- The complex Normal, Generalized Gaussian, *K*-distribution, complex *t*-distribution and all the compound-Gaussian pdf's belong to the CES class.

- The statistical behavior of high-resolution radar clutter can be accurately characterized by using the CES model ([24,25]).

Complex Elliptically Symmetric distributions

- A complex N-dimensional random vector $\mathbf{x}_m$ is Complex Elliptically Symmetric (CES) distributed if its pdf is [22]:

$$p_X(\mathbf{x}_m) = \left|\bar{\boldsymbol{\Sigma}}\right|^{-1} g\left(\left(\mathbf{x}_m - \boldsymbol{\gamma}\right)^H \bar{\boldsymbol{\Sigma}}^{-1}\left(\mathbf{x}_m - \boldsymbol{\gamma}\right)\right) \in CES_N\left(\boldsymbol{\gamma}, \bar{\boldsymbol{\Sigma}}, g\right)$$

- g is the density generator and γ the mean vector.

- $\bar{\boldsymbol{\Sigma}}$ is the full-rank scatter matrix, that is a scaled version of the covariance matrix $\bar{\mathbf{M}}$.

- A typical constraint is $\mathrm{tr}\left(\bar{\boldsymbol{\Sigma}}\right) = N$. As a consequence:

$$\bar{\mathbf{M}} \triangleq E\left\{(\mathbf{x}_m - \boldsymbol{\gamma})(\mathbf{x}_m - \boldsymbol{\gamma})^H\right\} = \sigma_X^2 \bar{\boldsymbol{\Sigma}}, \qquad \sigma_X^2 \triangleq \frac{E\left\{\mathbf{x}_m^H \mathbf{x}_m\right\}}{N} = \frac{\mathrm{tr}\left(\bar{\mathbf{M}}\right)}{N}$$

Estimation of the Scatter Matrix for CES data

- **Problem**: estimate the **scatter matrix** of a zero-mean CES distributed random vector in the presence of misspecified modeling, as described below.

- Set of *M* independent CES-distributed data: $\mathbf{x} = \{\mathbf{x}_m\}_{m=1}^{M} \in \mathbb{C}^N$

- True (unknown) data pdf: $p_X(\mathbf{x}_m) \in CES_N\left(\mathbf{0}, \bar{\boldsymbol{\Sigma}}, g\right)$

- Assumed data pdf: $f_X(\mathbf{x}_m; \boldsymbol{\theta}) \in CES_N\left(\mathbf{0}, \boldsymbol{\Sigma}, h_\zeta\right)$

$$\boldsymbol{\theta} = \left(\mathrm{vecs}(\boldsymbol{\Sigma})^T \quad \boldsymbol{\zeta}^T\right)^T$$

- The misspecification is in the choice of the density generator (which is parametrized by ζ).

A misspecified scenario

- A possible mismatched scenario in coherent radar [12]:
  - the **true** data pdf is a **complex *t*-distribution**;
  - the ML estimator of the scatter matrix (and of the statistical power) is derived under the **Gaussian model assumption**.

- The ML estimator of the scatter matrix under the Gaussian model assumption is the well-known **Sample Covariance Matrix (SCM)**.

1. Is the SCM a (misspecified) consistent estimator?
2. Is it efficient wrt the MCRB?
3. How large is its performance loss wrt the matched case?

Misspecified scatter matrix estimation

- **True model**: the *heavy-tailed* complex *t*-distribution.

$$p_X\left(\mathbf{x}_m \middle| \bar{\boldsymbol{\Sigma}}, \lambda, \eta\right) \triangleq \frac{1}{\pi^N \left|\bar{\boldsymbol{\Sigma}}\right|} \cdot \frac{\Gamma(N+\lambda)}{\Gamma(\lambda)} \left(\frac{\lambda}{\eta}\right)^{\lambda} \left(\frac{\lambda}{\eta} + \mathbf{x}_m^H \bar{\boldsymbol{\Sigma}}^{-1} \mathbf{x}_m\right)^{-(N+\lambda)}$$

- λ and η are the shape and scale parameters; they should be jointly estimated with the scatter matrix.

