Proximal Gradient Algorithms: Applications in Signal Processing

Niccoló Antonello, Lorenzo Stella, Panos Patrinos, Toon van Waterschoot

niccolo.antonello@idiap.ch, stellalo@amazon.com, panos.patrinos@esat.kuleuven.be, toon.vanwaterschoot@esat.kuleuven.be

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Proximal Gradient Algorithms: Applications in Signal Processing

Part I: Introduction

Toon van Waterschoot

toon.vanwaterschoot@esat.kuleuven.be

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Optimization in Signal Processing

Inference problems such as ...

- Signal estimation,
- Parameter estimation,
- Signal detection,
- Data classification

... naturally lead to **optimization problems**

$$x_\star = \underset{x}{\mathbf{argmin}} \underbrace{\mathrm{loss}(x) + \mathrm{prior}(x)}_{\mathrm{cost}(x)}$$

Variables x could be signal samples, model parameters, algorithm tuning parameters, etc.

Modeling and Inverse Problems

Many inference problems lead to optimization problems in which signal models need to be inverted, i.e. **inverse problems**:

> Given set of observations y, infer unknown signal or model parameters x

- Inverse problems are often ill-conditioned or underdetermined
- Large-scale problems may suffer more easily from ill-conditioning
- Including **prior** in cost function then becomes crucial (e.g. regularization)

Modeling and Inverse Problems

Choice of suitable cost function often depends on adoption of application-specific **signal model**, e.g.

- Dictionary model, e.g. sum of sinusoids $y = Dx$ with DFT matrix D
- Filter model, e.g. linear FIR filter $y = Hx$ with convolution matrix H
- Black-box model, e.g. neural network $y = f(x)$ with feature transformation function f

In this tutorial, we will often represent signal models as **operators**, i.e.

$$y = Ax \text{ with } A = \text{linear operator}$$
$$y = A(x) \text{ with } A = \text{nonlinear operator}$$

Motivating Examples

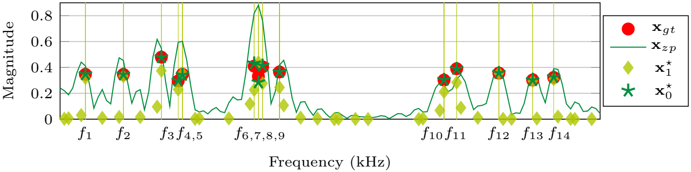
Example 1: **Line spectral estimation**

- DFT dictionary model with selection matrix S and inverse DFT matrix F_i

$$y = SF_i x$$

- Underdetermined inverse problem: $\dim(y) \ll \dim(x)$
- Spectral sparsity prior for line spectrum

$$x_\star = \operatorname*{\mathbf{argmin}}_{x} \underbrace{\text{DFT model output error}(x)}_{\text{loss}(x)} + \underbrace{\text{spectral sparsity}(x)}_{\text{prior}(x)}$$



Motivating Examples

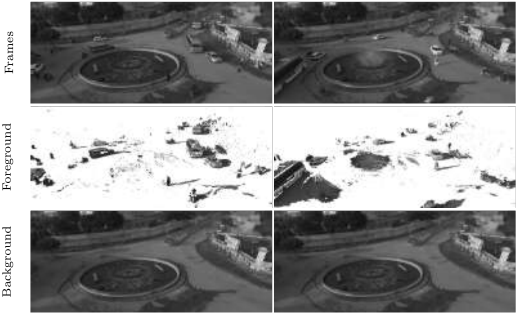
Example 2: **Video background removal**

- Static background + dynamic foreground decomposition model

$$Y = L + S$$

- Underdetermined inverse problem: $\dim(Y) = \frac{1}{2}\left(\dim(L) + \dim(S)\right)$
- Rank-1 prior for static BG + sparse prior for FG changes (robust PCA)

$$x_\star = \mathop{\mathbf{argmin}}_{x} \underbrace{\text{BG + FG model output error}(x)}_{\text{loss}(x)} + \underbrace{\text{BG rank + FG sparsity}(x)}_{\text{prior}(x)}$$



Motivating Examples

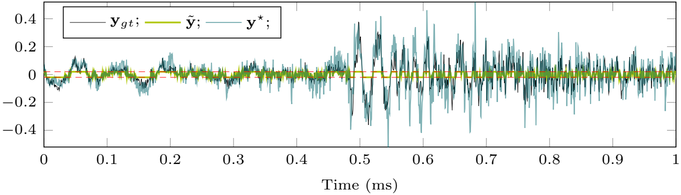
Example 3: **Audio de-clipping**

- DCT dictionary model with inverse DCT matrix $F_{i,c}$

$$y = F_{i,c}x$$

- Underdetermined inverse problem: missing data (clipped samples) in y
- Spectral sparsity for audio signal + amplitude prior for clipped samples

$$x_\star = \operatorname*{\mathbf{argmin}}_x \underbrace{\text{DCT model output error}(x)}_{\text{loss}(x)} + \underbrace{\text{spectral sparsity + clipping}(x)}_{\text{prior}(x)}$$



Challenges in Optimization

Linear vs. nonlinear optimization

- Linear: closed-form solution
- Nonlinear: iterative numerical optimization algorithms

Convex vs. nonconvex optimization

- Convex: unique optimal point (global minimum)
- Nonconvex: multiple optimal points (local minima)

