The Misspecified and Semiparametric lower bounds and their application to inference problems with Complex Elliptically Symmetric (CES) distributed data

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Part II - Outline of the talk

- Why semiparametric models?
- CRB in parametric models with finite-dimensional nuisance parameters: classical approach.
- CRB in parametric models with finite-dimensional nuisance parameters: "Hilbert-space-based" approach.
- Extension to semiparametric models.
- Semiparametric interpretation of Real and Complex ES distributions.
- Examples.

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Why semiparametric models?

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Examples

Parametric models

- A parametric model $\mathcal{P}_{\boldsymbol{\theta}}$ is defined as a set of pdfs that are parametrized by a finite-dimensional parameter vector $\boldsymbol{\theta}$:

$$\mathcal{P}_{\boldsymbol{\theta}} \triangleq \left\{ p_X(\mathbf{x}_1, \ldots, \mathbf{x}_M | \boldsymbol{\theta}), \boldsymbol{\theta} \in \Theta \subseteq \mathbb{R}^q \right\}.$$

- The (lack of) knowledge about the phenomenon of interest is summarized in $\boldsymbol{\theta}$ that needs to be estimated.

- **Pros**: Parametric inference procedures are generally "simple" due to the finite dimensionality of $\boldsymbol{\theta}$.

- **Cons**: A parametric model could be too restrictive and a *misspecification problem*[1] may occur [1,2,3,4,5,6].

[1] S. Fortunati, F. Gini, M. S. Greco and C. D. Richmond, "Performance Bounds for Parameter Estimation under Misspecified Models: Fundamental Findings and Applications", *IEEE Signal Processing Magazine*, vol. 34, no. 6, pp. 142-157, Nov. 2017.

Non-parametric models

- A non-parametric model $\mathcal{P}_p$ is a collection of pdfs possibly satisfying some functional constraints (i.e. *symmetry*):

$$\mathcal{P}_p \triangleq \{p_X(\mathbf{x}_1, \dots, \mathbf{x}_M) \in \mathcal{K}\},$$

where $\mathcal{K}$ is some constrained set of pdfs.

- **Pros**: The risk of model misspecification is minimized.

- **Cons**: In non-parametric inference we have to face with infinite-dimensional estimation problem.

- **Cons**: Non-parametric inference may be a prohibitive task due to the large amount of required data.

Semiparametric models

- A semiparametric model[2] $\mathcal{P}_{\boldsymbol{\theta},g}$ is a set of pdfs characterized by a finite-dimensional parameter $\boldsymbol{\theta} \in \Theta$ along with a *function*, i.e. an infinite-dimensional parameter, $g \in \mathcal{L}$ [7]:

$$\mathcal{P}_{\boldsymbol{\theta},g} \triangleq \{p_X(\mathbf{x}_1, \ldots, \mathbf{x}_M | \boldsymbol{\theta}, g), \boldsymbol{\theta} \in \Theta \subseteq \mathbb{R}^q, g \in \mathcal{L}\}.$$

- Usually, $\boldsymbol{\theta}$ is the (finite-dimensional) parameter of interest while g can be considered as a nuisance parameter.

- **Pros**: All parametric signal models involving an unknown noise distribution are semiparametric models.

- **Cons**: Tools from functional analysis are needed.

[2] P.J. Bickel, C.A.J Klaassen, Y. Ritov and J.A. Wellner, *Efficient and Adaptive Estimation for Semiparametric Models*, Johns Hopkins University Press, 1993.

Examples: CES distributions

- A CES distributed random vector $\mathbf{x} \in \mathbb{C}^N$ admits a pdf [8]:

$$p_X(\mathbf{x}; \boldsymbol{\mu}, \boldsymbol{\Sigma}) = c_{N,g} |\boldsymbol{\Sigma}|^{-1} g((\mathbf{x} - \boldsymbol{\mu})^H \boldsymbol{\Sigma}^{-1} (\mathbf{x} - \boldsymbol{\mu})),$$

  - $c_{N,g}$ is a normalizing constant,
  - $g \in \mathcal{G}$, $g : \mathbb{R}_0^+ \to \mathbb{R}^+$ is the *density generator*,
  - $\boldsymbol{\mu} \in \mathbb{C}^N$ is the mean value,
  - $\boldsymbol{\Sigma} \in \mathcal{M}_N$ is the (full rank) scatter matrix.

- The set of all CES pdfs is a semiparametric model of the form:

$$\mathcal{P}_{\boldsymbol{\mu}, \boldsymbol{\Sigma}, g} \triangleq \left\{ p_X | p_X(\mathbf{x} | \boldsymbol{\mu}, \boldsymbol{\Sigma}, g), \boldsymbol{\mu} \in \mathbb{C}^N, \boldsymbol{\Sigma} \in \mathcal{M}_N, g \in \mathcal{G} \right\}.$$

- This semiparametric model is a particular instance of the more general set of *semiparametric group models* [9, Sec. 4.2].

Examples: Missing data

- Let $\mathbf{z} \triangleq (\mathbf{x}^T, \mathbf{y}^T)^T$ be a *complete* dataset, where:
  - $\mathbf{x}$ is the *observed* (available) dataset.
  - $\mathbf{y}$ is the *unobservable* (missing) dataset.

- **Problem**: Estimate $\boldsymbol{\theta} \in \Theta$ from the observed dataset $\mathbf{x}$ when the pdf p_Y of the missing data $\mathbf{y}$ is unknown.

- The pdf p_X of the observed dataset can be expressed as:

$$p_X(\mathbf{x}|\boldsymbol{\theta}) = \int_{\mathcal{Y}} p_{X,Y}(\mathbf{x}, \mathbf{y}|\boldsymbol{\theta}) d\mathbf{y} = \int_{\mathcal{Y}} p_{X|Y}(\mathbf{x}|\mathbf{y}, \boldsymbol{\theta}) p_Y(\mathbf{y}) d\mathbf{y}.$$

- The set of all the pdfs of the observed dataset $\mathbf{x}$ is a *semiparametric mixture model* of the form [9, Sec. 4.5], [10]:

$$\mathcal{P}_{\boldsymbol{\theta}, p_Z} \triangleq \{p_X | p_X(\mathbf{x}|\boldsymbol{\theta}, p_Y), \boldsymbol{\theta} \in \Theta, p_Y \in \mathcal{K}\}.$$

Examples: Non-linear regression

- Let us consider the general non-linear regression model:

$$\mathbf{x} = f(\mathbf{z}, \boldsymbol{\theta}) + \boldsymbol{\epsilon},$$

  - $\boldsymbol{\theta} \in \Theta$: parameter vector to be estimated,
  - $f \in \mathcal{F}$: possibly unknown non-linear function,
  - $\mathbf{z}$: random vector with possibly unknown pdf $p_Z \in \mathcal{K}$,
  - $\boldsymbol{\epsilon}$: random noise with possibly unknown pdf $p_\epsilon \in \mathcal{E}$

- The set of all pdfs for $\mathbf{x}$ is a semiparametric model of the form:

$$\mathcal{P}_{\boldsymbol{\theta}, f, p_Z, p_\epsilon} \triangleq \{ p_X(\mathbf{x}|\boldsymbol{\theta}, f, p_Z, p_\epsilon), \boldsymbol{\theta} \in \Theta, f \in \mathcal{F}, p_Z \in \mathcal{K}, p_\epsilon \in \mathcal{E} \}.$$

- This model is a general form of a *semiparametric regression model* [9, Sec. 4.3].

Examples: Autoregressive processes

- Consider the $\mathrm{AR}(p)$ process:

$$x_n = \sum_{i=1}^{p} \theta_i x_{n-i} + w_n, \quad n \in (-\infty, \infty)$$

  - $\boldsymbol{\theta} \triangleq [\theta_1, \ldots, \theta_p]$: parameter vector to be estimated.
  - w_n: i.i.d. innovations with unknown pdf $p_w \in \mathcal{W}$,

- Let $\mathbf{x} \in \mathbb{R}^N$ a vector of N observations from an $\mathrm{AR}(p)$.

- The set of all possible pdfs for $\mathbf{x} \in \mathbb{R}^N$ is a semiparametric model [11,12]:

$$\mathcal{P}_{\boldsymbol{\theta},p_w} \triangleq \{p_X | p_X(\mathbf{x}|\boldsymbol{\theta}, p_w), \boldsymbol{\theta} \in \Theta, p_w \in \mathcal{W}\}.$$

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Score vectors in parametric models

- Let us consider the following *parametric model* involving a finite-dimensional vector of nuisance parameters:

$$\mathcal{P}_{\boldsymbol{\theta},\boldsymbol{\eta}} \triangleq \left\{ p_X(\mathbf{x}|\boldsymbol{\theta},\boldsymbol{\eta}), \boldsymbol{\theta} \in \Theta \subseteq \mathbb{R}^q, \boldsymbol{\eta} \in \Gamma \subseteq \mathbb{R}^d \right\},$$

  - $\boldsymbol{\theta} \in \Theta$: vector of the parameters of interest to be estimated,
  - $\boldsymbol{\eta} \in \Gamma$: vector of the (unknown) nuisance parameters.

- Denote with $\boldsymbol{\theta}_0$ and $\boldsymbol{\eta}_0$ the true value of $\boldsymbol{\theta} \in \Theta$ and $\boldsymbol{\eta} \in \Gamma$, respectively. Then $p_0(\mathbf{x}) \triangleq p_X(\mathbf{x}|\boldsymbol{\theta}_0,\boldsymbol{\eta}_0)$.

- **Score vectors** of the parametric model $\mathcal{P}_{\boldsymbol{\theta},\boldsymbol{\eta}}$ in $\boldsymbol{\theta}_0$ and $\boldsymbol{\eta}_0$:

$$\mathbf{s}_{\boldsymbol{\theta}_0} \triangleq \nabla_{\boldsymbol{\theta}} \ln p_X(\mathbf{x}|\boldsymbol{\theta}_0,\boldsymbol{\eta}_0), \quad \mathbf{s}_{\boldsymbol{\eta}_0} \triangleq \nabla_{\boldsymbol{\eta}} \ln p_X(\mathbf{x}|\boldsymbol{\theta}_0,\boldsymbol{\eta}_0).$$

The Fisher Information Matrix (FIM)

- The FIM for the parametric model $\mathcal{P}_{\boldsymbol{\theta},\boldsymbol{\eta}}$ is given by:

$$\mathbf{I}(\boldsymbol{\theta}_0, \boldsymbol{\eta}_0) \triangleq \begin{pmatrix} E_0\left\{\mathbf{s}_{\boldsymbol{\theta}_0}\mathbf{s}_{\boldsymbol{\theta}_0}^T\right\} & E_0\left\{\mathbf{s}_{\boldsymbol{\theta}_0}\mathbf{s}_{\boldsymbol{\eta}_0}^T\right\} \\ E_0\left\{\mathbf{s}_{\boldsymbol{\eta}_0}\mathbf{s}_{\boldsymbol{\theta}_0}^T\right\} & E_0\left\{\mathbf{s}_{\boldsymbol{\eta}_0}\mathbf{s}_{\boldsymbol{\eta}_0}^T\right\} \end{pmatrix}$$
$$= \begin{pmatrix} \mathbf{I}_{\boldsymbol{\theta}_0\boldsymbol{\theta}_0} & \mathbf{I}_{\boldsymbol{\theta}_0\boldsymbol{\eta}_0} \\ \mathbf{I}_{\boldsymbol{\theta}_0\boldsymbol{\eta}_0}^T & \mathbf{I}_{\boldsymbol{\eta}_0\boldsymbol{\eta}_0} \end{pmatrix},$$

  where $E_0\{h\} \triangleq \int h(\mathbf{x})p_0(\mathbf{x})d\mathbf{x}$.

- Let $\hat{\boldsymbol{\theta}}(\mathbf{x})$ be an *unbiased* estimator of $\boldsymbol{\theta}_0$: $E_0\{\hat{\boldsymbol{\theta}}(\mathbf{x})\} = \boldsymbol{\theta}_0$.

- How can we derive the CRB on the estimation of $\boldsymbol{\theta}_0$ in the presence of the unknown nuisance parameter vector $\boldsymbol{\eta}_0$?

Parametric CRB: classical approach

- The Cramér-Rao inequality provides us with a lower bound on the error covariance matrix of $\hat{\boldsymbol{\theta}}(\mathbf{x})$ when $\boldsymbol{\eta}_0$ is unknown (see e.g. [13, Sec. 10.7]):

$$E_0\left\{(\hat{\boldsymbol{\theta}}(\mathbf{x})-\boldsymbol{\theta}_0)(\hat{\boldsymbol{\theta}}(\mathbf{x})-\boldsymbol{\theta}_0)^T\right\} \geq \mathrm{CRB}(\boldsymbol{\theta}_0|\boldsymbol{\eta}_0).$$

- *Classical approach*: $\mathrm{CRB}(\boldsymbol{\theta}_0|\boldsymbol{\eta}_0)$ can be obtained from the FIM using the Matrix Inversion Lemma [14]:

$$\mathrm{CRB}(\boldsymbol{\theta}_0|\boldsymbol{\eta}_0) \triangleq \left(\mathbf{I}_{\boldsymbol{\theta}_0\boldsymbol{\theta}_0} - \mathbf{I}_{\boldsymbol{\theta}_0\boldsymbol{\eta}_0}\mathbf{I}^{-1}_{\boldsymbol{\eta}_0\boldsymbol{\eta}_0}\mathbf{I}^{T}_{\boldsymbol{\theta}_0\boldsymbol{\eta}_0}\right)^{-1}.$$

- It is possible to obtain this same result by using a geometrical, "**Hilbert-space-based**" approach [7].

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Hilbert spaces

Definition ([9, A.1, A.2],[15])

A Hilbert space $\mathcal{F}$ is a *normed vector space*

1. equipped with an *inner product* $\langle \cdot, \cdot \rangle$ and,
2. *complete* with respect to the norm $|| \cdot || = \sqrt{\langle \cdot, \cdot \rangle}$.

- A normed (metric) space is complete when every Cauchy sequences in $\mathcal{F}$ converges to an element of $\mathcal{F}$.