- The *statistical power* of $\mathbf{x}_m$ is: $\bar{\sigma}^2 = \dfrac{\lambda}{\eta(\lambda-1)}$

- The true, or "matched", parameter vector:

$$\boldsymbol{\tau} \triangleq [\mathrm{vecs}(\bar{\boldsymbol{\Sigma}})^T \quad \underbrace{\lambda \quad \eta}_{\zeta}]^T, \quad \mathrm{tr}(\bar{\boldsymbol{\Sigma}}) = N$$

Misspecified scatter matrix estimation

- **Assumed model**: the complex Gaussian distribution.

$$f_X\left(\mathbf{x}_m \middle| \boldsymbol{\theta}\right) \triangleq f_X\left(\mathbf{x}_m \middle| \boldsymbol{\Sigma}, \sigma^2\right) = \frac{1}{\left(\pi\sigma^2\right)^N \left|\boldsymbol{\Sigma}\right|} \exp\left(-\frac{\mathbf{x}_m^H \boldsymbol{\Sigma}^{-1} \mathbf{x}_m}{\sigma^2}\right)$$

- The "assumed" parameter vector to be estimated is:

$$\boldsymbol{\theta} = [\mathrm{vecs}(\boldsymbol{\Sigma})^T \quad \sigma^2]^T$$

- under the <u>constraint</u>:

$$\mathbf{f}\left(\boldsymbol{\theta}\right) = \mathbf{0} \quad \Rightarrow \quad \mathrm{tr}\left(\boldsymbol{\Sigma}\right) - N = 0$$

- Without such (or similar) constraint, $\boldsymbol{\Sigma}$ and σ^2 cannot be jointly estimated $\rightarrow$ there is an ambiguity.

Misspecified scatter matrix estimation

- The **constrained MML (CMML) estimator** is derived as:

$$\hat{\boldsymbol{\theta}}_{CMML}(\mathbf{x}) \triangleq \underset{\boldsymbol{\theta}\in\Theta}{\arg\max} \ln f_X(\mathbf{x};\boldsymbol{\theta})$$

→ assumed complex Gaussian distribution

$$= \underset{\boldsymbol{\theta}\in\Theta}{\arg\max} \sum_{m=1}^{M} \ln f_X(\mathbf{x}_m;\boldsymbol{\theta}), \qquad \mathbf{x}_m \sim p_X(\mathbf{x}_m)$$

true (unknown) complex t-distribution

- The closed form expression is:

$$\begin{cases} \hat{\boldsymbol{\Sigma}}_{CMML} = \dfrac{N}{Tr\{\mathbf{SCM}\}}\mathbf{SCM} = \dfrac{N}{Tr\left\{\dfrac{1}{M}\sum_{m=1}^{M}\mathbf{x}_m\mathbf{x}_m^H\right\}} \cdot \dfrac{1}{M}\sum_{m=1}^{M}\mathbf{x}_m\mathbf{x}_m^H \\ \hat{\sigma}^2_{CMML} = \dfrac{1}{M}\sum_{m=1}^{M}\dfrac{\mathbf{x}_m^H\hat{\boldsymbol{\Sigma}}_{CMML}^{-1}\mathbf{x}_m}{N} \end{cases}$$

Misspecified scatter matrix estimation

- The **CMML estimator** under the Gaussian assumption:

$$\begin{cases} \hat{\boldsymbol{\Sigma}}_{CMML} = \dfrac{N}{\sum_{m=1}^{M} \mathbf{x}_m^H \mathbf{x}_m} \sum_{m=1}^{M} \mathbf{x}_m \mathbf{x}_m^H \\ \hat{\sigma}_{CMML}^2 = \dfrac{1}{NM} \sum_{m=1}^{M} \mathbf{x}_m^H \hat{\boldsymbol{\Sigma}}_{CMML}^{-1} \mathbf{x}_m \end{cases}$$

Which is the convergence point of the CMML estimator?

$$\hat{\boldsymbol{\theta}}_{CMML}(\mathbf{x}) \underset{M\to\infty}{\overset{a.s.}{\to}} \boldsymbol{\theta}_0 = ?$$

$$\hat{\boldsymbol{\theta}}_{CMML} = \left[\mathrm{vecs}\left(\hat{\boldsymbol{\Sigma}}_{CMML}\right)^T \quad \hat{\sigma}_{CMML}^2 \right]^T$$