Smooth vs. non-smooth optimization

- Smooth: Newton-type methods using first- and second-order derivatives
- Non-smooth: first-order methods using (sub)gradients

Challenges in Optimization

Trends and observations:

- Loss is often linear/convex/smooth but prior is often not
- Even if non-convex problems are hard to solve globally, iterating from good initialization may yield local minimum close enough to global minimum
- Non-smooth problems are typically tackled with first-order methods, showing slower convergence than Newton-type methods

Key message of this tutorial:

> Also for **non-smooth** optimization problems,
> Newton-type methods showing fast convergence can be derived

- This greatly broadens variety of loss functions and priors that can be used
- Theory, software implementation, and signal processing examples will be presented in next 2.5h

Tutorial Outline

1. Introduction
2. Proximal Gradient (PG) algorithms
   - Proximal mappings and proximal gradient method
   - Dual and accelerated proximal gradient methods
   - Newton-type proximal gradient algorithms
3. Software Toolbox
   - Short introduction to Julia language
   - Structured Optimization package ecosystem
4. Demos and Examples
   - Line spectral estimation
   - Video background removal
   - Audio de-clipping
5. Conclusion

Proximal Gradient Algorithms: Applications in Signal Processing

Part II

Lorenzo Stella

stellalo@amazon.com

EUSIPCO 2019

AWS AI Labs (Amazon Research)

About me

- Applied Scientist at AWS AI Labs (Amazon Research)
- Deep learning, probabilistic time series models
- Time series forecasting, classification, anomaly detection...
- **We're hiring!**

Gluon Time Series: `github.com/awslabs/gluon-ts`

- Previously: Ph.D. at IMT Lucca and KU Leuven with Panos Patrinos
- The work presented here was done prior to joining Amazon

Outline

1. Preliminary concepts, composite optimization, proximal mappings

2. Proximal gradient method

3. Duality

4. Accelerated proximal gradient

5. Newton-type proximal gradient methods

6. Concluding remarks

Blanket assumptions

In this presentation:

- Underlying space is the Euclidean space $\mathbb{R}^n$ equipped with
  - Inner product $\langle \cdot, \cdot \rangle$, e.g. dot product)
  - Induced norm $\| \cdot \| = \sqrt{\langle \cdot, \cdot \rangle}$
- Linear mappings will be identified by their matrices and adjoints will be denoted by transpose $^\top$
- Most algorithms will be matrix-free: can view matrices and their transposes are linear mappings and their adjoints
- All results carry over to general Euclidean spaces, most of them even to Hilbert spaces

The space $\mathbb{R}^n$

- n-dimensional column vectors with real components endowed with

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \\ \vdots \\ x_n + y_n \end{bmatrix}, \qquad \alpha \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} \alpha x_1 \\ \alpha x_2 \\ \vdots \\ \alpha x_n \end{bmatrix}$$

- Standard inner product: $\langle x, y \rangle = x^\top y = \sum_{i=1}^n x_i y_i$ **dot product**
- Induced norm: $\|x\| = \|x\|_2 = \sqrt{\langle x, x \rangle} = \sqrt{\sum_{i=1}^n x_i^2}$ **Euclidean norm**

Alternative inner product and induced norm ($Q \succ 0$ is $n \times n$)

$$\langle x, y \rangle = \langle x, y \rangle_Q = x^\top Q y$$
$$\|x\| = \|x\|_Q = \sqrt{x^\top Q x}$$

The space $\mathbb{R}^{m \times n}$

- $m \times n$ real **matrices**

$$X = \begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{m1} & x_{m2} & \cdots & x_{mn} \end{bmatrix}$$

- Standard inner product

$$\langle X, Y \rangle = \textbf{trace}(X^\top Y) = \sum_{i=1}^{m} \sum_{j=1}^{n} X_{ij} Y_{ij}$$

- Induced norm

$$\|X\| = \|X\|_F = \sqrt{\langle X, X \rangle} = \sqrt{\sum_{i=1}^{n} \sum_{j=1}^{m} X_{ij}^2}$$

Frobenius norm

Extended-real-valued functions

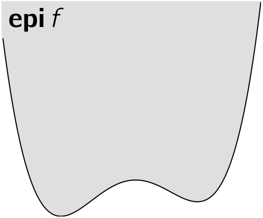
- **Extended real line** $\overline{\mathbb{R}} = \mathbb{R} \cup \{\infty\} = (-\infty, \infty]$
- **Extended-real-valued functions** $f : \mathbb{R}^n \to \overline{\mathbb{R}}$
- **Effective domain** $\textbf{dom}\, f = \{x \in \mathbb{R}^n \mid f(x) < \infty\}$
- f is called **proper** if $f(x) < \infty$ for some x ($\textbf{dom}\, f$ is nonempty)
- Offer a unified view of optimization problems

Main example: indicator of set $C \subseteq \mathbb{R}^n$

$$\delta_C(x) = \begin{cases} 0 & \text{if } x \in C \\ +\infty & \text{otherwise} \end{cases}$$

Epigraph

Epigraph: $\mathbf{epi}\, f = \{(x, \alpha) \in \mathbb{R}^n \times \mathbb{R} \mid f(x) \le \alpha\}$

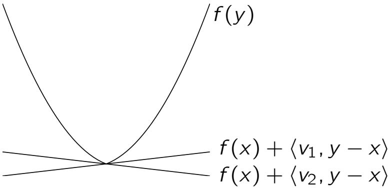


- f is **closed** iff $\mathbf{epi}\, f$ is a closed set.
- f is **convex** iff $\mathbf{epi}\, f$ is a convex set.