- $f_1, f_2, \cdots$ is a Cauchy sequence if, for every $\varepsilon > 0$ there is a positive integer N such that for all $i, j > N$, we have that:

$$||f_i - f_j|| < \varepsilon.$$

The square-integrable functions

- Let $(\mathcal{X}, \mathfrak{F}, \mu)$ be a measure space where $\mathcal{X} \subseteq \mathbb{R}^N$, $\mathfrak{F}$ is the Borel σ-algebra on $\mathcal{X}$ and μ is a measure on $\mathfrak{F}$. [3]

- Then, $L_2(\mu)$ is the space of all the measurable functions s. t.

$$L_2(\mu) = \left\{ f : \mathcal{X} \to \mathbb{R} \,\middle|\, \int_{\mathcal{X}} |f(\mathbf{x})|^2 d\mu(\mathbf{x}) < \infty \right\}.$$

- The $L_2(\mu)$ space is an Hilbert space with the following inner product:

$$\langle f_1, f_2 \rangle \triangleq \int_{\mathcal{X}} f_1(\mathbf{x}) f_2(\mathbf{x}) d\mu(\mathbf{x}).$$

- For the standard Lebesgue measure: $d\mu(\mathbf{x}) = d\mathbf{x}$.

[3] Some additional definitions are given in the backup slides.

The space of scalar zero-mean functions

- Let $(\mathcal{X}, \mathfrak{F}, P_X)$ be a probability space where $\mathcal{X} \subseteq \mathbb{R}^N$ is the sample space, $\mathfrak{F}$ is the Borel σ-algebra on $\mathcal{X}$ and P_X is a probability measure. [4]

- Let $\mathcal{H}$ be the Hilbert space defined as [10, Ch. 2]:

$$\mathcal{H} = \left\{ h : \mathcal{X} \to \mathbb{R} \,\middle|\, E_X\{h\} = 0, E_X\{|h|^2\} < \infty \right\}.$$

- The expectation operator $E_X\{\cdot\}$ is

$$E_X\{h\} \triangleq \int_{\mathcal{X}} h(\mathbf{x}) dP_X(\mathbf{x}) = \int_{\mathcal{X}} h(\mathbf{x}) p_X(\mathbf{x}) d\mathbf{x},$$

where p_X is the probability density function (pdf).

- The inner product in $\mathcal{H}$ is: $\langle h_1, h_2 \rangle \triangleq E_X\{h_1 h_2\}$.

[4] Some additional definitions are given in the backup slides.

The projection theorem (1/2)

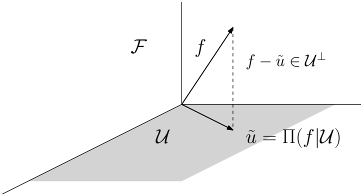
Theorem

Let $\mathcal{U}$ be a closed subspace of an Hilbert space $\mathcal{F}$ and take $f \in \mathcal{F}$. We call

$$d(f, \mathcal{U}) \triangleq \inf_{u \in \mathcal{U}} ||f - u||, \quad f \in \mathcal{F},$$

the distance of f to $\mathcal{U}$. Then there exists a unique element $\tilde{u} \in \mathcal{U}$ for which

$$||f - \tilde{u}|| = d(f, \mathcal{U}).$$



The projection theorem (2/2)

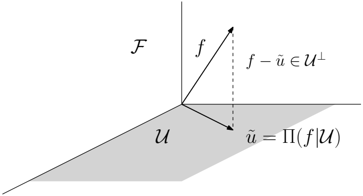
- f can be uniquely written as:

$$f = \tilde{u} + (f - \tilde{u}),$$

where $\tilde{u} \triangleq \Pi(f|\mathcal{U}) \in \mathcal{U}$ and $f - \tilde{u} \in \mathcal{U}^{\perp}$.

- $\tilde{u}$ is uniquely determined by the orthogonality constraint:

$$\langle f - \tilde{u}, u \rangle = \langle f - \Pi(f|\mathcal{U}), u \rangle = 0, \quad \forall u \in \mathcal{U}.$$



The linear span

- A *q-replicating* Hilbert space $\mathcal{F}^q$ is obtained by the Cartesian product of the q copies of $\mathcal{F}$ as $\mathcal{F}^q \triangleq \mathcal{F} \times \cdots \times \mathcal{F}$, then:

$$\mathcal{F}^q \ni \mathbf{f} = (f_1, f_2, \cdots, f_q)^T, \quad f_i \in \mathcal{F}.$$

- The inner product of $\mathcal{F}^q$ is induced by the one in $\mathcal{F}$:

$$\langle \mathbf{f}, \mathbf{g} \rangle = \sum_{i=1}^{q} \langle f_i, g_i \rangle .$$

- **Linear span**: Let $\mathbf{u} = (u_1, \cdots, u_k)^T$ be a column vector of k elements of $\mathcal{F}$. The *linear span* of the vector $\mathbf{u}$, defined as:

$$\mathcal{V} \triangleq \{\mathbf{v} | \mathbf{v} = \mathbf{A}\mathbf{u}, \mathbf{A} \text{ is any matrix in } \mathbb{R}^{q \times k}\},$$

is a *finite-dimensional* subspace of $\mathcal{F}^q$.

Projection onto a finite-dimensional subspace

$$\mathcal{V} \triangleq \{\mathbf{v} | \mathbf{v} = \mathbf{A}\mathbf{u}, \mathbf{A} \text{ is any matrix in } \mathbb{R}^{q \times k}\}.$$

- If $u_1, \ldots, u_k$ are linearly independent in $\mathcal{F}$, $\dim(\mathcal{V}) = kq$. [5]
- The projection of a generic element $\mathbf{f} \in \mathcal{F}^q$ onto the subspace $\mathcal{V}$ is given by [9, A.2], [10, Sec. 2.4]:

$$\Pi(\mathbf{f}|\mathcal{V}) = \left\langle \mathbf{f}, \mathbf{u}^T \right\rangle \left\langle \mathbf{u}, \mathbf{u}^T \right\rangle^{-1} \mathbf{u},$$

where

$$\left[\left\langle \mathbf{f}, \mathbf{u}^T \right\rangle\right]_{i,j} \triangleq \langle f_i, u_j \rangle, \quad \begin{array}{l} i = 1, \ldots, q, \\ j = 1, \ldots, k, \end{array}$$

$$\left[\left\langle \mathbf{u}, \mathbf{u}^T \right\rangle\right]_{i,j} \triangleq \langle u_i, u_j \rangle, \quad i, j = 1, \ldots, k.$$

[5] The proof of this result is in the backup slides (see also [10, Sec. 2.4]).

The vector-valued zero-mean functions

- Let $(\mathcal{X}, \mathfrak{F}, P_X)$ be a probability space.
- Let $\mathcal{H}^q$ be the q-replicating Hilbert space [10, Ch. 2]:

$$\begin{aligned}\mathcal{H}^q &= \mathcal{H} \times \cdots \times \mathcal{H} \\ &= \left\{ \mathbf{h} : \mathcal{X} \to \mathbb{R}^q \,\middle|\, E_X\{\mathbf{h}\} = \mathbf{0}, E_X\{\mathbf{h}^T\mathbf{h}\} < \infty \right\},\end{aligned}$$

- The induced inner product is:

$$\langle \mathbf{h}_1, \mathbf{h}_2 \rangle \triangleq E_X\{\mathbf{h}_1^T \mathbf{h}_2\}.$$

- The *covariance matrix* of $\mathbf{h} \in \mathcal{H}^q$ is:

$$\mathbf{C}_X(\mathbf{h}) \triangleq E_X\{\mathbf{h}\mathbf{h}^T\}.$$

Projection onto finite-dimensional subspaces

- Let $\mathbf{u} = (u_1, \cdots, u_k)^T$ be a column vector of k arbitrary elements of $\mathcal{H}$ and let $\mathcal{V}$ be its linear span.

- The orthogonal projection of an arbitrary element $\mathbf{h} \in \mathcal{H}^q$ onto $\mathcal{V}$ is unique and it is given by [9, A.2], [10, Sec. 2.4]:

$$\begin{aligned}\Pi(\mathbf{h}|\mathcal{V}) &= E_X\{\mathbf{h}\mathbf{u}^T\}E_X\{\mathbf{u}\mathbf{u}^T\}^{-1}\mathbf{u} \\ &= E_X\{\mathbf{h}\mathbf{u}^T\}\mathbf{C}_X(\mathbf{u})^{-1}\mathbf{u}.\end{aligned}$$

- Linear Minimum Mean Square Error (LMMSE) estimator:
  1. $\mathrm{MSE} \triangleq ||\mathbf{h} - \mathbf{A}\mathbf{u}||^2$ is minimized by $\Pi(\mathbf{h}|\mathcal{V})$, then $\hat{\mathbf{h}}_{LMMSE} = E_X\{\mathbf{h}\mathbf{u}^T\}\mathbf{C}_X(\mathbf{u})^{-1}\mathbf{u}$.
  2. The "orthogonality principle" is nothing but the Projection Theorem.

Score vectors as elements of $\mathcal{H}^r$ (1/2)

- Let us go back to the *parametric model*:

$$\mathcal{P}_{\boldsymbol{\theta},\boldsymbol{\eta}} \triangleq \left\{ p_X(\mathbf{x}|\boldsymbol{\theta},\boldsymbol{\eta}), \boldsymbol{\theta} \in \Theta \subseteq \mathbb{R}^q, \boldsymbol{\eta} \in \Gamma \subseteq \mathbb{R}^d \right\},$$

  - $\boldsymbol{\theta} \in \Theta$ is the vector of the parameters of interest,
  - $\boldsymbol{\eta} \in \Gamma$ is the vector of the (unknown) nuisance parameters,
  - $\boldsymbol{\gamma} \triangleq \left(\boldsymbol{\theta}^T, \boldsymbol{\eta}^T\right)^T \in \mathbb{R}^r, \; r = q + d$.
  - $p_0(\mathbf{x}) \triangleq p_X(\mathbf{x}|\boldsymbol{\theta}_0, \boldsymbol{\eta}_0)$ is the "true" pdf.

- The **score vector** for the true parameter vector γ_0 is:

$$\mathbf{s}_{\gamma_0} \triangleq \nabla_{\gamma} \ln p_X(\mathbf{x}|\gamma_0) = \begin{pmatrix} \nabla_{\boldsymbol{\theta}} \ln p_X(\mathbf{x}|\boldsymbol{\theta}_0, \boldsymbol{\eta}_0) \\ \nabla_{\boldsymbol{\eta}} \ln p_X(\mathbf{x}|\boldsymbol{\theta}_0, \boldsymbol{\eta}_0) \end{pmatrix} \triangleq \begin{pmatrix} \mathbf{s}_{\boldsymbol{\theta}_0} \\ \mathbf{s}_{\boldsymbol{\eta}_0} \end{pmatrix}$$

  - $\mathbf{s}_{\boldsymbol{\theta}_0}$ is $q \times 1$ the score vector of the parameters of interest,
  - $\mathbf{s}_{\boldsymbol{\eta}_0}$ is $d \times 1$ the nuisance score vector.

Score vectors as elements of $\mathcal{H}^r$ (2/2)

- Under standard regularity conditions [16]:

$$E_0\left\{\mathbf{s}_{\gamma_0}\right\} = \int_{\mathcal{X}} \nabla_\gamma \ln p_X(\mathbf{x}|\gamma_0) dP_0(\mathbf{x})$$
$$= \int_{\mathcal{X}} \frac{\nabla_\gamma p_X(\mathbf{x}|\gamma_0)}{p_0(\mathbf{x})} p_0(\mathbf{x}) d\mathbf{x} = \nabla_\gamma \int_{\mathcal{X}} p_X(\mathbf{x}|\gamma_0) d\mathbf{x} = 0,$$

and $E_0\left\{\mathbf{s}_{\gamma_0}^T \mathbf{s}_{\gamma_0}\right\} < \infty$.

- Then, by definition[6] of $\mathcal{H}^r$:

$$\mathcal{H}^r \ni \mathbf{s}_{\gamma_0} = \begin{pmatrix} \mathbf{s}_{\boldsymbol{\theta}_0} \\ \mathbf{s}_{\boldsymbol{\eta}_0} \end{pmatrix} \quad \Rightarrow \quad \mathbf{s}_{\boldsymbol{\theta}_0} \in \mathcal{H}^q, \quad \mathbf{s}_{\boldsymbol{\eta}_0} \in \mathcal{H}^d.$$

[6] $\mathcal{H}^r = \left\{\mathbf{h} : \mathcal{X} \to \mathbb{R}^r \,\middle|\, E_0\{\mathbf{h}\} = \mathbf{0}, E_0\{\mathbf{h}^T\mathbf{h}\} < \infty \right\}$.

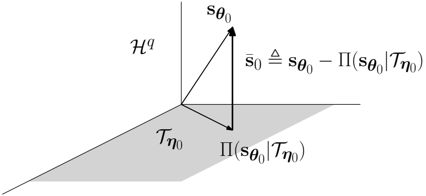
The efficient score vector

- The *nuisance tangent space*[7] $\mathcal{T}_{\boldsymbol{\eta}_0}$ is defined as the linear span of $\mathbf{s}_{\boldsymbol{\eta}_0}$ in $\mathcal{H}^q$ [10, Ch. 3]:

$$\mathcal{T}_{\boldsymbol{\eta}_0} \triangleq \{\mathbf{t} | \mathbf{t} = \mathbf{A}\mathbf{s}_{\boldsymbol{\eta}_0}, \mathbf{A} \text{ is any matrix in } \mathbb{R}^{q \times d}\} \subset \mathcal{H}^q.$$

- Let us define the **efficient score vector** as [9, Ch. 2]:

$$\begin{aligned} \bar{\mathbf{s}}_0 &\triangleq \mathbf{s}_{\boldsymbol{\theta}_0} - \Pi(\mathbf{s}_{\boldsymbol{\theta}_0} | \mathcal{T}_{\boldsymbol{\eta}_0}) \\ &= \mathbf{s}_{\boldsymbol{\theta}_0} - E\{\mathbf{s}_{\boldsymbol{\theta}_0}\mathbf{s}_{\boldsymbol{\eta}_0}^T\}\mathbf{I}_{\boldsymbol{\eta}_0\boldsymbol{\eta}_0}^{-1}\mathbf{s}_{\boldsymbol{\eta}_0}. \end{aligned}$$



[7] The geometrical intuition behind this terminology is given in the backup slides.