Misspecified scatter matrix estimation

- The pseudo-true parameter vector is:

$$\boldsymbol{\theta}_0 = \arg\min_{\boldsymbol{\theta}\in\tilde{\Theta}} \left\{ -E_p \left\{ \ln f_X (\mathbf{x}_m; \boldsymbol{\theta}) \right\} \right\}$$

- For the case study at hand, it can be shown that:

$$\boldsymbol{\theta}_0 = [\text{vecs}(\bar{\boldsymbol{\Sigma}})^T \quad \bar{\sigma}^2]^T \qquad \text{and} \qquad \det\left(\mathbf{A}_{\boldsymbol{\theta}_0}\right) \neq 0$$

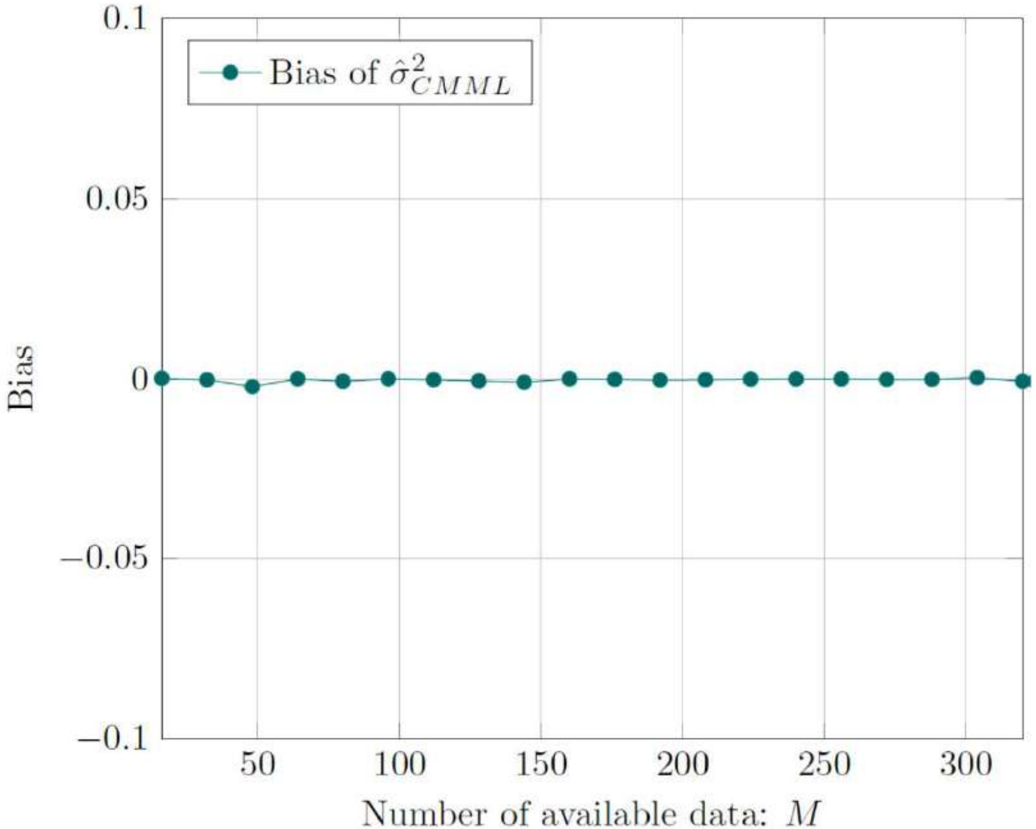
The **CMMLE** converges ***a.s.*** to:

$$\hat{\sigma}^2_{CMML}(\mathbf{x}) \underset{M\to\infty}{\overset{a.s.}{\rightarrow}} \bar{\sigma}^2 = \frac{\lambda}{\eta(\lambda-1)}, \quad \hat{\boldsymbol{\Sigma}}_{CMML}(\mathbf{x}) \underset{M\to\infty}{\overset{a.s.}{\rightarrow}} \bar{\boldsymbol{\Sigma}}$$

Hence, it provides **consistent** estimates for both the statistical power and the scatter matrix.