Subdifferential

Subdifferential of a proper, convex function $f : \mathbb{R}^n \to \overline{\mathbb{R}}$:

$$\partial f(x) = \{v | f(y) \geq f(x) + \langle v, y - x\rangle \ \forall y \in \mathbb{R}^n\}$$



- $\partial f(x)$ is a convex set
- $\partial f(x) = \{v\}$ iff f is differentiable at x with $\nabla f(x) = v$
- $\bar{x}$ minimizes f iff $0 \in \partial f(\bar{x})$
- Definition above can be extended to **nonconvex** f

Composite optimization problems

$$\textbf{minimize} \;\; \varphi(x) = f(x) + g(x)$$

Assumptions

- $f : \mathbb{R}^n \to \mathbb{R}$ differentiable with L-Lipschitz gradient (L-smooth)

$$\|\nabla f(x) - \nabla f(y)\| \leq L\|x - y\| \qquad \forall x, y \in \mathbb{R}^n$$

- $g : \mathbb{R}^n \to \overline{\mathbb{R}}$ proper, closed
- Set of optimal solutions **argmin** $f + g$ is nonempty

Proximal mapping (or operator)

Assume $g : \mathbb{R}^n \to \overline{\mathbb{R}}$ closed, proper

$$\mathbf{prox}_{\gamma g}(x) = \operatorname*{\mathbf{argmin}}_{z \in \mathbb{R}^n} \left\{ g(z) + \frac{1}{2\gamma}\|z - x\|^2 \right\}, \qquad \gamma > 0$$

If g is convex:

- for all $x \in \mathbb{R}^n$, function $z \mapsto g(z) + \frac{1}{2\gamma}\|z - x\|^2$ is strongly convex
- $\mathbf{prox}_{\gamma g}(x)$ is unique for all $x \in \mathbb{R}^n$, i.e., $\mathbf{prox}_{\gamma g} : \mathbb{R}^n \to \mathbb{R}^n$

Examples

- $f(x) = 0$: $\mathbf{prox}_{\gamma f}(x) = x$
- $f(x) = \delta_C(x)$: $\mathbf{prox}_{\gamma f}(x) = \Pi_C(x)$

Proximal mapping: generalization of Euclidean projection

Proximal mapping (or operator)

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Examples

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- $f(x) = \delta_C(x)$: $\mathbf{prox}_{\gamma f}(x) = \Pi_C(x)$

Proximal mapping: generalization of Euclidean projection

Properties

$$\mathbf{prox}_{\gamma g}(x) = \mathop{\mathbf{argmin}}_{z \in \mathbb{R}^n} \left\{ g(z) + \tfrac{1}{2\gamma}\|z - x\|^2 \right\}, \qquad \gamma > 0$$

- If g is convex, from the optimality conditions:

$$\begin{aligned} p \in \mathop{\mathbf{argmin}}_{z}\, g(z) + \tfrac{1}{2\gamma}\|z - x\|^2 &\iff -\gamma^{-1}(p - x) \in \partial g(p) \\ &\iff x \in p + \gamma \partial g(p) \end{aligned}$$

- In other words

$$p \in x - \gamma \partial g(p) \tag{$\spadesuit$}$$

- Equivalent to **implicit subgradient** step
- Analogous to implicit Euler method for ODEs
- From ($\spadesuit$), any fixed-point $\bar{x} = \mathbf{prox}_{\gamma g}(\bar{x})$ satisfies $0 \in \partial g(\bar{x})$

Fixed-points of $\mathbf{prox}_{\gamma g}$ $\equiv$ minimizers of g

Properties

$$\mathbf{prox}_{\gamma g}(x) = \underset{z \in \mathbb{R}^n}{\mathbf{argmin}} \left\{ g(z) + \tfrac{1}{2\gamma} \|z - x\|^2 \right\}, \qquad \gamma > 0$$

- For convex g, mapping $\mathbf{prox}_{\gamma g} : \mathbb{R}^n \to \mathbb{R}^n$ is **firmly nonexpansive (FNE)**

$$\| \mathbf{prox}_{\gamma g}(x) - \mathbf{prox}_{\gamma g}(y) \|^2 \leq \langle \mathbf{prox}_{\gamma g}(x) - \mathbf{prox}_{\gamma g}(y), x - y \rangle \quad \forall x, y \in \mathbb{R}^n$$

Properties

$$\mathbf{prox}_{\gamma g}(x) = \underset{z \in \mathbb{R}^n}{\mathbf{argmin}} \left\{ g(z) + \tfrac{1}{2\gamma} \|z - x\|^2 \right\}, \qquad \gamma > 0$$

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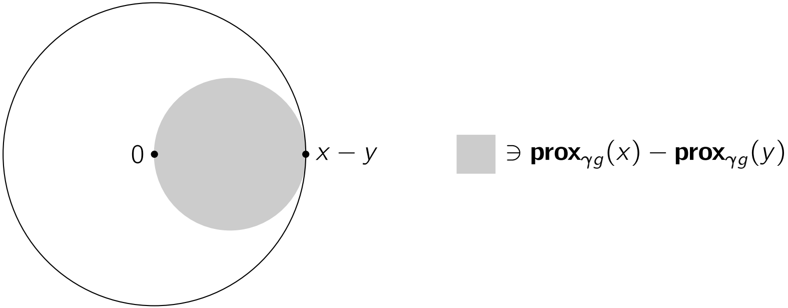