Evaluation of the CRB using $\bar{\mathbf{s}}_0$

- $\bar{\mathbf{s}}_0$ is the residual of $\mathbf{s}_{\boldsymbol{\theta}_0}$ after projecting it onto the nuisance tangent space $\mathcal{T}_{\boldsymbol{\eta}_0}$.

- Let us define the efficient FIM as:

$$\bar{\mathbf{I}}(\boldsymbol{\theta}_0|\boldsymbol{\eta}_0) \triangleq E_0\left\{\bar{\mathbf{s}}_0\bar{\mathbf{s}}_0^T\right\}.$$

- Through direct calculation, we get:

$$\bar{\mathbf{I}}(\boldsymbol{\theta}_0|\boldsymbol{\eta}_0) = \mathbf{I}_{\boldsymbol{\theta}_0\boldsymbol{\theta}_0} - \mathbf{I}_{\boldsymbol{\theta}_0\boldsymbol{\eta}_0}\mathbf{I}_{\boldsymbol{\eta}_0\boldsymbol{\eta}_0}^{-1}\mathbf{I}_{\boldsymbol{\theta}_0\boldsymbol{\eta}_0}^T.$$

- The inverse of $\bar{\mathbf{I}}(\boldsymbol{\theta}_0|\boldsymbol{\eta}_0)$ is exactly the $\mathrm{CRB}(\boldsymbol{\theta}_0|\boldsymbol{\eta}_0)$ previously derived by means of the Matrix Inversion Lemma:

$$\left[E\left\{\bar{\mathbf{s}}_0\bar{\mathbf{s}}_0^T\right\}\right]^{-1} \triangleq \left[\bar{\mathbf{I}}(\boldsymbol{\theta}_0|\boldsymbol{\eta}_0)\right]^{-1} = \mathrm{CRB}(\boldsymbol{\theta}_0|\boldsymbol{\eta}_0).$$

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The three basic ingredients

- In summary, to derive the $\mathrm{CRB}(\boldsymbol{\theta}_0|\boldsymbol{\eta}_0)$, we only need:

  1. The Hilbert space $\mathcal{H}^q$,
  2. The nuisance tangent space $\mathcal{T}_{\boldsymbol{\eta}_0} \subset \mathcal{H}^q$ of the parametric model $\mathcal{P}_{\boldsymbol{\theta},\boldsymbol{\eta}}$ at $\boldsymbol{\eta}_0$,
  3. The projection operator onto $\mathcal{T}_{\boldsymbol{\eta}_0}$: $\Pi(\mathbf{s}_{\boldsymbol{\theta}_0}|\mathcal{T}_{\boldsymbol{\eta}_0})$.

- Important fact: None of them require the finite dimensionality of the nuisance parameters [7].

- This alternative way to calculate the CRB can be extended to semiparametric models.

- To make this extension possible, we have to introduce the concept of *parametric submodel*.

Parametric submodels (1/3)

- Let us recall the semiparametric model:

$$\mathcal{P}_{\boldsymbol{\theta},g} \triangleq \{p_X(\mathbf{x}|\boldsymbol{\theta},g), \boldsymbol{\theta} \in \Theta \subseteq \mathbb{R}^q, g \in \mathcal{L}\}.$$

- The **i-th parametric submodel**[8] of $\mathcal{P}_{\boldsymbol{\theta},g}$ is defined as [10, Sec. 4.2], [9, Sec. 3.1], [17,18,11], :

$$\mathcal{P}_{\boldsymbol{\theta},\nu_i} = \{p_X(\mathbf{x}|\boldsymbol{\theta},\nu_i(\mathbf{x},\boldsymbol{\eta})), \boldsymbol{\theta} \in \Theta, \boldsymbol{\eta} \in \Gamma_i\},$$

where:

$$\begin{aligned} \nu_i : \Gamma_i &\to \mathcal{L} \\ \boldsymbol{\eta} &\mapsto \nu_i(\cdot,\boldsymbol{\eta}), \end{aligned}$$

- The function $\nu_i \in \mathcal{L}$ is a *known* function parametrized by a vector of *unknown* parameters.

[8] An explicit example of parametric submodel is given in the backup slides.

Parametric submodels (2/3)

- Denote the "true semiparametric vector" and the related true pdf as $(\boldsymbol{\theta}_0^T, g_0)^T$ and $p_0(\mathbf{x}) \triangleq p_X(\mathbf{x}|\boldsymbol{\theta}_0, g_0)$, respectively.
- For every $i \in \mathcal{I}$, the *i-th parametric submodel*:

$$\mathcal{P}_{\boldsymbol{\theta},\nu_i} = \left\{ p_X(\mathbf{x}|\boldsymbol{\theta}, \nu_i(\mathbf{x}, \boldsymbol{\eta}), \boldsymbol{\theta} \in \Theta, \boldsymbol{\eta} \in \Gamma_i \right\},$$

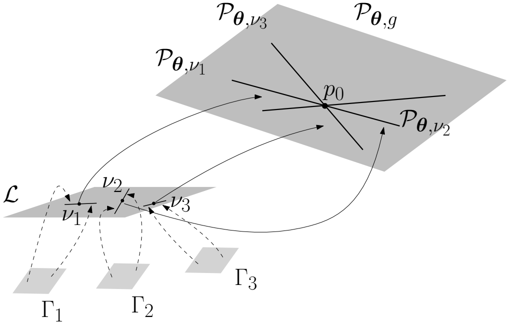
has to satisfy the following three conditions [10, Sec. 4.2]:

C0) $\nu_i : \Gamma_i \to \mathcal{L}$ is a smooth parametric map,

C1) $\mathcal{P}_{\boldsymbol{\theta},\nu_i} \subseteq \mathcal{P}_{\boldsymbol{\theta},g}$,

C2) $p_0(\mathbf{x}) \in \mathcal{P}_{\boldsymbol{\theta},\nu_i}$, i.e. there exists a vector $(\boldsymbol{\theta}_0^T, \boldsymbol{\eta}_0^T)^T$ such that $p_X(\mathbf{x}|\boldsymbol{\theta}_0, \nu_i(\mathbf{x}, \boldsymbol{\eta}_0)) = p_X(\mathbf{x}|\boldsymbol{\theta}_0, g_0) \triangleq p_0(\mathbf{x})$.

Parametric submodels (3/3)



- The generalization to the semiparametric framework can be done in two steps:

  1. Exploit the obtained results in the set of (artificial) parametric submodels $\{\mathcal{P}_{\boldsymbol{\theta},\nu_i}\}_{i\in\mathcal{I}}$,
  2. "Take the limit" to generalize them in the infinite-dimensional semiparametric framework.

Semiparametric nuisance tangent space (1/2)

- For every parametric submodel:

$$\mathcal{P}_{\boldsymbol{\theta},\nu_i} = \{p_X(\mathbf{x}|\boldsymbol{\theta},\nu_i(\mathbf{x},\boldsymbol{\eta})), \boldsymbol{\theta} \in \Theta, \boldsymbol{\eta} \in \Gamma_i\},$$

we have a relevant nuisance tangent space:

$$\mathcal{T}_{\boldsymbol{\eta}_{0,i}} \triangleq \{\mathbf{t}_i | \mathbf{t}_i = \mathbf{A}_i \mathbf{s}_{\boldsymbol{\eta}_{0,i}} : \mathbf{A}_i \text{ is any matrix in } \mathbb{R}^{q \times d_i}\},$$

where $\mathbf{s}_{\boldsymbol{\eta}_{0,i}} \triangleq \nabla_{\boldsymbol{\eta}} \ln p_X(\mathbf{x}|\boldsymbol{\theta}_0,\nu_i(\mathbf{x},\boldsymbol{\eta}_0))$.

- The **semiparametric nuisance tangent space** is defined as:[9]

$$\mathcal{T}_{g_0} \triangleq \overline{\bigcup_{\{\mathcal{P}_{\boldsymbol{\theta},\nu_i}\}_{i \in \mathcal{I}}} \mathcal{T}_{\boldsymbol{\eta}_{0,i}}} \subseteq \mathcal{H}^q.$$

[9] The closure $\overline{\mathcal{A}}$ of a set $\mathcal{A}$ is defined as the smallest closed set that contains $\mathcal{A}$, or equivalently, as the set of all elements in $\mathcal{A}$ together with all the limit points of $\mathcal{A}$.

Semiparametric nuisance tangent space (2/2)

- Recall that the Hilbert space $\mathcal{H}^q$ is a complete normed space with norm:

$$||\mathbf{h}_1 - \mathbf{h}_2|| = \sqrt{E_0\{(\mathbf{h}_1 - \mathbf{h}_2)^T(\mathbf{h}_1 - \mathbf{h}_2)\}}, \quad \forall \mathbf{h}_1, \mathbf{h}_2 \in \mathcal{H}^q.$$

- The semiparametric nuisance tangent space $\mathcal{T}_{g_0} \subseteq \mathcal{H}^q$ can be expressed as [10, Sec. 4.4],[19],[18]: [10]

$$\mathcal{T}_{g_0} \triangleq \left\{\mathbf{h} \in \mathcal{H}^q | \forall \varepsilon > 0, \exists i \in \mathcal{I} : ||\mathbf{h} - \mathbf{A}_i \mathbf{s}_{\boldsymbol{\eta}_{0,i}}|| < \varepsilon\right\}$$

- Unlike $\mathcal{T}_{\boldsymbol{\eta}_{0,i}}$ that has finite dimension, $\mathcal{T}_{g_0}$ is in general an infinite-dimensional subspace of $\mathcal{H}^q$.

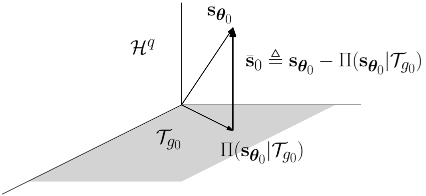
[10] A more explicit definition of the nuisance tangent space requires the notion of *Hellinger differentiability* [19],[9, Sec. 3.2]. See also the backup slides.

The projection operator $\Pi(\cdot|\mathcal{T}_{g_0})$

- The *existence* and the *uniqueness* of the projection operator $\Pi(\cdot|\mathcal{T}_{g_0})$ is guaranteed by the Projection Theorem.

- The **semiparametric efficient score vector** for the estimation of $\boldsymbol{\theta}_0 \in \Theta$ in the presence of the nuisance function $g_0 \in \mathcal{L}$ is [9, Sec. 3.3]:

$$\bar{\mathbf{s}}_0 \triangleq \mathbf{s}_{\boldsymbol{\theta}_0} - \Pi(\mathbf{s}_{\boldsymbol{\theta}_0}|\mathcal{T}_{g_0}).$$



The Semiparametric CRB (SCRB) (1/2)

Theorem ([9, Sec. 3.4], [19], [10, Theo. 4.2], [18]):
A lower bound on the MSE of "any" [11] robust estimator of $\boldsymbol{\theta}_0$ in the presence of the nuisance function $g_0 \in \mathcal{L}$ is given by:

$$\mathrm{SCRB}(\boldsymbol{\theta}_0|g_0) = [\bar{\mathbf{I}}(\boldsymbol{\theta}_0|g_0)]^{-1},$$

where $\bar{\mathbf{I}}(\boldsymbol{\theta}_0|g_0) \triangleq E_0\{\bar{\mathbf{s}}_0\bar{\mathbf{s}}_0^T\}$ is the *semiparametric FIM* (SFIM) and:

$$\bar{\mathbf{s}}_0 \triangleq \mathbf{s}_{\boldsymbol{\theta}_0} - \Pi(\mathbf{s}_{\boldsymbol{\theta}_0}|\mathcal{T}_{g_0}).$$

[10] J. M. Begun, W. J. Hall, W.-M. Huang, and J. A. Wellner, "Information and asymptotic efficiency in parametric-nonparametric models", *The Annals of Statistics*, vol. 11, no. 2, pp. 432-452, 1983.

[9, Sec. 3.4] P. Bickel, C. Klaassen, Y. Ritov, and J. Wellner, *Effient and Adaptive Estimation for Semiparametric Models*. Johns Hopkins University Press, 1993.

[11] The class of estimators to which the SCRB applies is discussed ahead.

The Semiparametric CRB (SCRB) (2/2)

- The expression of $\mathrm{SCRB}(\boldsymbol{\theta}_0|g_0)$ is formally equivalent to $\mathrm{CRB}(\boldsymbol{\theta}_0|\boldsymbol{\eta}_0)$ derived for finite-dimensional nuisance vectors.
- The Hilbert-space-based approach allows to handle both finite and infinite-dimensional nuisance parameters.
- The $\mathrm{SCRB}(\boldsymbol{\theta}_0|g_0)$ is higher than any $\mathrm{CRB}(\boldsymbol{\theta}_0|\boldsymbol{\eta}_{0,i})$ derived in the *i-th* parametric submodel.
- A semiparametric model contains less information on $\boldsymbol{\theta}_0$ than any of its possible parametric submodel.

A bound for any robust estimator

- The SCRB is a lower bound for the MSE of any *Regular and Asymptotically Linear (RAL) estimator* [9, Sec. 2.2 and Ch. 7], [10, Ch.3], [20, Ch. 4] [21,18,22,23].

- All the robust M-, S-, L- estimators belong to this class [24]:

- It can be shown that every RAL estimator is:
  1. Consistent: $\hat{\boldsymbol{\theta}}(\mathbf{x}_1, \ldots, \mathbf{x}_M) \triangleq \hat{\boldsymbol{\theta}}_M \underset{M \to \infty}{\longrightarrow} \boldsymbol{\theta}_0$,
  2. Asymptotically normal: $\sqrt{M}(\hat{\boldsymbol{\theta}}_M - \boldsymbol{\theta}_0) \underset{M \to \infty}{\sim} \mathcal{N}(\mathbf{0}, \boldsymbol{\Xi}(\boldsymbol{\theta}_0, g_0))$.

- Consequently, the following inequality holds [9, Ch. 2 and 3]:

$$\boldsymbol{\Xi}(\boldsymbol{\theta}_0, g_0) \geq \mathrm{SCRB}(\boldsymbol{\theta}_0 | g_0).$$

- Note that efficient estimators may not exist [25].