Misspecified scatter matrix estimation

- Bias of the estimator $\hat{\sigma}^2_{CMML}$: $b_{\hat{\sigma}^2_{CMML}} \triangleq E\left\{\hat{\sigma}^2_{CMML} - \bar{\sigma}^2\right\}$



Simulation parameters:

$$[\bar{\mathbf{\Sigma}}]_{i,j} = \rho^{|i-j|}$$

$$\rho = 0.8$$

$$N = 16$$

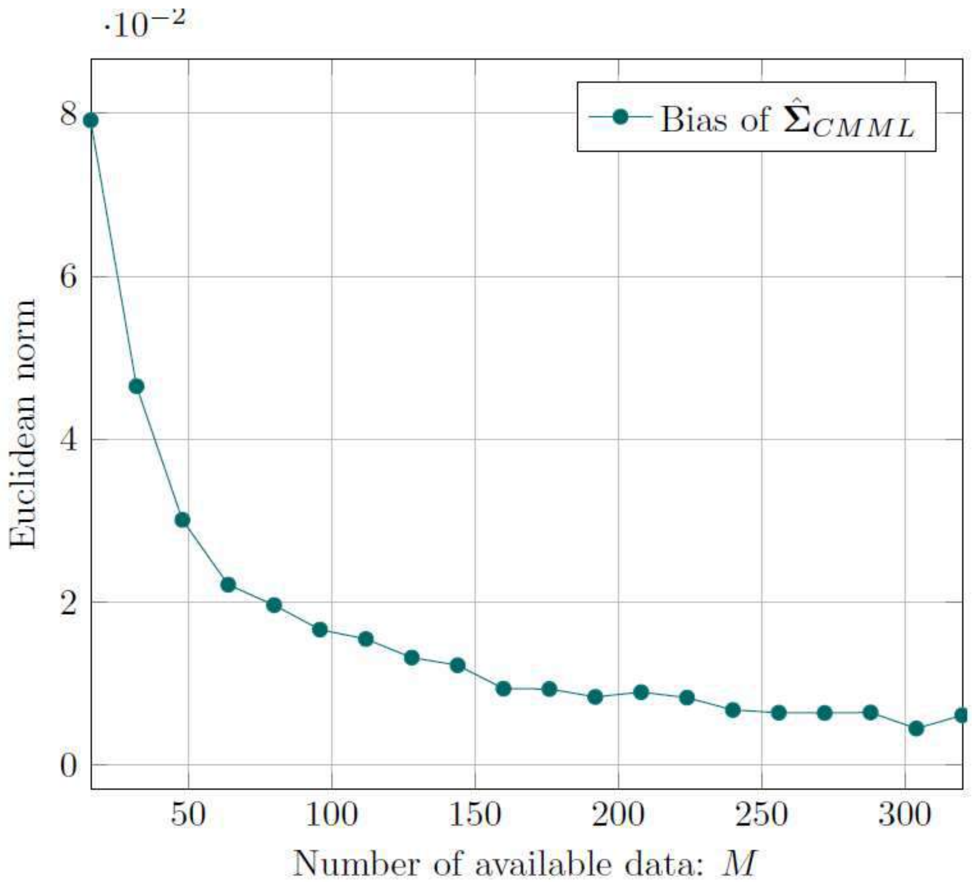
$$\lambda = 3$$

$$\eta = 1$$

$$\bar{\sigma}^2 = \frac{\lambda}{\eta(\lambda - 1)} = \frac{3}{2}$$

Misspecified scatter matrix estimation

- Bias of the estimator $\hat{\boldsymbol{\Sigma}}_{CMML}$: $b_{\hat{\boldsymbol{\Sigma}}_{CMML}} \triangleq \left\| E\left\{ \text{vecs}(\hat{\boldsymbol{\Sigma}}_{CMML} - \bar{\boldsymbol{\Sigma}}) \right\} \right\|_2$