Properties

$$\mathbf{prox}_{\gamma g}(x) = \underset{z \in \mathbb{R}^n}{\mathbf{argmin}} \left\{ g(z) + \frac{1}{2\gamma}\|z - x\|^2 \right\}, \qquad \gamma > 0$$

- For convex g, mapping $\mathbf{prox}_{\gamma g} : \mathbb{R}^n \to \mathbb{R}^n$ is **firmly nonexpansive (FNE)**

$$\|\mathbf{prox}_{\gamma g}(x) - \mathbf{prox}_{\gamma g}(y)\|^2 \le \langle \mathbf{prox}_{\gamma g}(x) - \mathbf{prox}_{\gamma g}(y), x - y \rangle \quad \forall x, y \in \mathbb{R}^n$$

- FNE implies $\mathbf{prox}_{\gamma g}$ **nonexpansive** (Cauchy-Schwarz)

$$\|\mathbf{prox}_{\gamma g}(x) - \mathbf{prox}_{\gamma g}(y)\| \le \|x - y\| \quad \forall x, y \in \mathbb{R}^n$$

Examples of proximal mappings

- Convex quadratic function

$$g(x) = \frac{1}{2}\langle x, Qx\rangle + \langle q, x\rangle \qquad \mathbf{prox}_{\gamma g}(x) = (I + \gamma Q)^{-1}(x - \gamma q)$$

- Euclidean norm

$$g(x) = \|x\| \qquad \mathbf{prox}_{\gamma g}(x) = \begin{cases} (1 - \gamma/\|x\|)x, & \|x\| > \gamma, \\ 0, & \text{otherwise} \end{cases}$$

- L_1-norm

$$g(x) = \|x\|_1 = \sum_i |x_i| \qquad [\mathbf{prox}_{\gamma g}(x)]_i = \begin{cases} x_i + \gamma & x_i < -\gamma \\ 0 & |x_i| \le \gamma \\ x_i - \gamma & x_i > \gamma \end{cases}$$

- Nuclear norm

$$g(X) = \sum \mathbf{diag}\,\Sigma \qquad \mathbf{prox}_{\gamma g}(X) = U\hat{\Sigma}V^T$$

where $X = U\Sigma V^T$ where $\mathbf{diag}\,\hat{\Sigma} = \mathbf{prox}_{\gamma\|\cdot\|_1}(\mathbf{diag}\,\Sigma)$

Proximal calculus rules

- **Separable sum**: $f(x_1, x_2) = f_1(x_1) + f_2(x_2)$

$$\mathbf{prox}_{\gamma f}(x_1, x_2) = (\mathbf{prox}_{\gamma f_1}(x_1), \mathbf{prox}_{\gamma f_2}(x_2))$$

- **Scaling and translation**: $f(x) = \phi(\alpha x + \beta)$, $\alpha \neq 0$

$$\mathbf{prox}_{\gamma f}(x) = \tfrac{1}{\alpha}(\mathbf{prox}_{\alpha^2 \lambda \phi}(\alpha x + \beta) - \beta)$$

- **Postcomposition**: $f(x) = \alpha \phi(x) + \beta$, $\alpha > 0$

$$\mathbf{prox}_{\gamma f}(x) = \mathbf{prox}_{\alpha \gamma \phi}(x)$$

- **Orthogonal composition**: $f(x) = \phi(Qx)$, $Q^\top Q = QQ^\top = I$

$$\mathbf{prox}_{\gamma f}(x) = Q^\top \mathbf{prox}_{\gamma \phi}(Qx)$$

(e.g.: $Q = \mathrm{DCT}, \mathrm{DFT}$)

Properties

$$\mathbf{prox}_{\gamma g}(x) = \underset{z \in \mathbb{R}^n}{\mathbf{argmin}} \left\{ g(z) + \tfrac{1}{2\gamma} \|z - x\|^2 \right\}, \qquad \gamma > 0$$

- If g is convex $\mathbf{prox}_{\gamma g}$ is single-valued
- If g is **non**convex $\mathbf{prox}_{\gamma g}$ is **set-valued** in general
  - Can be empty, can be multi-valued
  - If g is **lower bounded** then $\mathbf{prox}_{\gamma g}(x)$ nonempty for all x
  - Algorithms will work by taking any $p \in \mathbf{prox}_{\gamma g}(x)$

Outline

Composite optimality conditions

$$\textbf{minimize}\ \ \varphi(x) = f(x) + g(x)$$

- If $x_\star$ is a local minimum of φ then

$$-\nabla f(x_\star) \in \partial g(x_\star) \tag{1}$$

- Moreover, we have shown already that for any $x \in \mathbb{R}^n$

$$p \in \mathbf{prox}_{\gamma g}(x) \iff x \in p + \gamma \partial g(p) \tag{2}$$

- We can reformulate (1) as follows, using (2)

$$\begin{aligned} -\nabla f(x_\star) \in \partial g(x_\star) \ &\iff\ x_\star - \gamma \nabla f(x_\star) \in x_\star + \gamma \partial g(x_\star) \\ &\iff\ x_\star = \mathbf{prox}_{\gamma g}(x_\star - \gamma \nabla f(x_\star)) \end{aligned}$$