Evaluation of $\Pi(\cdot|\mathcal{T}_{g_0})$

- The crucial step to evaluate $\mathrm{SCRB}(\boldsymbol{\theta}_0|g_0)$ is in determining the semiparametric efficient score vector:

$$\bar{\mathbf{s}}_0 \triangleq \mathbf{s}_{\boldsymbol{\theta}_0} - \Pi(\mathbf{s}_{\boldsymbol{\theta}_0}|\mathcal{T}_{g_0}).$$

- To this end, we need to:
  1. Calculate $\mathbf{s}_{\boldsymbol{\theta}_0} = \nabla_{\boldsymbol{\theta}} \ln p_X(\mathbf{x}|\boldsymbol{\theta}_0, g_0)$ (easy task),
  2. Evaluate the projection $\Pi(\mathbf{s}_{\boldsymbol{\theta}_0}|\mathcal{T}_{g_0})$ (difficult task).

- Two possible approaches:
  1. Least Favourable Submodel (if it exists) [12],
  2. **Projection as a conditional expectation**.

[12] Some additional details are given in the backup slides.

Projection and conditional expectation (1/3)

- We defined $\mathcal{H}^q$ as the Hilbert space of the q-dimensional zero-mean function on the probability space $(\mathcal{X}, \mathfrak{F}, P_X)$:

$$\mathbf{h} \equiv \mathbf{h}(\mathbf{x}), \quad \mathbf{x} \in \mathcal{X} \subseteq \mathbb{R}^N.$$

- Let $f : \mathbb{R}^N \to \mathbb{R}$ be a measurable function. We define a *statistic* V of the random vector $\mathbf{x}$ as:

$$V =_d f(\mathbf{x}), \quad \forall \mathbf{x} \in \mathcal{X}.$$

- Let $\mathfrak{G}(V) \subseteq \mathfrak{F}$ be the sub-σ algebra generated by V. [13]

- The set of the q-dim zero-mean functions on $(\mathcal{X}, \mathfrak{G}(V), P_X)$ is a closed linear subspace, say $\mathcal{V}$, of $\mathcal{H}^q$ [26, Theo. 23.2].

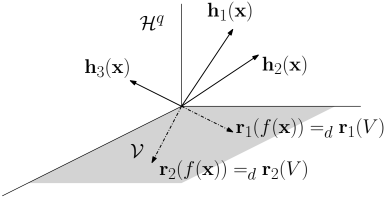
[13] Additional details are given in the backup slides.

Projection and conditional expectation (2/3)

- Let $\mathbf{r} \in \mathcal{H}^q$ be a zero-mean function of $\mathbf{x} \in \mathcal{X}$ through the function f, i.e.: [14]

$$\mathbf{r} \equiv \mathbf{r}(f(\mathbf{x})) =_d \mathbf{r}(V) \in \mathcal{V} \subseteq \mathcal{H}^q.$$

- Consequently, $\mathbf{r} \in \mathcal{H}^q$ can be considered as a q-dimensional function defined on $(\mathcal{X}, \mathfrak{G}(V), P_X)$ with $\mathfrak{G}(V) \subseteq \mathfrak{F}$.



[14] The symbol "$=_d$" means "has the same distribution as".

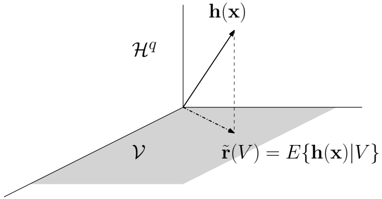
Projection and conditional expectation (3/3)

- The conditional expectation $E\{\mathbf{h}|V\}$ is the unique element in $\mathcal{V}$, such that [26, Def. 23.3, Theo. 23.3][15]:

$$\langle \mathbf{h} - E\{\mathbf{h}|V\}, \mathbf{r}\rangle \triangleq E\left\{\left(\mathbf{h} - E\{\mathbf{h}|V\}\right)^T \mathbf{r}\right\} = 0, \quad \forall \mathbf{r} \in \mathcal{V}.$$

Given the Projection Theorem, the previous definition implies:

$$\Pi(\cdot|\mathcal{V}) = E\{\cdot|V\}.$$



[15] This definition is consistent with the classical one [26, Ch. 32]. See the proof in the backup slides.

Part II - Outline of the talk

Why semiparametric models?

CRB in parametric models with finite-dimensional nuisance parameters: classical approach

CRB in parametric models with finite-dimensional nuisance parameters: "Hilbert-space-based" approach

Extension to semiparametric models

Semiparametric interpretation of Real and Complex ES distributions

Examples

Spherically Symmetric (SS) distributions

- Let $\mathbf{z} \in \mathbb{R}^N$ be a real-valued random vector.
- Let $\mathcal{O}$ be the set of all unitary transformations:

$$\begin{aligned} \mathcal{O} \ni O : \mathbb{R}^N &\to \mathbb{R}^N \\ \mathbf{z} &\mapsto O(\mathbf{z}) = \mathbf{O}\mathbf{z}, \end{aligned}$$

  for any unitary matrix $\mathbf{O}$, i.e $\mathbf{O}^T\mathbf{O} = \mathbf{O}\mathbf{O}^T = \mathbf{I}$.

- Then, $\mathbf{z}$ is said to be SS-distributed if its distribution is invariant to any unitary transformations $\mathbf{O} \in \mathcal{O}$, i.e.

$$\mathbf{z} =_d \mathbf{O}\mathbf{z}.$$

- We indicate with $\mathcal{S}$ the class of all SS-distributions.

Properties of the (SS) distributions (1/4)

Property P1 [16]

- The SS-distributed random vector $\mathbf{z} \sim SS(g)$ has a pdf:

$$p_Z(\mathbf{z}) = 2^{-N/2} g\left(||\mathbf{z}||^2\right),$$

where $\mathcal{G} \ni g$, is a function, called *density generator* and

$$\mathcal{G} = \left\{ g : \mathbb{R}_0^+ \to \mathbb{R}^+ \,\middle|\, \int_0^\infty t^{N/2-1} g(t) dt < \infty \right\}.$$

- The set of all SS pdfs can be described as:

$$\mathcal{S} = \left\{ p_Z | p_Z(\mathbf{z}) = 2^{-N/2} g\left(||\mathbf{z}||^2\right), \forall g \in \mathcal{G} \right\}.$$

[16] See [27] or [28, Ch. 3] for the proofs of these properties. A comprehensive list is also summarized in [29].

Property P2

- Let $s_N \triangleq 2\pi^{N/2}/\Gamma(N/2)$ be the surface area of the unit sphere $\mathbb{R}S^N$ in $\mathbb{R}^N$.

- The pdf of $\mathcal{Q} =_d ||\mathbf{z}||^2$, called *2nd-order modular variate*, is:

$$p_{\mathcal{Q}}(q) = s_N 2^{-N/2-1} q^{N/2-1} g\left(q\right).$$

- The pdf of $\mathcal{R} \triangleq \sqrt{\mathcal{Q}} =_d ||\mathbf{z}||$, called *modular variate*, is:

$$p_{\mathcal{R}}(r) = s_N 2^{-N/2} r^{N-1} g\left(r^2\right).$$

Properties of the (SS) distributions (3/4)

Property P3: *Stochastic Representation Theorem*

- Let $\mathbf{u} \sim \mathcal{U}(\mathbb{R}S^N)$ be a random vector uniformly distributed on $\mathbb{R}S^N$, i.e. $||\mathbf{u}|| = 1$.

- If $\mathbf{z} \in \mathbb{R}^N$ is SS-distributed, i.e. $\mathbf{z} \sim SS(g)$, then:

$$\mathbf{z} =_d \sqrt{\mathcal{Q}}\mathbf{u} =_d \mathcal{R}\mathbf{u},$$

- Moreover, $\mathcal{Q}$ and $\mathbf{u}$ (or $\mathcal{R}$ and $\mathbf{u}$) are independent.

- P2 and P3 imply that, not knowing the density generator g has an impact only on the pdf of the r.v. $\mathcal{R}$ (or $\mathcal{Q}$).

Properties of the (SS) distributions (4/4)

Property P4: *Invariant statistic*

- By definition of SS distributions, $||\cdot||$ is an *invariant statistic* since [30, Ch. 6]
$$||\mathbf{z}|| =_d ||\mathbf{O}\mathbf{z}||,$$
for every unitary matrix $\mathbf{O} \in \mathcal{O}$.

- Moreover, given two SS-distributed r.v. $\mathbf{z}_1$ and $\mathbf{z}_2$, we have:
$$||\mathbf{z}_1|| =_d ||\mathbf{z}_2|| \;\Rightarrow\; \mathbf{z}_1 =_d \mathbf{O}\mathbf{z}_2, \quad \forall \mathbf{O} \in \mathcal{O}.$$

- Then, the modular variate $\mathcal{R} =_d ||\mathbf{z}||$ is a *maximal invariant statistic* for the set of the SS-distributed random vectors.

Tangent space and invariance

- Let $\mathcal{A}$ be a group of transformations from $\mathbb{R}^N$ into itself:

$$\begin{aligned} \mathcal{A} \ni \alpha : \mathbb{R}^N &\to \mathbb{R}^N \\ \mathbf{z} &\mapsto \alpha(\mathbf{z}), \end{aligned}$$

- Suppose that $\mathcal{P}$ is a set of pdfs which are invariant with respect to $\mathcal{A}$, i.e.:

$$\mathcal{P} = \left\{ p_Z | p_Z(\alpha(\mathbf{z})) = p_Z(\mathbf{z}); \forall \alpha \in \mathcal{A}, \forall \mathbf{z} \in \mathbb{R}^N \right\}.$$

- Then, the tangent space $\mathcal{T}$ of $\mathcal{P}$ is given by [9, App. 3]: [17]

$$\mathcal{T} = \left\{ h \in \mathcal{H} | h(\alpha(\mathbf{z})) = h(\mathbf{z}), \forall \alpha \in \mathcal{A}, \forall \mathbf{z} \in \mathbb{R}^N \right\}$$

[17] Remember that $\mathcal{H} = \left\{ h : \mathcal{X} \to \mathbb{R} \middle| E_X\{h\} = 0, E_X\{|h|^2\} < \infty \right\}$.

Projection and invariance

If there exists an invariant statistic D for $\mathbf{z} \sim p_Z$ s.t.:

$$D =_d D(\alpha(\mathbf{z})), \quad \forall \alpha \in \mathcal{A},$$

then the projection operator on $\mathcal{T}$ can be calculated as [9, App. 3]:

$$\Pi(\cdot|\mathcal{T}) = E\{\cdot|D\}.$$

Example: SS distributions

- The tangent space $\mathcal{T}_{\mathcal{S}}$ is given by:

$$\mathcal{T}_{\mathcal{S}} = \left\{ h \in \mathcal{H} | h(||\mathbf{z}||) = h(\mathbf{z}), \forall \mathbf{z} \in \mathbb{R}^N \right\},$$

- $\Pi(\cdot|\mathcal{T}_{\mathcal{S}}) = E\{\cdot|\mathcal{R}\}$ where $\mathcal{R} =_d ||\mathbf{z}||$ is the modular variate.

Parametric group models (1/2)

- Let $\mathcal{A}$ be a group of *parametric* transformations from $\mathbb{R}^N$ into itself:

$$\mathcal{A} = \{\alpha | \alpha(\cdot; \boldsymbol{\theta}) \triangleq \alpha_{\boldsymbol{\theta}}(\cdot); \boldsymbol{\theta} \in \Theta \subseteq \mathbb{R}^q\}.$$

  - $\alpha_{\boldsymbol{\theta}}^{-1}(\cdot)$ defines the inverse of $\alpha_{\boldsymbol{\theta}}(\cdot)$,
  - $(\alpha_{\boldsymbol{\theta}_2} \circ \alpha_{\boldsymbol{\theta}_1})(\cdot) \triangleq \alpha_{\boldsymbol{\theta}_2}(\alpha_{\boldsymbol{\theta}_1}(\cdot))$ denotes the composition,
  - $\boldsymbol{\theta}_e$ indicates the parameter vector that characterizes the identity transformation $\alpha_{\boldsymbol{\theta}_e}$, s.t. $\alpha_{\boldsymbol{\theta}_e}(\cdot) = \cdot$.

Example: Let us define $\boldsymbol{\theta} \triangleq [\mu, \sigma]^T$, then:

$$\alpha_{\boldsymbol{\theta}}(z) \triangleq \mu + \sigma z,$$

$$\alpha_{\boldsymbol{\theta}}^{-1}(z) = (z - \mu)/\sigma, \qquad \boldsymbol{\theta}_e \triangleq [0, 1]^T.$$

Parametric group models (2/2)

- Let $\mathbf{z} \in \mathbb{R}^N$ be a random vector s.t. $\mathbf{z} \sim p_Z(\mathbf{z})$.
- The *parametric group model*, generated by the action of $\mathcal{A}$ on $\mathbf{z}$ can be expressed as:

$$\mathcal{P}_{\boldsymbol{\theta}} = \left\{ p_X | p_X(\mathbf{x}|\boldsymbol{\theta}) = |\mathbf{J}(\alpha_{\boldsymbol{\theta}}^{-1})(\mathbf{x})| p_Z(\alpha_{\boldsymbol{\theta}}^{-1}(\mathbf{x})); \boldsymbol{\theta} \in \Theta \right\},$$

where:

  - $[\mathbf{J}(\alpha_{\boldsymbol{\theta}}^{-1})(\mathbf{x})]_{i,j} \triangleq \partial[\alpha^{-1}(\mathbf{x}; \boldsymbol{\theta})]_i / \partial\theta_j$ is the Jacobian matrix of the inverse transformation $\alpha_{\boldsymbol{\theta}}^{-1}$,
  - $|\cdot|$ defines the (absolute value of the) determinant of the Jacobian matrix.

Semiparametric group models (1/2)

- If p_Z is allowed to vary in a function set $\mathcal{L}$, we get a *semiparametric group model*:

$$\mathcal{P}_{\boldsymbol{\theta},p_Z} = \{p_X | \, p_X(\mathbf{x}|\boldsymbol{\theta}, p_Z) = |\mathbf{J}(\alpha_{\boldsymbol{\theta}}^{-1})(\mathbf{x})| p_Z(\alpha_{\boldsymbol{\theta}}^{-1}(\mathbf{x})); \\ \boldsymbol{\theta} \in \Theta, p_Z \in \mathcal{L}\} \, .$$

- The calculation of the projection operator can be greatly simplified!
  1. Evaluate the projection on the semiparametric nuisance tangent space at the identity $\alpha_{\boldsymbol{\theta}_e}$.
  2. "Translate" the projection in any other $\boldsymbol{\theta}$ of the parameter space Θ.