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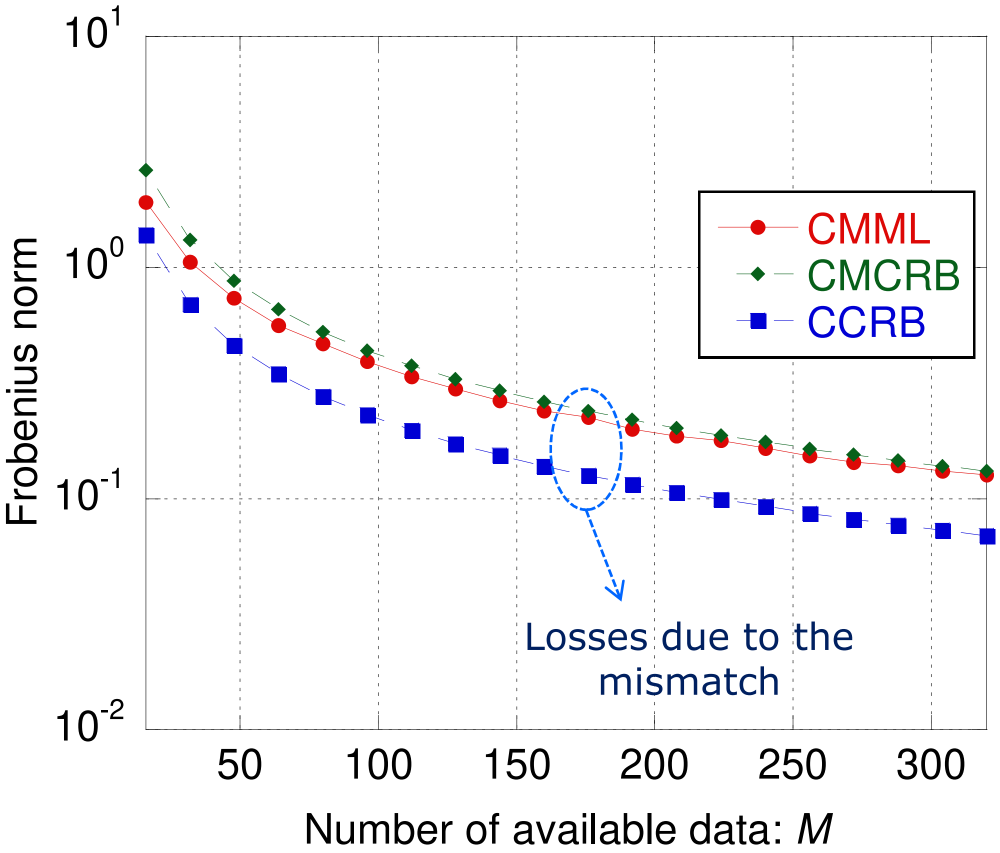
$$\lambda = 3$$

$$\eta = 1$$

$$\bar{\sigma}^2 = \frac{\lambda}{\eta(\lambda - 1)} = \frac{3}{2}$$

Misspecified scatter matrix estimation

$$\varepsilon_{\hat{\Sigma}_{CMML}} \triangleq \left\| E\left\{ \mathrm{vecs}(\hat{\boldsymbol{\Sigma}}_{CMML} - \bar{\boldsymbol{\Sigma}})\mathrm{vecs}(\hat{\boldsymbol{\Sigma}}_{CMML} - \bar{\boldsymbol{\Sigma}})^T \right\} \right\|_F$$