- We have shown that $x_\star$ satisfies (1) iff it is a **fixed point** of mapping

$$T(x) = \mathbf{prox}_{\gamma g}(x - \gamma \nabla f(x))$$

Proximal gradient method

To minimize $f + g$ iterate

$$x^{k+1} = \mathbf{prox}_{\gamma g}(x^k - \gamma \nabla f(x^k))$$

- Reduces to gradient method if $g = 0$

$$x^{k+1} = x^k - \gamma \nabla f(x^k)$$

- Reduces to gradient projection when $g = \delta_C$

$$x^{k+1} = \Pi_C(x^k - \gamma \nabla f(x^k))$$

- Reduces to proximal point method when $f = 0$

$$x^{k+1} = \mathbf{prox}_{\gamma g}(x^k)$$

Interpretations

$$x^{k+1} = \mathbf{prox}_{\gamma g}(x^k - \gamma \nabla f(x^k))$$

- Proximal gradient step can be expressed as linearized (in f) sub-problem

$$x^{k+1} = \underset{u}{\mathbf{argmin}}\{\underbrace{f(x^k) + \langle \nabla f(x^k), u - x^k \rangle}_{\ell_f(u;x^k)} + g(u) + \tfrac{1}{2\gamma}\|u - x^k\|^2\}$$

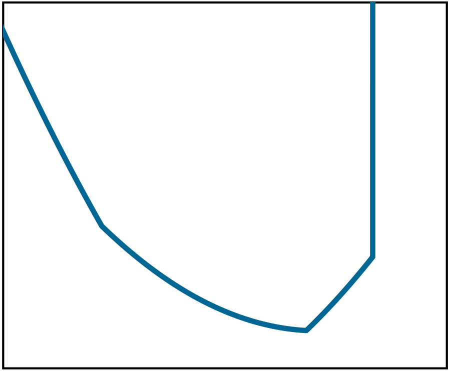
- Since ∇f is Lipschitz, for $\gamma \leq 1/L$:

$$f(u) \leq \ell_f(u; x^k) + \tfrac{1}{2\gamma}\|u - x^k\|^2 \quad \text{for all } u \in \mathbb{R}^n$$

- Thus $\ell_f(u; x^k) + g(u) + \frac{1}{2\gamma}\|u - x^k\|^2$ **majorizes** $\varphi(u)$
- Proximal gradient as a **majorization minimization algorithm**

Interpretations

$$x^{k+1} = \underset{u}{\mathbf{argmin}}\{\underbrace{f(x^k) + \langle \nabla f(x^k), u - x^k \rangle}_{\ell_f(u;x^k)} + g(u) + \frac{1}{2\gamma}\|u - x^k\|^2\}$$



$\varphi = f + g$

Interpretations

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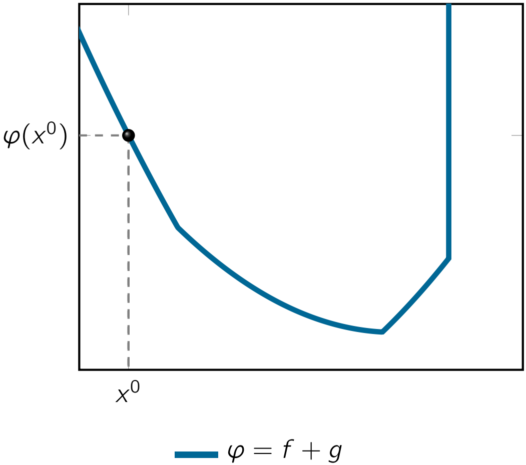
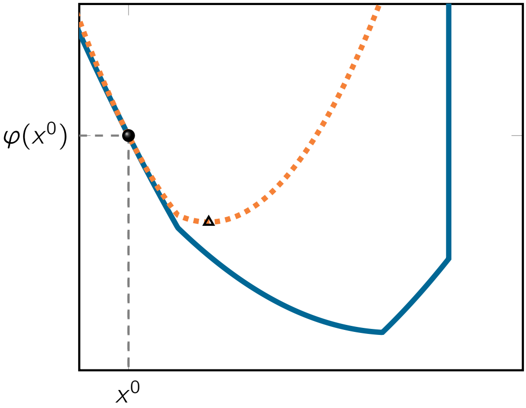


Interpretations

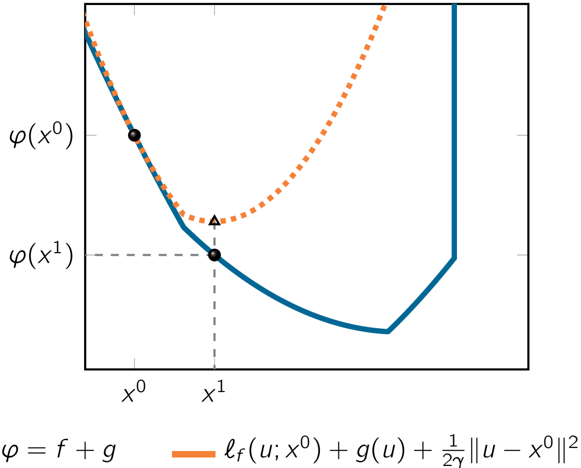
$$x^{k+1} = \operatorname*{\mathbf{argmin}}_u \{\underbrace{f(x^k) + \langle \nabla f(x^k), u - x^k \rangle}_{\ell_f(u;x^k)} + g(u) + \tfrac{1}{2\gamma}\|u - x^k\|^2\}$$