Semiparametric group models (2/2)

- $\mathcal{T}_{p_{Z,0}}(\boldsymbol{\theta}_e) \subseteq \mathcal{H}^q$: Semiparametric nuisance tangent space at the identity $\boldsymbol{\theta}_e$.
- $\mathcal{T}_{p_{Z,0}}(\boldsymbol{\theta}) \subseteq \mathcal{H}^q$: Semiparametric nuisance tangent space at a generic $\boldsymbol{\theta} \in \Theta$.

The projection operator on $\mathcal{T}_{p_{Z,0}}(\boldsymbol{\theta})$ can be obtained as [9, Sec. 4.2, Lemma 3]:

$$\Pi(\cdot|\mathcal{T}_{p_{Z,0}}(\boldsymbol{\theta})) = \Pi(\cdot \circ \alpha_{\boldsymbol{\theta}}|\mathcal{T}_{p_{Z,0}}(\boldsymbol{\theta}_e)) \circ \alpha_{\boldsymbol{\theta}}^{-1}, \quad \forall \boldsymbol{\theta} \in \Theta.$$

From SS to RES distributions (1/2)

- Let us define the parameter space $\Theta \subseteq \mathbb{R}^q$ as:

$$\Theta = \{\boldsymbol{\theta} \in \mathbb{R}^q | \boldsymbol{\theta} = [\boldsymbol{\mu}^T, \mathrm{vecs}(\boldsymbol{\Sigma})^T]^T; \boldsymbol{\mu} \in \mathbb{R}^N, \boldsymbol{\Sigma} \in \mathcal{M}_N\}.$$

- We can define the group of parametric transformations $\mathcal{A}$ as:

$$\begin{aligned} \mathcal{A} \ni \alpha_{\boldsymbol{\theta}} : \mathbb{R}^N &\to \mathbb{R}^N, \ \forall \boldsymbol{\theta} \in \Theta \\ \mathbf{z} &\mapsto \alpha_{\boldsymbol{\theta}}(\mathbf{z}) = \boldsymbol{\mu} + \boldsymbol{\Sigma}^{1/2}\mathbf{z}. \end{aligned}$$

  - The identity $\alpha_{\boldsymbol{\theta}_e}$ is parametrized by $\boldsymbol{\theta}_e = [\mathbf{0}^T, \mathrm{vecs}(\mathbf{I})^T]^T$,
  - The inverse is simply given by:

$$\alpha_{\boldsymbol{\theta}}^{-1}(\cdot) = \boldsymbol{\Sigma}^{-1/2}(\cdot - \boldsymbol{\mu}).$$

From SS to RES distributions (2/2)

- A random vector $\mathbf{x} \in \mathbb{R}^N$ is said to be RES-distributed if it can be expressed as:

$$\mathbf{x} = \alpha_{\boldsymbol{\theta}}(\mathbf{z}) = \boldsymbol{\mu} + \boldsymbol{\Sigma}^{1/2}\mathbf{z} =_d \boldsymbol{\mu} + \mathcal{R}\boldsymbol{\Sigma}^{1/2}\mathbf{u},$$

  - $\mathbf{z} \sim SS(g)$ is an SS-distributed random vector,
  - $\mathbf{u} \sim \mathcal{U}(\mathbb{R}S^N)$ and $\mathcal{R} = \sqrt{\mathcal{Q}}$ is the modular variate, s.t.:

$$\mathcal{Q} =_d ||\mathbf{z}||^2 = ||\alpha_{\boldsymbol{\theta}}^{-1}(\mathbf{x})||^2 = (\mathbf{x} - \boldsymbol{\mu})^T \boldsymbol{\Sigma}^{-1}(\mathbf{x} - \boldsymbol{\mu}).$$

- RES distributions represent a semiparametric group model:

$$\mathcal{P}_{\boldsymbol{\theta},g} = \left\{ p_X | p_X(\mathbf{x}|\boldsymbol{\theta}, g) = 2^{-N/2}|\boldsymbol{\Sigma}|^{-1/2} g(||\alpha_{\boldsymbol{\theta}}^{-1}(\mathbf{x})||^2); \right. \\ \left. \boldsymbol{\theta} \in \Theta, g \in \mathcal{G} \right\},$$

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Examples

Evaluation of the SCRB for the RES class

$$p(\mathbf{x}|\boldsymbol{\theta}_0, g_0) = 2^{-N/2}|\boldsymbol{\Sigma}_0|^{-1/2} g((\mathbf{x}-\boldsymbol{\mu}_0)^T \boldsymbol{\Sigma}_0^{-1}(\mathbf{x}-\boldsymbol{\mu}_0)),$$

$$\boldsymbol{\theta}_0 = [\boldsymbol{\mu}_0^T, \mathrm{vecs}(\boldsymbol{\Sigma}_0)^T]^T.$$

- **Problem**: Find the (Constrained) SCRB on the estimation of the mean vector $\boldsymbol{\mu}_0$ and of the scatter matrix $\boldsymbol{\Sigma}_0$ when the density generator g_0 is unknown.

- To avoid the ambiguity between $\boldsymbol{\Sigma}_0$ and g_0, we put a constraint on the scatter matrix:

$$\mathbf{c}(\boldsymbol{\Sigma}_0) = \mathbf{0}.$$

- All the details can be found in [29].

Evaluation of the SCRB for the RES class

Step A: Evaluation of the score vector $\mathbf{s}_{\boldsymbol{\theta}_0}$

- By definition:

$$\mathbf{s}_{\boldsymbol{\theta}_0} = \nabla_{\boldsymbol{\theta}} \ln p_X(\mathbf{x}|\boldsymbol{\theta}_0, g_0) = \begin{pmatrix} \mathbf{s}_{\boldsymbol{\mu}_0} \\ \mathbf{s}_{\mathrm{vecs}(\boldsymbol{\Sigma}_0)} \end{pmatrix}$$

- Through direct calculation, we get:

$$\mathbf{s}_{\boldsymbol{\mu}_0} =_d -2\sqrt{\mathcal{Q}}\psi_0(\mathcal{Q})\boldsymbol{\Sigma}_0^{-1/2}\mathbf{u},$$

$$\mathbf{s}_{\mathrm{vecs}(\boldsymbol{\Sigma}_0)} =_d -\mathbf{D}_N^T \Big(2^{-1}\mathrm{vec}(\boldsymbol{\Sigma}_0^{-1})+ \\ +\mathcal{Q}\psi_0(\mathcal{Q})\boldsymbol{\Sigma}_0^{-1/2} \otimes \boldsymbol{\Sigma}_0^{-1/2}\mathrm{vec}(\mathbf{u}\mathbf{u}^T)\Big),$$

  - $\psi_0(t) \triangleq d \ln g_0(t)/dt,$
  - *Duplication matrix*: $\mathbf{D}_N \mathrm{vecs}(\mathbf{A}) = \mathrm{vec}(\mathbf{A})$, $\forall \mathbf{A}$ symmetric.

Evaluation of the SCRB for the RES class

Step B: Evaluation of the projection operator $\Pi(\mathbf{s}_{\boldsymbol{\theta}_0}|\mathcal{T}_{g0})$

- Due to the group structure underlying the RES class, $\mathcal{T}_{g0}$ evaluated at the group identity $\boldsymbol{\theta}_e$ is given by:

$$\mathcal{T}_{g0}(\boldsymbol{\theta}_e) = \{\mathbf{l}|\mathbf{l} = h\mathbf{a}; h \in \mathcal{T}_{\mathcal{S}}, \mathbf{a} \in \mathbb{R}^q\};$$

where $\mathcal{T}_{\mathcal{S}}$ is the tangent space of the SS distributions:

$$\mathcal{T}_{\mathcal{S}} = \left\{h \in \mathcal{H}|h(||\mathbf{x}||) = h(\mathbf{x}), \forall \mathbf{x} \in \mathbb{R}^N\right\},$$

- Using the property of the semiparametric group model:

$$\begin{aligned}\Pi(\mathbf{s}_{\boldsymbol{\theta}_0}|\mathcal{T}_{g0}(\boldsymbol{\theta}_0)) &= \Pi\left(\mathbf{s}_{\boldsymbol{\theta}_0} \circ \alpha_{\boldsymbol{\theta}_0}|\mathcal{T}_{g0}(\boldsymbol{\theta}_e)\right) \circ \alpha_{\boldsymbol{\theta}_0}^{-1} \\ &= E\left\{\mathbf{s}_{\boldsymbol{\theta}_0} \circ \alpha_{\boldsymbol{\theta}_0}|\mathcal{R}\right\} \circ \alpha_{\boldsymbol{\theta}_0}^{-1}.\end{aligned}$$

Evaluation of the SCRB for the RES class

- Through direct calculation (see [29] for the details):

$$\Pi(\mathbf{s}_{\boldsymbol{\theta}_0}|\mathcal{T}_{g_0}) = \begin{pmatrix} \Pi(\mathbf{s}_{\boldsymbol{\mu}_0}|\mathcal{T}_{g_0}) \\ \Pi(\mathbf{s}_{\mathrm{vecs}(\boldsymbol{\Sigma}_0)}|\mathcal{T}_{g_0}) \end{pmatrix}$$

$$=_d \begin{pmatrix} \mathbf{0} \\ -\mathbf{D}_N^T \left(\frac{1}{2} + \frac{1}{N}\mathcal{Q}\psi_0(\mathcal{Q})\right) \mathrm{vec}(\boldsymbol{\Sigma}_0^{-1}) \end{pmatrix}.$$

- The score function $\mathbf{s}_{\boldsymbol{\mu}_0}$ of the mean value is orthogonal to the nuisance tangent space $\mathcal{T}_{g_0}$,

- Not knowing the true g_0 does not have any impact in the (asymptotic) estimation performance of $\boldsymbol{\mu}_0$ [21].

Evaluation of the SCRB for the RES class

Step C: Evaluation of the semiparametric FIM $\bar{\mathbf{I}}(\boldsymbol{\theta}_0, g_0)$

- The efficient score vector $\bar{\mathbf{s}}_0$ can then be expressed as:

$$\bar{\mathbf{s}}_0 = \mathbf{s}_{\boldsymbol{\theta}_0} - \Pi(\mathbf{s}_{\boldsymbol{\theta}_0}(\mathbf{x})|\mathcal{T}_{g_0})$$
$$=_d \begin{pmatrix} -2\sqrt{\mathcal{Q}}\psi_0(\mathcal{Q})\boldsymbol{\Sigma}_0^{-1/2}\mathbf{u} \\ -\mathbf{D}_N^T\mathcal{Q}\psi_0(\mathcal{Q})\left(\boldsymbol{\Sigma}_0^{-1/2}\otimes\boldsymbol{\Sigma}_0^{-1/2}\mathrm{vec}(\mathbf{u}\mathbf{u}^T) - \frac{\mathrm{vec}(\boldsymbol{\Sigma}_0^{-1})}{N}\right) \end{pmatrix}.$$

- Finally the SFIM $\bar{\mathbf{I}}(\boldsymbol{\theta}_0|g_0)$ can be obtained as:

$$\bar{\mathbf{I}}(\boldsymbol{\theta}_0|g_0) = E_0\{\bar{\mathbf{s}}_0\bar{\mathbf{s}}_0^T\}$$
$$= \begin{pmatrix} \mathbf{C}_0(\bar{\mathbf{s}}_{\boldsymbol{\mu}_0}) & \mathbf{0} \\ \mathbf{0}^T & \mathbf{C}_0(\bar{\mathbf{s}}_{\mathrm{vecs}(\boldsymbol{\Sigma}_0)}) \end{pmatrix},$$

where $\mathbf{C}_0(\mathbf{h}) \triangleq E_0\{\mathbf{h}\mathbf{h}^T\}$, $\forall \mathbf{h} \in \mathcal{H}^q$.

Evaluation of the SCRB for the RES class

- Through direct calculation of the expectation, we get:

$$\mathbf{C}_0(\bar{\mathbf{s}}_{\boldsymbol{\mu}_0}) = \frac{4E\{\mathcal{Q}\psi_0(\mathcal{Q})^2\}}{N}\boldsymbol{\Sigma}_0^{-1},$$

and

$$\mathbf{C}_0(\bar{\mathbf{s}}_{\mathrm{vecs}(\boldsymbol{\Sigma}_0)}) = \frac{2E\{\mathcal{Q}^2\psi_0(\mathcal{Q})^2\}}{N(N+2)}\times$$
$$\times\, \mathbf{D}_N^T\left(\boldsymbol{\Sigma}_0^{-1}\otimes\boldsymbol{\Sigma}_0^{-1} - \frac{1}{N}\mathrm{vec}(\boldsymbol{\Sigma}_0^{-1})\mathrm{vec}(\boldsymbol{\Sigma}_0^{-1})^T\right)\mathbf{D}_N.$$

- The block-diagonal structure of $\bar{\mathbf{I}}(\boldsymbol{\theta}_0|g_0)$ implies that the estimates of vector $\boldsymbol{\mu}_0$ and $\boldsymbol{\Sigma}_0$ are asymptotically decoupled.

- $\boldsymbol{\mu}_0$ can be substituted with any consistent estimator without affecting the asymptotic performance of the scatter matrix estimator.

Evaluation of the SCRB for the RES class

Step D: Evaluation of the constrained $\mathrm{SCRB}(\boldsymbol{\theta}_0|g_0)$

- To avoid the scale-ambiguity problem, we need to put a constraint on $\boldsymbol{\Sigma}_0$, i.e. $\mathbf{c}(\boldsymbol{\Sigma}_0) = \mathbf{0}$.