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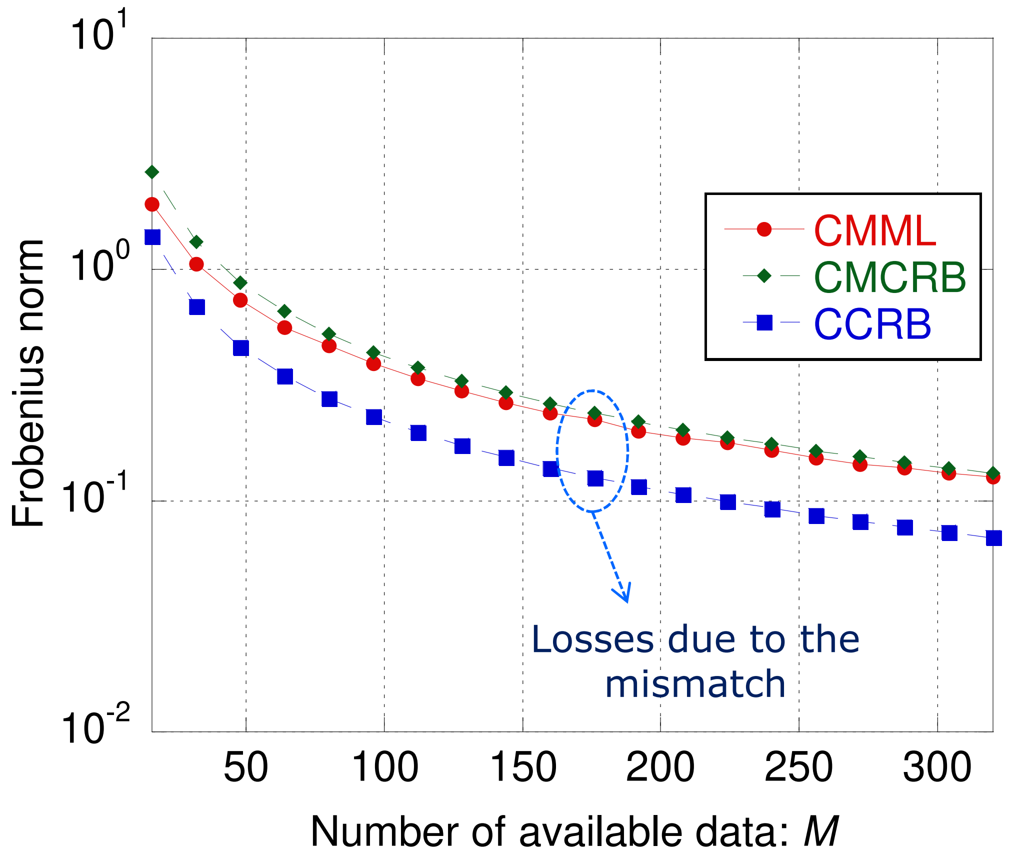
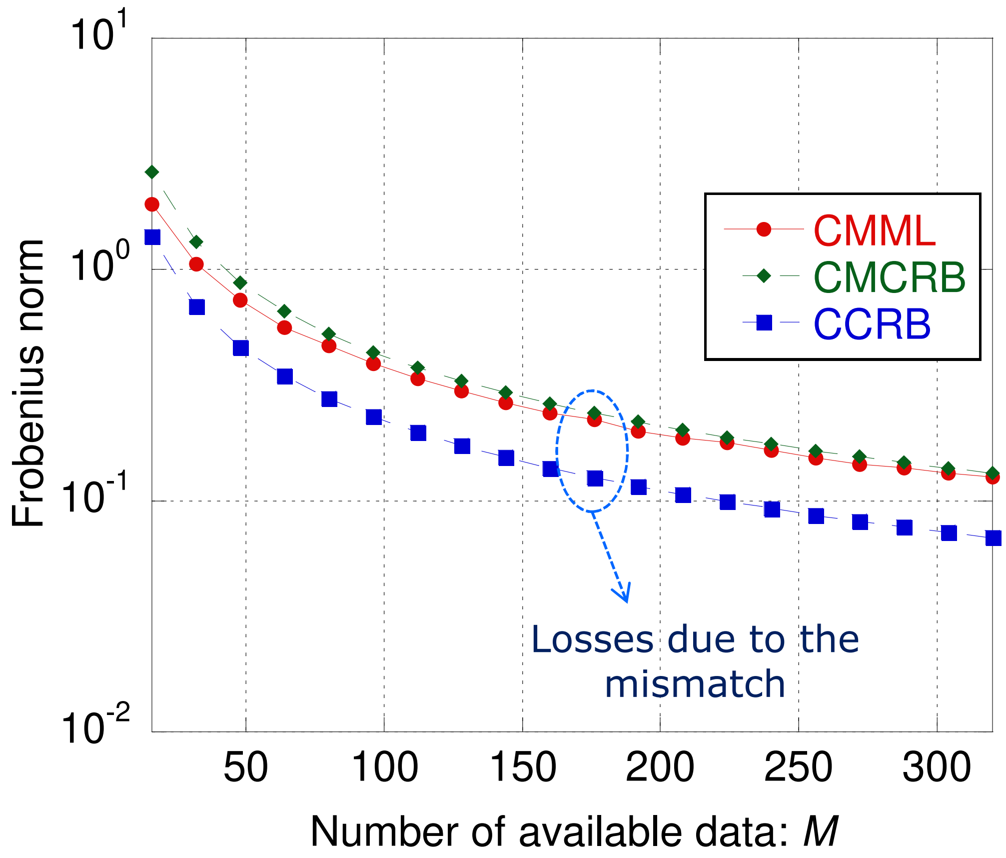
Misspecified scatter matrix estimation