$\varphi = f + g$ $\quad$ $\ell_f(u; x^0) + g(u) + \frac{1}{2\gamma}\|u - x^0\|^2$

Interpretations

$$x^{k+1} = \underset{u}{\mathbf{argmin}}\{\underbrace{f(x^k) + \langle \nabla f(x^k), u - x^k\rangle}_{\ell_f(u;x^k)} + g(u) + \tfrac{1}{2\gamma}\|u - x^k\|^2\}$$



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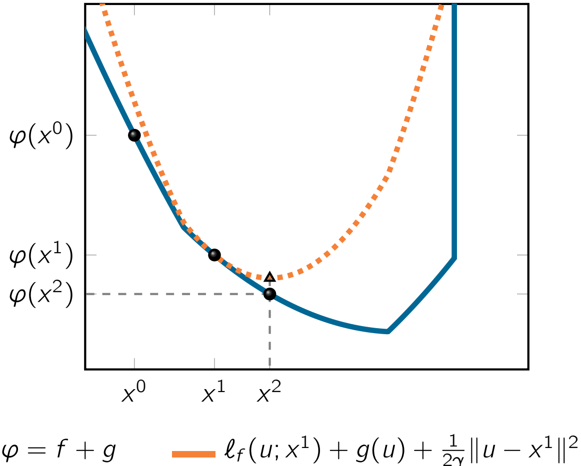

Interpretations

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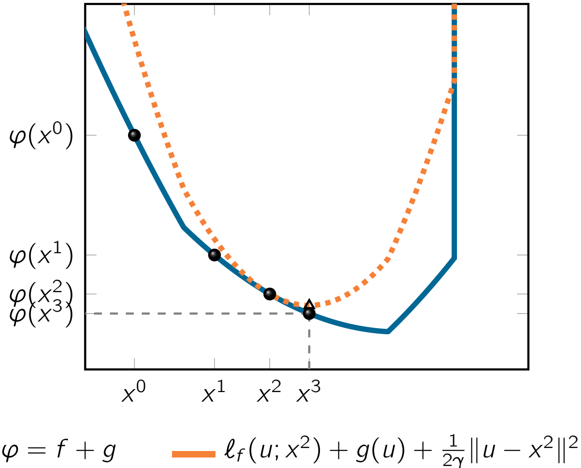

Convergence rate (convex case)

Theorem (Convergence rate – convex case)

The iterates of proximal gradient method with $\gamma \in (0, 1/L]$ *satisfy*

$$\varphi(x^k) - \varphi(x_\star) \leq \frac{\|x_0 - x_\star\|^2}{2\gamma k}$$

- **Conclusion**: to reach $\varphi(x^k) - \varphi(x_\star) \leq \epsilon$, proximal gradient needs

$$k = \left\lceil \frac{\|x_0 - x_\star\|^2}{2\gamma\epsilon} \right\rceil \qquad \text{iterations}$$

Convergence rate (strongly convex case)

$$\|x^{k+1} - x_\star\|^2 \leq (1 - \gamma\mu)\|x^k - x_\star\|^2 \qquad (\spadesuit)$$

- if f **strongly convex** ($\mu > 0$), then linear convergence

$$\|x^k - x_\star\|^2 \leq c^k\|x^0 - x_\star\|^2 \qquad c = 1 - \gamma\mu$$

- for $\gamma = \frac{1}{L}$ contraction factor is $c = 1 - \frac{\mu}{L}$
- for small $\frac{\mu}{L}$ convergence is slow
- $\mu = 0$: ($\spadesuit$) shows that distance from solution set is nonincreasing

$$\|x^{k+1} - x_\star\| \leq \|x^k - x_\star\|$$

- sequences with this property are called Fejér monotone
- Fejér monotonicity: convergence of the sequence of iterates to some $x_\star$

Convergence (nonconvex case)

- If f is **nonconvex** and $\gamma \leq 1/L$

$$\lim_{k\to\infty} \|R(x^k)\| = 0 \qquad R(x) = x - \mathbf{prox}_{\gamma g}(x - \gamma \nabla f(x))$$

- $R : \mathbb{R}^n \to \mathbb{R}^n$ is sufficiently regular, e.g. it's Lipschitz if g is convex
- This implies that every cluster point $\bar{x}$ of $(x^k)_{k\in\mathbb{N}}$ satisfies

$$R(\bar{x}) = 0 \iff -\nabla f(\bar{x}) \in \partial g(\bar{x})$$

- Convergence of the sequence using *Kurdyka-Lojasiewicz* assumption

Proximal gradient with line search

- In practice Lipschitz constant L is not known, how to select γ?
- Can do backtracking: start with γ_0 large and at every iteration run

---

Algorithm 1: Line search to determine γ

Input: x^k, γ_{k-1} and $\beta \in (0,1)$

$\gamma \leftarrow \gamma_{k-1}$

while $f(z) > f(x^k) + \langle \nabla f(x^k), z - x^k \rangle + \frac{1}{2\gamma}\|z - x^k\|^2$ **do**

  $z \leftarrow \mathbf{prox}_{\gamma g}(x^k - \gamma \nabla f(x^k))$

  $\gamma \leftarrow \beta\gamma$

end

- Requires one evaluation of $\mathbf{prox}_{\gamma g}$ and f per line search iteration
- Only a finite number of backtrackings will be necessary
- Preserves convergence properties of the algorithms