- Let $\mathbf{J_c}(\boldsymbol{\Sigma}_0)$ be the Jacobian matrix of the constraint, then there exists a matrix $\mathbf{U}$ s.t. [31,32]:

$$\mathbf{J_c}(\boldsymbol{\Sigma}_0)\mathbf{U} = \mathbf{0}, \qquad \mathbf{U}^T\mathbf{U} = \mathbf{I}.$$

- The constrained $\mathrm{SCRB}(\boldsymbol{\theta}_0|g_0)$ can be expressed as:

$$\mathrm{CSCRB}(\boldsymbol{\theta}_0|g_0) = \begin{pmatrix} \frac{N}{4E\{\mathcal{Q}\psi_0(\mathcal{Q})^2\}}\boldsymbol{\Sigma}_0 & \mathbf{0} \\ \mathbf{0}^T & \mathbf{U}\left(\mathbf{U}^T\mathbf{C}_0(\bar{\mathbf{s}}_{\mathrm{vecs}(\boldsymbol{\Sigma}_0)})\mathbf{U}\right)^{-1}\mathbf{U}^T \end{pmatrix}.$$

Numerical results

- Let $\{\mathbf{x}_m\}_{m=1}^{M}$ be a set of M i.i.d. RES-distributed data, s.t.:

$$\mathbf{x}_m \sim RES_N(\mathbf{x}; \boldsymbol{\mu}_0, \boldsymbol{\Sigma}_0, g_0), \quad m = 1, \dots, M.$$

- Let us define $\{\bar{\mathbf{x}}_m\}_{m=1}^{M}$ as the set of M vectors such that:

$$\bar{\mathbf{x}}_m = \mathbf{x}_m - \hat{\boldsymbol{\mu}}, \quad m = 1, \dots, M,$$

and $\hat{\boldsymbol{\mu}}$ is the sample mean estimator, i.e.

$$\hat{\boldsymbol{\mu}} \triangleq M^{-1} \sum_{m=1}^{M} \mathbf{x}_m.$$

- $\hat{\boldsymbol{\mu}}$ is a consistent and unbiased estimator.

Three "semiparametric" estimators (1/3)

- The efficiency w.r.t. the CSCRB of three estimators is investigated:
  - the constrained Sample Covariance matrix (CSCM),
  - the constrained Tyler's estimator (C-Tyler),
  - the constrained Huber's estimator (C-Hub).

- We impose a constraint on the trace: $\mathrm{tr}(\boldsymbol{\Sigma}_0) = N$.

- The CSCM is given by:

$$\begin{cases} \hat{\boldsymbol{\Sigma}}_{SCM} \triangleq \dfrac{1}{M} \displaystyle\sum_{m=1}^{M} \bar{\mathbf{x}}_m \bar{\mathbf{x}}_m^T \\ \hat{\boldsymbol{\Sigma}}_{CSCM} \triangleq \dfrac{N}{\mathrm{tr}(\hat{\boldsymbol{\Sigma}}_{SCM})} \hat{\boldsymbol{\Sigma}}_{SCM} \end{cases},$$

Three “semiparametric” estimators (2/3)

- The C-Tyler and the C-Hub are given by the convergence point of the following recursion:

$$\begin{cases} \mathbf{S}_T^{(k+1)} = \dfrac{1}{M}\displaystyle\sum_{m=1}^{M} \varphi(t^{(k)})\bar{\mathbf{x}}_m\bar{\mathbf{x}}_m^T \\ \hat{\Sigma}_T^{(k+1)} = N\mathbf{S}_T^{(k+1)}/\mathrm{tr}(\mathbf{S}_T^{(k+1)}) \end{cases},$$

where $t^{(k)} = \bar{\mathbf{x}}_m^T(\hat{\Sigma}_T^{(k)})^{-1}\bar{\mathbf{x}}_m$ and the starting point is $\hat{\Sigma}_T^{(0)} = \mathbf{I}$.

- The weight function $\varphi(t)$ for Tyler’s estimator is [33,8]:

$$\varphi_{Tyler}(t) = N/t,$$

Three "semiparametric" estimators (3/3)

- The weight function for Huber's estimator is given by [24,34]

$$\varphi_{Hub}(t) = \begin{cases} 1/b & t \leqslant \delta^2 \\ \delta^2/(tb) & t > \delta^2 \end{cases},$$

  and
  - $\delta = F_{\chi^2_N}(u)$, [18]
  - $b = F_{\chi^2_{N+2}}(\delta^2) + \delta^2(1 - F_{\chi^2_N}(\delta^2))/N$ [8], [34].

- u is a tuning parameter that controls the trade-off between robustness and efficiency.

- For $u \to 1$ Huber's estimator is equal to the SCM, while for $u \to 0$ Huber's estimator tends to Tyler's estimator.

[18] $F_{\chi^2_N}(\cdot)$ indicates the distribution of a chi-squared random variable with N degrees of freedom.

Simulation setup

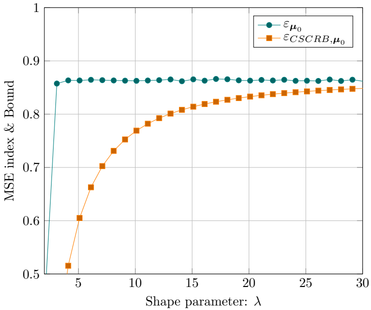
- Two different "true" distributions are considered:
  1. The t-distribution,
  2. The Generalized Gaussian (GG) distribution.

- Simulation parameters
  - $[\boldsymbol{\Sigma}_0]_{i,j} = \rho^{|i-j|}$, $\rho = 0.8$ $i, j = 1, \ldots, N$. Moreover $N = 8$,
  - The data power is chosen to be $\sigma_X^2 = E_{\mathcal{Q}}\{\mathcal{Q}\}/N = 4$,
  - The data mean value is chosen to be $[\boldsymbol{\mu}_0]_i = 1$, $i = 1, \ldots, N$,
  - The number of the available i.i.d. data vectors is $M = 3N = 24$,
  - The tuning parameter u of Huber's estimator $u = 0.5$.

- The MSE of the scatter matrix estimators is compared with:
  1. The $\mathrm{CSCRB}(\boldsymbol{\theta}_0|g_0)$ previously derived,
  2. The classical constrained CRB, i.e. $\mathrm{CCRB}(\boldsymbol{\theta}_0)$, evaluated under perfect knowledge of the density generator [35,36].

t-distribution - Mean vector

$$\varepsilon_{\boldsymbol{\mu}_0} \triangleq ||E\{(\hat{\boldsymbol{\mu}} - \boldsymbol{\mu}_0)(\hat{\boldsymbol{\mu}} - \boldsymbol{\mu}_0)^T\}||_F, \quad \varepsilon_{CSCRB,\boldsymbol{\mu}_0} \triangleq ||[\mathrm{CSCRB}(\boldsymbol{\theta}_0|g_0)]_{\boldsymbol{\mu}_0}||_F.$$

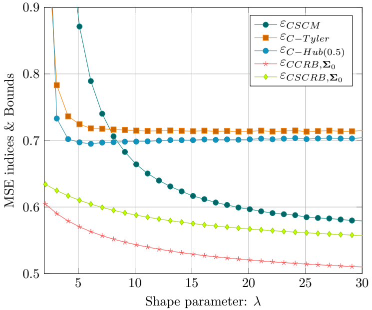


- For the estimation of $\boldsymbol{\mu}_0$, CSCRB coincides with CCRB.
- When the shape parameter λ goes to infinity, the t-distribution tends to a Gaussian one.
- Then, for $\lambda \to \infty$, the sample mean tends to be efficient.

t-distribution - Scatter matrix

$$\varepsilon_\alpha \triangleq ||E\{(\mathrm{vecs}(\hat{\mathbf{\Sigma}}_\alpha) - \mathrm{vecs}(\mathbf{\Sigma}_0))(\mathrm{vecs}(\hat{\mathbf{\Sigma}}_\alpha) - \mathrm{vecs}(\mathbf{\Sigma}_0))^T\}||_F,$$

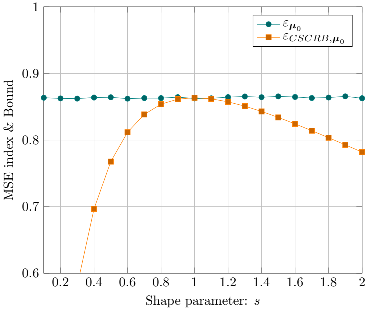
$$\varepsilon_{CSCRB,\mathbf{\Sigma}_0} \triangleq ||[\mathrm{CSCRB}(\boldsymbol{\theta}_0|g_0)]_{\mathbf{\Sigma}_0}||_F, \quad \varepsilon_{CCRB,\mathbf{\Sigma}_0} \triangleq ||[\mathrm{CCRB}(\boldsymbol{\theta}_0)]_{\mathbf{\Sigma}_0}||_F.$$



- The CSCM tends to be efficient w.r.t. the CSCRB as $\lambda \to \infty$.
- Both C-Tyler's and C-Huber's estimators are not efficient with respect to the CSCRB.

GG distribution - Mean vector

$$\varepsilon_{\boldsymbol{\mu}_0} \triangleq ||E\{(\hat{\boldsymbol{\mu}} - \boldsymbol{\mu}_0)(\hat{\boldsymbol{\mu}} - \boldsymbol{\mu}_0)^T\}||_F, \quad \varepsilon_{CSCRB,\boldsymbol{\mu}_0} \triangleq ||[\mathrm{CSCRB}(\boldsymbol{\theta}_0|g_0)]_{\boldsymbol{\mu}_0}||_F.$$

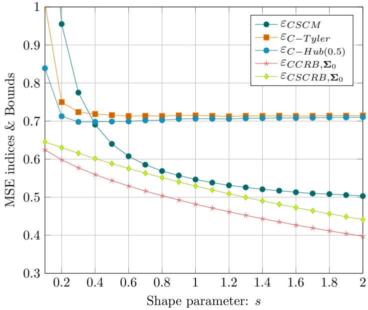


- When $s = 1$, the GG distribution is exactly Gaussian one.
- Hence, for $s = 1$, the sample mean is an efficient estimator.

GG distribution - Scatter matrix

$$\varepsilon_\alpha \triangleq ||E\{(\text{vecs}(\hat{\boldsymbol{\Sigma}}_\alpha) - \text{vecs}(\boldsymbol{\Sigma}_0))(\text{vecs}(\hat{\boldsymbol{\Sigma}}_\alpha) - \text{vecs}(\boldsymbol{\Sigma}_0))^T\}||_F,$$

$$\varepsilon_{CSCRB,\boldsymbol{\Sigma}_0} \triangleq ||[\text{CSCRB}(\boldsymbol{\theta}_0|g_0)]_{\boldsymbol{\Sigma}_0}||_F, \quad \varepsilon_{CCRB,\boldsymbol{\Sigma}_0} \triangleq ||[\text{CCRB}(\boldsymbol{\theta}_0)]_{\boldsymbol{\Sigma}_0}||_F.$$



- The lack of knowledge of the particular density generator has an higher impact when the tails of the true distribution become lighter [37].

The SCRB for the CES class

- The derivation of:[19]
  - SCRB for the estimation of the mean vector and of the scatter matrix in CES distributed random vectors,
  - The Semiparametric Slepian-Bangs formula,
  - The Semiparametric Stochastic CRB (SSCRB),

  can be found in [38]:

  S. Fortunati, F. Gini, M. S. Greco, A. M. Zoubir, and M. Rangaswamy, "Semiparametric CRB and Slepian-Bangs formulas for Complex Elliptically Symmetric distributions," *accepted in IEEE Transactions on Signal Processing*, 2019. [Online]. Available: http://arxiv.org/abs/1902.09541.

- The application of these theoretical results to Direction of Arrival (DOA) estimation problems is discussed in [39]:

  S. Fortunati, F. Gini, M. S. Greco, "Semiparametric stochastic CRB for DOA estimation in elliptical data model," in 2019 27th European Signal Processing Conference, *EUSIPCO*, Sep. 2019.

[19] Additional details are given in the backup slides.

Conclusions

- We provided a fresh look to the Semiparametric Cramér-Rao Bound (SCRB) by showing its relations with the classical (parametric) CRB [7].
- The link between parametric and semiparametric framework is given by the Hilbert-space geometry underling any inference problem.
- The application of the SCRB to the scatter matrix estimation in RES and CES distributed data has been discussed.
- Future works will explore possible applications of the semiparametric inference to well-known signal processing problems, in particular the *semiparametric detection*.

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Backup slides

σ-algebras and measures

- Let $\mathcal{X}$ be some set and let $2^{\mathcal{X}}$ represent its power set. Then a subset $\mathfrak{F} \subseteq 2^{\mathcal{X}}$ is called a σ-algebra if (see e.g. [26, Ch. 2]):
  1. $\mathcal{X} \in \mathfrak{F}$,
  2. If $A \in \mathcal{X}$ is in $\mathfrak{F}$, then so is its complement, $\mathcal{X} \setminus A$,
  3. If $\{A_i\}_{i \in \mathbb{N}} \in \mathfrak{F}$, then so $\bigcup_{i=1}^{\infty} A_i \in \mathfrak{F}$.
- A function $\mu : \mathfrak{F} \to [0, \infty)$ is called a measure if:
  1. $\mu(\emptyset) = 0$ (*Null empty set*),
  2. For all countable collections $\{A_i\}_{i=1}^{\infty}$ of pairwise disjoint sets in $\mathfrak{F}$, $\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(A_i)$ (*Countable additivity*).
- The couple $(\mathcal{X}, \mathfrak{F})$ is a *measurable space*, while the triplet $(\mathcal{X}, \mathfrak{F}, \mu)$ is a *measure space*.

Probability spaces and random variables

- A probability space is a measure space $(\Omega, \mathfrak{D}, P)$ where:
    1. Ω is the *sample space* that represents the set of all possible outcomes of a random experiment,
    2. $\mathfrak{D}$ is the σ-algebra on Ω,
    3. P is a probability measure, that is a measure $P : \mathfrak{D} \to [0, 1]$ satisfying $P(\Omega) = 1$.

- Let $(\Omega, \mathfrak{D}, P)$ be a probability space and $(\mathcal{X}, \mathfrak{F})$ a measurable space.

A *random variable* (r.v.) X is a measurable function $X : \Omega \to \mathcal{X}$, that is for every subset $A \in \mathfrak{F}$, its preimage

$$X^{-1}(A) \triangleq \{\omega \in \Omega | X(\omega) \in A\},$$

is an element of the σ-algebra $\mathfrak{D}$, i.e. $X^{-1}(B) \in \mathfrak{D}$.

Distribution and density functions

- A r.v. allows us to "transport" the probability structure, defined in the abstract space $(\Omega, \mathfrak{D}, P)$, in $(\mathcal{X}, \mathfrak{F})$.

- Specifically, a new probability measure can be defined on $(\mathcal{X}, \mathfrak{F})$ as follows:

$$P_X(A) \triangleq P\left(\{\omega \in \Omega | X(\omega) \in A\}\right) = P(X^{-1}(A)), \quad A \in \mathfrak{F}.$$

- Consequently, the triplet $(\mathcal{X}, \mathfrak{F}, P_X)$ is a probability space.