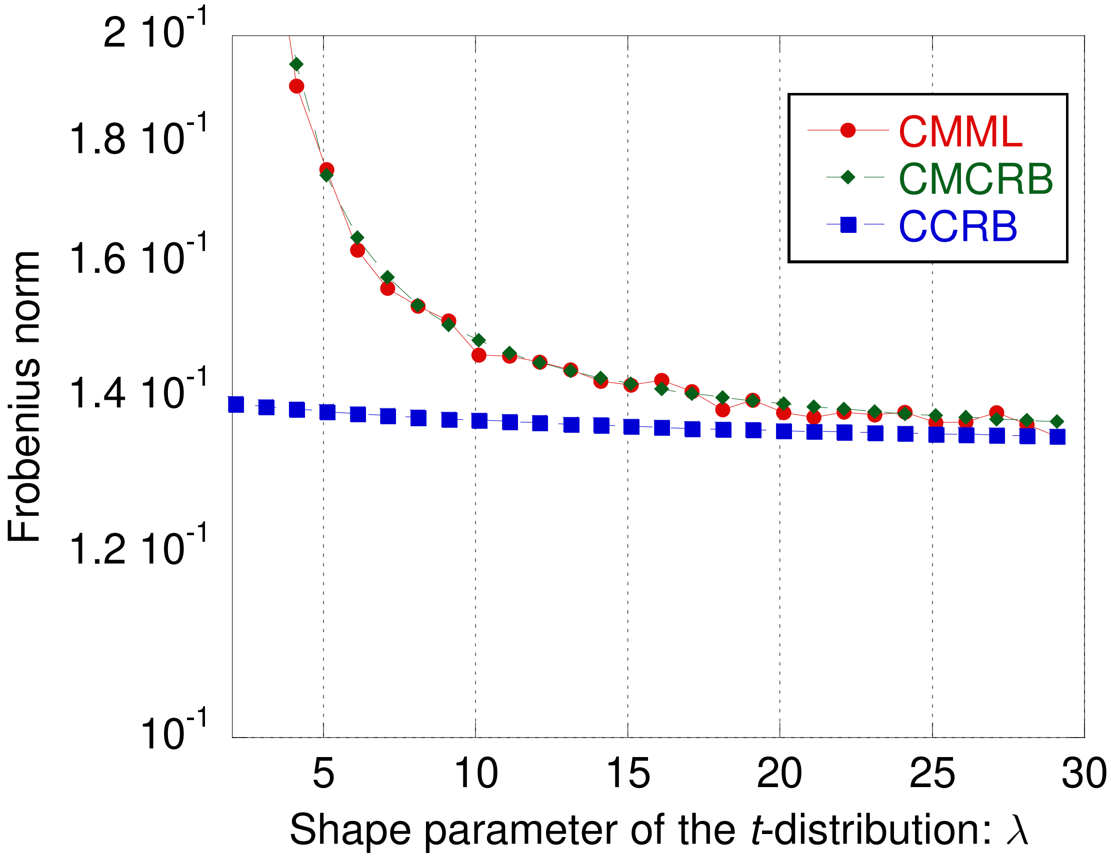
- **CCRB** is the **constrained** "matched" **CRB** [30,31,32] on the joint estimation of $\bar{\Sigma}$, λ, and η (for t-distributed data).