Outline

Duality

$$\textbf{minimize} \;\; f(x) + g(Ax)$$

- f and g are proper, closed, convex
- A matrix (e.g. data) or linear operator (e.g. finite differencing)

Note: computing $\mathbf{prox}_{\gamma(g \circ A)}$ is much more complex than $\mathbf{prox}_{\gamma g}$

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Example: simple **bound** constraints become **polyhedral** constraints

$$g = \delta_{\{z : z \le b\}} \qquad \Rightarrow \qquad \mathbf{prox}_{\gamma g} = \Pi_{\{z : z \le b\}} = \mathbf{min}(\cdot, b)$$

$$(g \circ A) = \delta_{\{x : Ax \le b\}} \qquad \Rightarrow \qquad \mathbf{prox}_{\gamma (g \circ A)} = \Pi_{\{x : Ax \le b\}} = \textbf{?}$$

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Reformulate problem in **separable form** and solve the dual:

$$\begin{aligned} \underset{x,z}{\textbf{minimize}} \;\; & f(x) + g(z) \\ \textbf{subject to} \;\; & Ax = z \end{aligned}$$

Duality

Primal

$$\underset{x,z}{\textbf{minimize}}\; f(x) + g(z)$$
$$\textbf{subject to}\; Ax = z$$

Dual

$$\underset{y}{\textbf{minimize}}\;\; f^*(-A^T y) + g^*(y)$$

- Functions f^* and g^* are the **Fenchel conjugates** of f and g

$$f^*(u) = \sup_x \{\langle u, x\rangle - f(x)\} \qquad \text{(similarly for } g\text{)}$$

- If f is μ-strongly convex then f^* has μ^{-1}-Lipschitz gradient

$$\nabla f^*(u) = \underset{x}{\textbf{argmax}}\{\langle u, x\rangle - f(x)\}$$

- **Moreau identity:** $y = \textbf{prox}_{\gamma g}(y) + \gamma\, \textbf{prox}_{\gamma^{-1} g^*}(\gamma^{-1} y)$

We can apply (accelerated) proximal gradient method to the dual

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Outline

1. Preliminary concepts, composite optimization, proximal mappings
2. Proximal gradient method
3. Duality
4. **Accelerated proximal gradient**
5. Newton-type proximal gradient methods
6. Concluding remarks

Accelerated proximal gradient (APG)

$$\textbf{minimize } f(x) + g(x)$$

- When f and g are convex, convergence rate of proximal gradient is $O(1/k)$
- Proximal gradient reduces to gradient method whenever $g \equiv 0$
- Gradient method **not optimal** for smooth convex problems
- Optimal convergence rate is $O(1/k^2)$
- **Nesterov (1983)** suggested simple modification that attains optimal rate
- Beck & Teboulle (2009) extended the method to composite problems

Accelerated proximal gradient (APG)

Start with $x^{-1} = x^0$, repeat

$$\beta_k = \begin{cases} 0 & \text{if } k = 0, \\ \frac{k-1}{k+2} & \text{if } k = 1, 2, \ldots \end{cases}$$

$$y^k = x^k + \beta_k(x^k - x^{k-1}) \qquad \textbf{extrapolation step}$$

$$x^{k+1} = \mathbf{prox}_{\gamma g}\left(y^k - \gamma \nabla f(y^k)\right) \qquad \textbf{proximal gradient step}$$

Theorem (Convergence rate of APG – convex case)

The iterates of APG with $\gamma \in (0, 1/L]$ satisfy

$$\varphi(x^{k+1}) - \varphi_\star \leq \frac{2L}{(k+2)^2}\|x^0 - x_\star\|^2$$

- APG faster than PG in theory (and practice!)
- Convergent extensions to nonconvex problems exist

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Example: lasso (or basis pursuit)

$$\underset{x}{\textbf{minimize}} \ \ \frac{1}{2}\|y - Ax\|^2 + \lambda\|x\|_1 \qquad A \in \mathbb{R}^{1000\times 2500}$$

- Original signal $\hat{x}$ is sparse with 100 nonzeros
- Output $y = A\hat{x} + \mathcal{N}(0, \sigma)$ (SNR = 10)
- $f(x) = \frac{1}{2}\|y - Ax\|^2$, $g(x) = \lambda\|x\|_1$
- Lipschitz constant of ∇f is $\|A^\top A\|$

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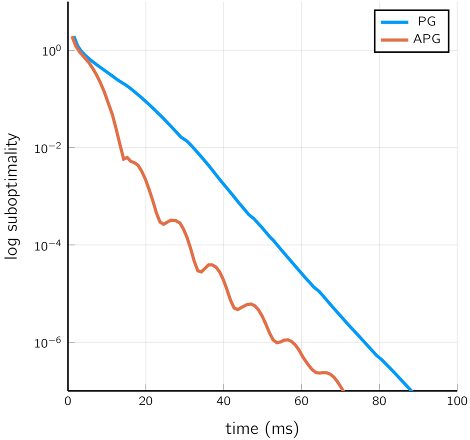

$$\lambda = 0.05\lambda_{\mathbf{max}}$$

$$nnz(x_\star) = 90$$

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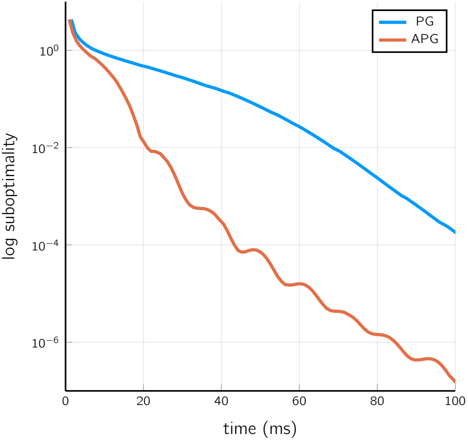

$$\lambda = 0.02\lambda_{\mathbf{max}}$$

$$nnz(x_\star) = 187$$

Example: lasso (or basis pursuit)