- **Example:** If $\mathcal{X} \equiv \mathbb{R}$ and $\mathfrak{F}$ is the Borel σ-algebra on $\mathbb{R}$, then P_X is the *distribution* of X [26, Ch. 11].

- The *density* p_X of X is a measurable function satisfying:

$$P_X\left((-\infty, x]\right) = \int_{-\infty}^{x} p_X(a)da, \quad \forall x \in \mathbb{R}.$$

Sub-σ-algebra generated by a transformation

- Let $(\mathcal{X}, \mathfrak{F}, P_X)$ be a probability space as previously defined.
- Let $T : (\mathcal{X}, \mathfrak{F}) \to (\mathcal{Y}, \mathfrak{L})$ a measurable transformation on $\mathcal{X}$.
- The preimage of T, i.e.:

$$\mathfrak{G}(T) \triangleq \left\{G \in \mathfrak{F} | G = T^{-1}(A),\ A \in \mathfrak{L}\right\}$$

  may be a coarser subset of $\mathfrak{F}$!
- It can be shown that $\mathfrak{G}(T)$ is a σ-algebra [26, Theo. 8.1] and, clearly, $\mathfrak{G}(T) \subseteq \mathfrak{F}$.
- $\mathfrak{G}(T)$ is then indicated as the sub-σ-algebra generated by the transformation T [26, Def. 23.3].

Proof: Finite-dimensionality of the linear span

Theorem

Let $\mathbf{u} = (u_1, \cdots, u_k)^T$ be a column vector of k arbitrary elements of an infinite-dimensional Hilbert space $\mathcal{F}$. The linear span of $\mathbf{u}$, defined as:

$$\mathcal{V} \triangleq \{\mathbf{v} | \mathbf{v} = \mathbf{A}\mathbf{u}, \mathbf{A} \text{ is any matrix in } \mathbb{R}^{q \times k}\},$$

is a *finite-dimensional* subspace of $\mathcal{F}^q$. Moreover, if $u_1, \cdots, u_k$ are linearly independent in $\mathcal{F}$, then $\dim(\mathcal{V}) = kq$.

Proof

- Assume that the entries of $\mathbf{u}$ are linearly independent.
- The dimension of a (finite-dimensional) space is equal to the minimum number of linearly independent vectors required to span it.

Proof: Finite-dimensionality of the linear span

- Then if $\mathcal{V}$ has dimension qk, there must exist qk linearly independent q-dimensional vectors such that $\mathcal{V} = \mathrm{span}\{\mathbf{v}_{11}, \ldots, \mathbf{v}_{1k}, \mathbf{v}_{q1}, \ldots, \mathbf{v}_{q \cdot k}\}$.

- Each vector $\mathbf{v}_{ij}$, $i = 1, \ldots, q; j = 1, \ldots, k$ can be constructed by putting all except the i-th entry equal to 0 and the i-th entry equal to $u_j \in \mathcal{F}$ for $j = 1, \ldots, k$, i.e:

$$\begin{array}{cccccccc} \mathbf{v}_{11} & \cdots & \mathbf{v}_{1k} & \mathbf{v}_{21} & \cdots & \mathbf{v}_{2k} & \\ \shortparallel & \shortparallel & \shortparallel & \shortparallel & \shortparallel & \shortparallel & \\ \begin{pmatrix} v_1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} & \cdots & \begin{pmatrix} v_k \\ 0 \\ \vdots \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ v_1 \\ \vdots \\ 0 \end{pmatrix} & \cdots & \begin{pmatrix} 0 \\ v_k \\ \vdots \\ 0 \end{pmatrix} & \cdots \end{array}$$

- By visual inspection, it is immediate to verify that they are linearly independent and this conclude the proof.

Parametric submodels of the CES model (1/3)

- A CES (zero-mean) random vector $\mathbf{x} \in \mathbb{C}^N$ admits a pdf [8]:

$$p_X(\mathbf{x}; \mathbf{\Sigma}) = c_{N,g}|\mathbf{\Sigma}|^{-1} g(\mathbf{x}^H \mathbf{\Sigma}^{-1} \mathbf{x}) \triangleq CES_N(\mathbf{x}; \mathbf{\Sigma}, g),$$

- $\mathcal{G} \ni g : \mathbb{R}_0^+ \to \mathbb{R}^+$ is the *density generator* and

$$\mathcal{G} \triangleq \left\{ g : \mathbb{R}_0^+ \to \mathbb{R}^+ | \int_0^\infty t^{N-1} g(t) dt < \infty \right\}$$

- The set of all CES pdfs is a semiparametric model of the form:

$$\mathcal{P}_{\mathbf{\Sigma},g} \triangleq \{ p_X | p_X(\mathbf{x}|\mathbf{\Sigma}, g), \mathbf{\Sigma} \in \mathcal{M}_N, g \in \mathcal{G} \}.$$

- How can we build a parametric submodel of $\mathcal{P}_{\mathbf{\Sigma},g}$?

Parametric submodels of the CES model (2/3)

- The set of all the density generator $\mathcal{G}$ is a convex set!

Proof

For every $g_0, g_1 \in \mathcal{G}$ and for every $\eta \in [0, 1]$, we have that:

1. $\eta g_1(t) + (1-\eta)g_0(t)$ is a function of $t \triangleq \mathbf{x}^H \mathbf{\Sigma}^{-1}\mathbf{x}$,
2. By linearity, $\int_0^\infty t^{N-1}[\eta g_1(t) + (1-\eta)g_0(t)]dt < \infty$,

then $\eta g_1 + (1-\eta)g_0 \in \mathcal{G}$ and consequently $\mathcal{G}$ is a convex set.

- Then it is immediate to verify that:

$$\begin{aligned} CES_N(\mathbf{x}; \mathbf{\Sigma}, g_0) &= CES_N(\mathbf{x}; \mathbf{\Sigma}, \eta g_1 + (1-\eta)g_0) \\ &= \eta CES_N(\mathbf{x}; \mathbf{\Sigma}, g_1) + (1-\eta)CES_N(\mathbf{x}; \mathbf{\Sigma}, g_0). \end{aligned}$$

- $\mathcal{P}_{\mathbf{\Sigma},g}$ is a convex set as well!

Parametric submodels of the CES model (3/3)

- Let us define a smooth parametric map as:

$$\begin{aligned}\nu_i &: [0,1] \to \mathcal{G} \\ &\eta \mapsto \nu_i(t,\eta) \triangleq \eta g_i(t) + (1-\eta) g_0(t),\end{aligned}$$

  where g_i is a generic density generator while g_0 is the true one.

- The relevant *i-th* parametric submodel is then given by:

$$\mathcal{P}_{\boldsymbol{\Sigma},\nu_{\eta_i}} = \left\{ p_X | p_X(\mathbf{x}|\boldsymbol{\Sigma}, \eta g_i + (1-\eta) g_0), \boldsymbol{\Sigma} \in \mathcal{M}_N, \eta \in [0,1] \right\}.$$

- It is immediate to verify that this submodel satisfies the conditions C0, C1 and C2 given in slide 32.

- In particular, Condition C2 is verified by choosing $\eta = 0$.

Hellinger differentiability

- Let $p_X(\mathbf{x}|\boldsymbol{\theta})$ be a parametric pdf with $\boldsymbol{\theta} \in \Theta \subset \mathbb{R}^d$.

- We indicate with $u_{\boldsymbol{\theta}}(\mathbf{x})$ the following parametric map:

$$\begin{aligned} u_{\boldsymbol{\theta}} : \Theta &\to L_2 \\ \boldsymbol{\theta} &\mapsto u_{\boldsymbol{\theta}}(\mathbf{x}) \triangleq \sqrt{p_X(\mathbf{x}|\boldsymbol{\theta})}, \end{aligned}$$

- $u_{\boldsymbol{\theta}}$ is Hellinger (Fréchet) differentiable in $\boldsymbol{\theta}_0$ if there exists a vector $\dot{\mathbf{u}}_{\boldsymbol{\theta}_0} \equiv \dot{\mathbf{u}}_{\boldsymbol{\theta}_0}(\mathbf{x})$ such that:

$$||u_{\boldsymbol{\theta}_0+\mathbf{h}} - u_{\boldsymbol{\theta}_0} - \dot{\mathbf{u}}_{\boldsymbol{\theta}_0}^T \mathbf{h}|| = o(\textstyle\sum_i h_i^2), \quad \mathbf{h} \to 0,$$

where $||u_{\boldsymbol{\theta}}||^2 = \langle u_{\boldsymbol{\theta}}, u_{\boldsymbol{\theta}} \rangle = \int u_{\boldsymbol{\theta}}^2(\mathbf{x}) d\mathbf{x}$.

- $\dot{\mathbf{u}}_{\boldsymbol{\theta}_0} \equiv \dot{\mathbf{u}}_{\boldsymbol{\theta}_0}(\mathbf{x})$ is the Hellinger derivative of $u_{\boldsymbol{\theta}}$ in $\boldsymbol{\theta}_0$.

A geometrical intuition (1/4)

- Since $u_{\boldsymbol{\theta}}(\mathbf{x}) \triangleq \sqrt{p_X(\mathbf{x}|\boldsymbol{\theta})}$, we have that:

$$||u_{\boldsymbol{\theta}}||^2 = \langle u_{\boldsymbol{\theta}}, u_{\boldsymbol{\theta}} \rangle = \int p_X(\mathbf{x}|\boldsymbol{\theta}) d\mathbf{x} = 1, \quad \forall \boldsymbol{\theta} \in \Theta.$$

- $u_{\boldsymbol{\theta}}$ can be interpreted as a differentiable map between Θ and (a subset of) the *surface* $S(L_2)$ of the unit sphere in L_2.

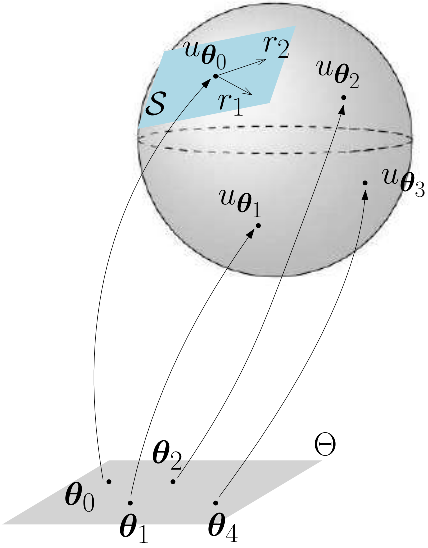
- Given a point on $S(L_2)$, say $u_{\boldsymbol{\theta}_0}$, the tangent space $\mathcal{S} \subseteq L_2$ of $\mathcal{S}_0$ at $u_{\boldsymbol{\theta}_0}$ is defined by the orthogonality condition:

$$\langle r, u_{\boldsymbol{\theta}_0} \rangle = 0 \quad \Leftrightarrow \quad r \in \mathcal{S}.$$

- Note that the tangent space $\mathcal{S}_0$ is a subset of L_2, while previously we defined it as a subset of $\mathcal{H}$.[20]

[20] Remember that $\mathcal{H} = \left\{ h : \mathcal{X} \to \mathbb{R} \middle| E_X\{h\} = 0, E_X\{|h|^2\} < \infty \right\}$.

A geometrical intuition (2/4)



A geometrical intuition (3/4)

- Are the two definition consistent?

- Let us define the (locally) one-to-one transformation:

$$\begin{aligned} H_0 : \mathcal{S} &\to \mathcal{H} \\ r &\mapsto H_0(r) \triangleq \frac{2r}{u_{\boldsymbol{\theta}_0}} = h. \end{aligned}$$

- Then, we have:

$$\begin{aligned} r \in \mathcal{S} \Rightarrow \langle r, u_{\boldsymbol{\theta}_0} \rangle &= \int r(\mathbf{x}) u_{\boldsymbol{\theta}_0}(\mathbf{x}) d\mathbf{x} = 0 \\ &\Rightarrow 2^{-1} \int h(\mathbf{x}) u^2_{\boldsymbol{\theta}_0}(\mathbf{x}) = 2^{-1} \int h(\mathbf{x}) p(\mathbf{x}|\boldsymbol{\theta}_0) d\mathbf{x} = 0 \\ &\Rightarrow E_X\{h\} = 0 \Rightarrow h \in \mathcal{H}. \end{aligned}$$

A geometrical intuition (4/4)

- The vice-versa is as follows:

$$h \in \mathcal{H} \Rightarrow E_X\{h\} = \int h(\mathbf{x})p(\mathbf{x}|\boldsymbol{\theta}_0)d\mathbf{x} = 0$$
$$\Rightarrow 2\int r(\mathbf{x})u_{\boldsymbol{\theta}_0}^{-1}(\mathbf{x})p(\mathbf{x}|\boldsymbol{\theta}_0)d\mathbf{x} = 2\int r(\mathbf{x})u_{\boldsymbol{\theta}_0}(\mathbf{x})d\mathbf{x} = 0$$
$$\Rightarrow \langle r, u_{\boldsymbol{\theta}_0}\rangle = 0 \Rightarrow r \in \mathcal{S}.$$

Then the two definition are consistent [9, Sec. 3.1, Prep. 3]:

$$\langle r, u_{\boldsymbol{\theta}_0}\rangle = 0,\ \forall r \in \mathcal{S} \quad \Leftrightarrow \quad E_X\{h\} = 0,\ \forall h \in \mathcal{H}.$$

Hellinger derivative and score vector

- Recall that the score vector of $p_X(\mathbf{x}|\boldsymbol{\theta})$ in $\boldsymbol{\theta}_0$ is defined as:

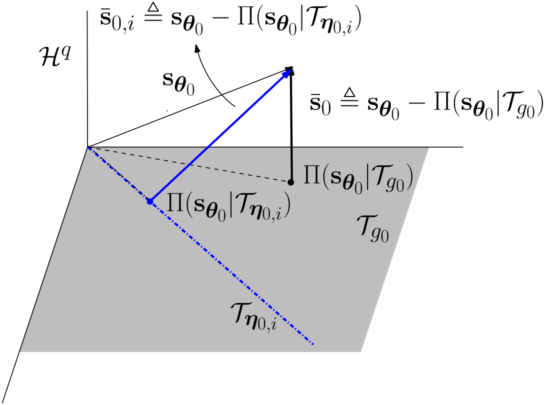
$$\mathbf{s}_{\boldsymbol{\theta}_0} \triangleq \nabla_{\boldsymbol{\theta}} \ln p_X(\mathbf{x}|\boldsymbol{\theta}_0).$$