Misspecified scatter matrix estimation



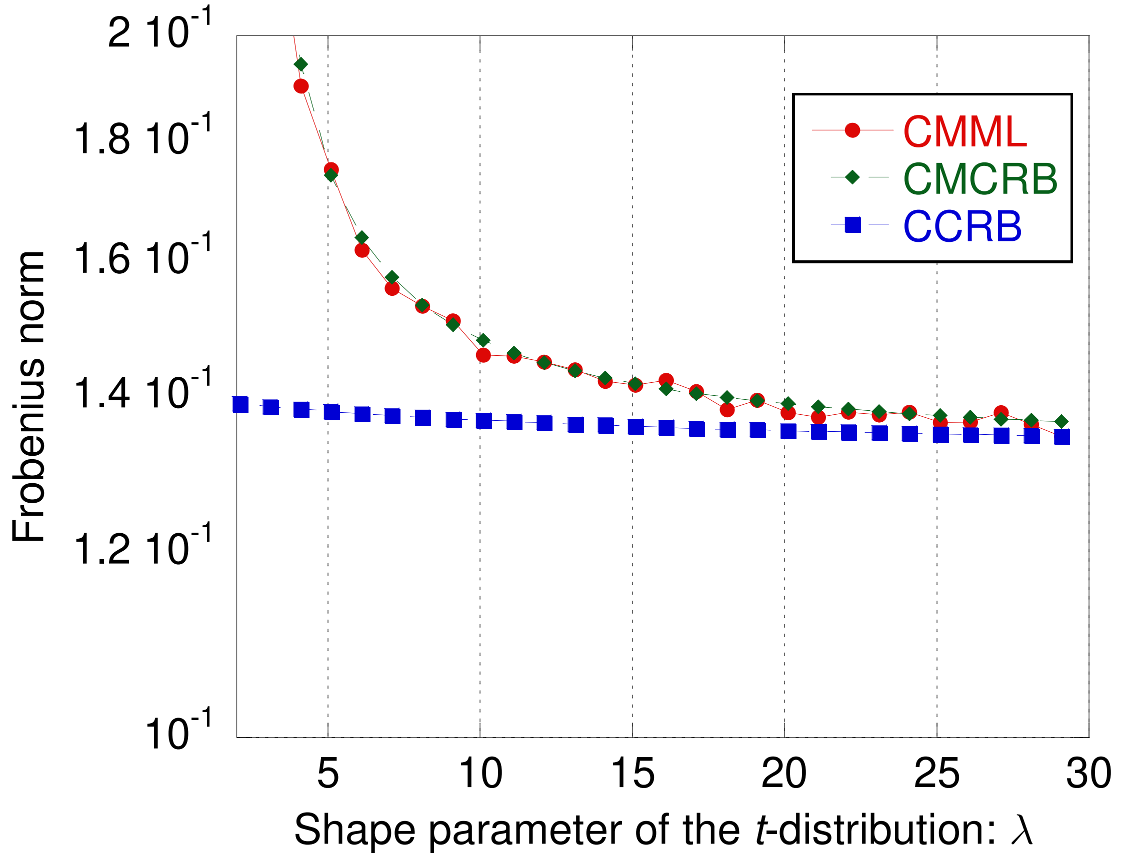
- **CMCRB** is the **constrained MCRB** [20] on the joint estimation of $\bar{\Sigma}$ and $\bar{\sigma}^2$(under model misspecification).

Misspecified scatter matrix estimation



- When $\lambda \rightarrow 0$ (extremely spiky data), the estimation losses due to model mismatching rapidly increase.

Misspecified scatter matrix estimation



- When $\lambda \to \infty$ the data tend to be Gaussian distributed and the MSE of the **CMMLE**, the **CMCRB**, and the **CCRB** tend to coincide.

Concluding remarks for Part I

- We summarized the fundamental concepts about lower bounds and efficient estimators in the presence of model misspecification.

- The MML estimator is asymptotically MS-unbiased and its error covariance matrix asymptotically equates the MCRB.

- We showed how to apply these theoretical findings to two well-known problems:
    1. Direction of Arrivals (DOAs) estimation with an array of antennas;
    2. Estimation of the the disturbance scatter matrix in complex *t*-distributed data.

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