- If for all $\boldsymbol{\theta} \in \Theta \subseteq \mathbb{R}^q$ [9, Sec. 2.1, Prep. 1]:
  - $p_X(\mathbf{x}|\boldsymbol{\theta})$ is continuously differentiable in $\boldsymbol{\theta}$ for almost all $\mathbf{x}$,
  - $\left(\sum_i [\mathbf{s}_{\boldsymbol{\theta}_0}]_i^2\right)^{1/2} \in L_2(P_0)$,
  - The FIM $\mathbf{I}(\boldsymbol{\theta}) \triangleq \int \mathbf{s}_{\boldsymbol{\theta}}(\mathbf{x})\mathbf{s}_{\boldsymbol{\theta}}^T(\mathbf{x})p_X(\mathbf{x}|\boldsymbol{\theta})d\mathbf{x}$ is non-singular and continuous in $\boldsymbol{\theta}$,

  then [9, Sec. 2.1], we have that:

$$\dot{\mathbf{u}}_{\boldsymbol{\theta}_0} = \frac{1}{2} u_{\boldsymbol{\theta}_0} \mathbf{s}_{\boldsymbol{\theta}_0}, \quad \dot{\mathbf{u}}_{\boldsymbol{\theta}_0} \in \mathcal{S}^q, \ \mathbf{s}_{\boldsymbol{\theta}_0} \in \mathcal{H}^q.$$

The Semiparametric CRB (SCRB)



$$\mathcal{T}_{\boldsymbol{\eta}_{0,i}} \subseteq \mathcal{T}_{g_0}, \forall i \in \mathcal{I} \quad \Rightarrow \quad ||\bar{\mathbf{s}}_{0,i}|| \geq ||\bar{\mathbf{s}}_0||, \forall i \in \mathcal{I}$$

$$\Rightarrow \quad E_0\{\bar{\mathbf{s}}_{0,i}\bar{\mathbf{s}}_{0,i}^T\} \geq E_0\{\bar{\mathbf{s}}_0\bar{\mathbf{s}}_0^T\} \triangleq \bar{\mathbf{I}}(\boldsymbol{\theta}_0|g_0)$$

The Least Favourable Submodel (1/2)

- The Least Favourable Submodel (LFS) (if it exists) is the $\bar{i}$*-th* parametric submodel of $\mathcal{P}_{\boldsymbol{\theta},g}$ s.t.:

$$\sup_{\{\mathcal{P}_{\boldsymbol{\theta},\nu_i}\}} \left[E_0\{\bar{\mathbf{s}}_{0,i}\bar{\mathbf{s}}_{0,i}^T\}\right]^{-1} = \max_{\{\mathcal{P}_{\boldsymbol{\theta},\nu_i}\}} \left[E_0\{\bar{\mathbf{s}}_{0,i}\bar{\mathbf{s}}_{0,i}^T\}\right]^{-1}$$
$$= \bar{\mathbf{I}}(\boldsymbol{\theta}_0|\nu_{\bar{i}})^{-1},$$

- Let us define as Least Favourable Direction (LFD) the score vector [9, Sec. 3.1], [11, Sec. 2.2]:

$$\mathbf{s}_{\boldsymbol{\eta}_{0,\bar{i}}}(\mathbf{x}) = \nabla_{\boldsymbol{\eta}} \ln p_X(\mathbf{x}|\gamma_0, \nu_{\bar{i}}(\mathbf{x}, \boldsymbol{\eta}_0)),$$

- Then, as shown previously, for the parametric case:

$$\Pi(\mathbf{s}_{\boldsymbol{\theta}_0}|\mathcal{T}_{\boldsymbol{\eta}_{0,\bar{i}}}) = E_0\{\mathbf{s}_{\boldsymbol{\theta}_0}\mathbf{s}_{\boldsymbol{\eta}_{0,\bar{i}}}^T\}\mathbf{C}_0(\mathbf{s}_{\boldsymbol{\eta}_{0,\bar{i}}})^{-1}\mathbf{s}_{\boldsymbol{\eta}_{0,\bar{i}}}.$$

The Least Favourable Submodel (2/2)

- The existence of a LFS depends on the "level of richness" of the set of the parametric submodels $\{\mathcal{P}_{\boldsymbol{\theta},\nu_i}\}_{i \in \mathcal{I}}$.
- Unfortunately, the existence of a LFS needs to be verified on a case-by-case basis.
- Moreover, if it exists, figuring out which such LFS is, is not an easy task (see [11] for some hints on this).
- We refer to [9] for an exhaustive list of semiparametric models that admits a LFS expressible in "closed-form".

Conditional expectation: a remark (1/2)

- Let $h \equiv h(X)$ be a function of the random variable (r.v.) X.
- We defined the conditional expectation as $E\{h(X)|Y\}$ as the unique function of the r.v. Y such that:

$$E\{[h(X) - E\{h(X)|Y\}]Y\} = 0.$$

- The explicit "operative definition" of $E\{h(X)|Y\}$ is:

$$\begin{aligned} E\{h(X)|Y\} &\triangleq \int_{\mathcal{X}} h(x) p_{X|Y}(x|y) dx \\ &= \int_{\mathcal{X}} h(x) \frac{p_{X,Y}(x,y)}{p_Y(y)} dx, \end{aligned}$$

where $p_{X,Y}$ is the joint pdf of X and Y, $p_{X|Y}$ is the conditional pdf of X given Y and p_Y is the pdf of Y.

Conditional expectation: a remark (2/2)

- Are the two definitions consistent?

$$E\{[h(X) - E\{h(X)|Y\}]Y\} = 0 \Rightarrow$$
$$\int_{\mathcal{X},\mathcal{Y}} [h(X) - E\{h(X)|Y = y\}]p_{X,Y}(x,y)dxdy = 0$$

$$\begin{aligned}
\int_{\mathcal{X},\mathcal{Y}} & h(x)p_{X,Y}(x,y)dxdy \\
&= \int_{\mathcal{X},\mathcal{Y}} E\{h(X)|Y = y\}p_{X,Y}(x,y)dxdy \\
&= \int_{\mathcal{Y}} E\{h(X)|Y = y\}p_Y(y)dy \\
&= \int_{\mathcal{Y}} \left[\int_{\mathcal{X}} h(x)\frac{p_{X,Y}(x,y)}{p_Y(y)}dx\right] p_Y(y)dy \\
&= \int_{\mathcal{X},\mathcal{Y}} h(x)p_{X,Y}(x,y)dxdy. \qquad \blacksquare
\end{aligned}$$

From RES to CES distributions (1/3)

Definition ([40], [28], [8] and [41, Ch. 4])

- Let $\mathbf{x}_R \in \mathbb{R}^N$ and $\mathbf{x}_I \in \mathbb{R}^N$ be two real random vectors.

- $\mathbf{z} \triangleq \mathbf{x}_R + j\mathbf{x}_I \in \mathbb{C}^N$ is said to be CES-distributed with mean vector $\boldsymbol{\mu}$ and scatter matrix $\boldsymbol{\Sigma}$:

$$\boldsymbol{\mu} = \boldsymbol{\mu}_R + j\boldsymbol{\mu}_I \in \mathbb{C}^N \quad \boldsymbol{\Sigma} = \mathbf{C}_1 + j\mathbf{C}_2 \in \mathbb{C}^{N\times N},$$

  iff $\tilde{\mathbf{x}} \triangleq (\mathbf{x}_R^T, \mathbf{x}_I^T)^T \in \mathbb{R}^{2N}$ is RES-distributed with mean vector $\tilde{\boldsymbol{\mu}} = (\boldsymbol{\mu}_R^T, \boldsymbol{\mu}_I^T)^T$ and scatter matrix $\tilde{\boldsymbol{\Sigma}}$ satisfying:

$$\tilde{\boldsymbol{\Sigma}} = \frac{1}{2}\begin{pmatrix} \mathbf{C}_1 & -\mathbf{C}_2 \\ \mathbf{C}_2 & \mathbf{C}_1 \end{pmatrix},$$

  where $\mathbf{C}_1$ is symmetric and $\mathbf{C}_2$ is skew-symmetric.

From RES to CES distributions (2/3)

- Let $\tilde{\mathbf{x}} \sim RES_{2N}(\tilde{\mathbf{x}}; \tilde{\boldsymbol{\mu}}, \tilde{\boldsymbol{\Sigma}}, g)$ be a RES-distributed random vector.

- When the scatter matrix $\tilde{\boldsymbol{\Sigma}}$ has full rank, we have that:

$$\begin{aligned} RES_{2N}(\tilde{\mathbf{x}}; \tilde{\boldsymbol{\mu}}, \tilde{\boldsymbol{\Sigma}}, g) &\triangleq p_{\tilde{X}}(\tilde{\mathbf{x}}; \tilde{\boldsymbol{\mu}}, \tilde{\boldsymbol{\Sigma}}, g) \\ &= 2^{-(2N)/2} |\tilde{\boldsymbol{\Sigma}}|^{-1/2} g\left((\tilde{\mathbf{x}} - \tilde{\boldsymbol{\mu}})^T \tilde{\boldsymbol{\Sigma}}^{-1} (\tilde{\mathbf{x}} - \tilde{\boldsymbol{\mu}})^T \right) \\ &= |\boldsymbol{\Sigma}|^{-1} g\left(2(\mathbf{z} - \boldsymbol{\mu})^H \boldsymbol{\Sigma}^{-1} (\mathbf{z} - \boldsymbol{\mu}) \right) \\ &= p_Z(\mathbf{z}; \boldsymbol{\mu}, \boldsymbol{\Sigma}, h) \triangleq CES_N(\mathbf{z}; \boldsymbol{\mu}, \boldsymbol{\Sigma}, h), \end{aligned}$$

where $h(t) \triangleq g(2t)$.

- The functional form of the density generator remains unchanged except for the scaling factor 2 of its argument.

From RES to CES distributions (2/3)

- There exists a one-to-one mapping between a subset of the RES distributions and the (circular) CES distributions.

- The semiparametric theory already developed for the RES class holds true for the CES class as well.

- In particular, CES distributions are a *semiparametric group model* generated by the set of Complex Spherically Symmetric (CSS) distributions [28, Sec. 3.5] through the action of:

$$\alpha_{(\boldsymbol{\mu},\boldsymbol{\Sigma})} : \mathbb{C}^N \to \mathbb{C}^N, \ \forall \boldsymbol{\mu}, \boldsymbol{\Sigma}$$
$$CSS(g) \sim \mathbf{z} \mapsto \alpha_{(\boldsymbol{\mu},\boldsymbol{\Sigma})}(\mathbf{z}) = \boldsymbol{\mu} + \boldsymbol{\Sigma}^{1/2}\mathbf{z}.$$

The SCRB for the CES class

- The steps to derive the SCRB for the CES class follow exactly the ones already discussed for the RES one.
- Difference: the mean vector $\boldsymbol{\mu}$ and the scatter matrix $\boldsymbol{\Sigma}$ are complex quantities!
- The *Wirtinger* or $\mathbb{CR}$ -calculus has to be used to evaluate the derivatives [42,43,44,45,46,47,48,49].
- All the details can be found in [38].

Slepian-Bangs (SB) formula

- Introduced by Slepian and Bangs in [50] and [51], the SB formula has been extensively used for many years in array processing.

- The "classic" SB formula is a compact expression of the Fisher Information Matrix (FIM) for parameter estimation under a Gaussian data model [13, Appendix 3C].

- Specifically:
  - $\boldsymbol{\theta} \in \Theta \subseteq \mathbb{R}^d$: deterministic parameter vector,
  - $\mathbf{z} \sim CN(\boldsymbol{\mu}(\boldsymbol{\theta}), \boldsymbol{\Sigma}(\boldsymbol{\theta}))$: complex Gaussian random vector.

- Then the SB formula provides us with a closed-form expression of the FIM for the estimation of $\boldsymbol{\theta} \in \Theta$.

Semiparametric Slepian-Bangs (SSB) formula

- Generalizations to:
  1. Non-circular complex Gaussian distributions [52],
  2. CES distributions [36],
  3. Non-circular CES distributions [53],
  4. Model misspecification under Gaussianity assumption [1],
  5. Model misspecification under CES assumption [54],
  6. Semiparametric model under CES assumption [38].

- Let $\mathbb{C}^N \ni \mathbf{z} \sim CES_N(\boldsymbol{\mu}(\boldsymbol{\theta}), \boldsymbol{\Sigma}(\boldsymbol{\theta}), h)$ be a CES-distributed random vector parameterized by $\boldsymbol{\theta} \in \Theta \subseteq \mathbb{R}^d$.

- The semiparametric SB (SSB) formula in [38] provides the efficient FIM for the estimation of $\boldsymbol{\theta}$ in the presence of an *unknown*, nuisance density generator $h \in \mathcal{G}$.

Semiparametric Stochastic CRB (SSCRB)

- Assume to have an array of N sensors and K narrowband sources impinging on the array from $\{\nu_1, \dots, \nu_K\}$ directions.
- Data snapshots $\mathbf{z}_m \sim CES_N(\mathbf{z}; \mathbf{0}, \boldsymbol{\Sigma}(\boldsymbol{\nu}, \boldsymbol{\Gamma}, \sigma^2), h_0)$, $\forall m$ whose density generator $h_0 \in \bar{\mathcal{G}}$ is unknown and [55]:

$$\boldsymbol{\Sigma} \equiv \boldsymbol{\Sigma}(\boldsymbol{\nu}, \boldsymbol{\Gamma}, \sigma^2) = \mathbf{A}(\boldsymbol{\nu})\boldsymbol{\Gamma}\mathbf{A}(\boldsymbol{\nu})^H + \sigma^2 \mathbf{I}_N.$$

- The $\mathrm{SSCRB}(\boldsymbol{\nu}_0|\boldsymbol{\zeta}_0, \sigma_0^2, h_0)$ [38,39] generalizes the classical, Gaussian-based, SCRB [56,57] since:
    1. The Gaussianity assumption is replaced by the more general CES assumption,
    2. The additional infinite-dimensional *nuisace* parameter h_0 is taken into account.