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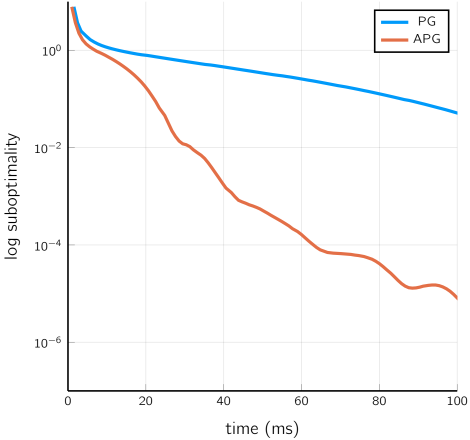

$$\lambda = 0.01\lambda_{\mathbf{max}}$$

$$nnz(x_\star) = 404$$

Outline

Newton-type methods (smooth case)

1. Solve **minimize** $f(x)$ using

$$x^{k+1} = \underset{x}{\textbf{argmin}}\, f(x^k) + \langle \nabla f(x^k), x - x^k \rangle + \tfrac{1}{2}\|x - x^k\|^2_{H_k} \qquad H_k \succ 0$$

2. Solve $\nabla f(x) = 0$

$$x^{k+1} = x^k - H_k^{-1} \nabla f(x^k) \qquad H_k \text{ nonsingular}$$

- **Equivalent approaches**
- Choose $H_k \approx \nabla^2 f(x^k) \equiv J\nabla f(x^k)$
- Gradient method corresponds to $H_k = I$
- **Damp** iterations using line search to guarantee convergence

Can we extend this to composite problems $f + g$?

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Variable metric proximal gradient

$$\textbf{minimize} \;\; f(x) + g(x)$$

$$\begin{aligned} d^k &= \underset{d}{\textbf{argmin}} \;\; f(x^k) + \langle \nabla f(x^k), d\rangle + \tfrac{1}{2}\|d\|_{H_k}^2 + g(x^k + d) && H_k \succ 0 \\ x^{k+1} &= x^k + \tau_k d^k && \tau_k > 0 \end{aligned}$$

where $H_k \approx \nabla^2 f(x^k)$. Define the **scaled proximal mapping**

$$\textbf{prox}_g^H(x) = \underset{z}{\textbf{argmin}} \left\{ g(z) + \tfrac{1}{2}\|z - x\|_H^2 \right\}, \quad H \succ 0$$

Then the above is equivalent to

$$\begin{aligned} d^k &= \textbf{prox}_g^{H_k}(x^k - H_k^{-1}\nabla f(x^k)) - x^k \\ x^{k+1} &= x^k + \tau_k d^k && \tau_k > 0 \end{aligned}$$

Variable metric proximal gradient

- Becker, Fadili, 2012: f, g both convex, uses a modified SR1 method to approximate $H_k \approx \nabla^2 f$
- Lee et al., 2014: f, g both convex, show superlinear convergence when H_k is computed using quasi-Newton formulas, and solving subproblem inexactly
- Chouzenoux et al., 2014: f can be nonconvex, analyzes convergence of an inexact method under KL assumption
- Frankel et al., 2015: f, g can both be nonconvex
- ...

Variable metric proximal gradient

$$\mathbf{prox}_g^H(x) = \underset{z}{\mathbf{argmin}} \left\{ g(z) + \tfrac{1}{2}\|z - x\|_H^2 \right\}, \quad H \succ 0$$

Major **practical** drawback:

- No closed-form for computing $\mathbf{prox}_g^H$ in general, even for very simple g
- Closed form for $\mathbf{prox}_g^H$ if:
  - if $g = \|\cdot\|_1$ then H must be **diagonal**
  - if $g = \|\cdot\|_2$ then H must have **constant diagonal**
- Otherwise, need **inner iterative procedure** to compute $\mathbf{prox}_g^H$
- Change in oracle
  - before: ∇f and $\mathbf{prox}_\gamma\, g$ (often very simple to compute)
  - after: ∇f and $\mathbf{prox}_g^H$ (much harder in general)
- Not as easily implementable as original proximal gradient method

Newton-type method for optimality conditions

$$R(x) = x - \mathbf{prox}_{\gamma g}(x - \gamma \nabla f(x)) = 0 \qquad (\spadesuit)$$

- Any local minimum $x_\star$ satisfies $R(x_\star) = 0$
- System of nonlinear, nonsmooth equations
- **Idea:** apply Newton-type method to solve ($\spadesuit$)

$$\begin{aligned} z^k &= \mathbf{prox}_{\gamma g}(x^k - \gamma \nabla f(x^k)) \\ d^k &= B_k(z^k - x^k) \qquad B_k \in \mathbb{R}^{n\times n} \text{ nonsingular} \\ x^{k+1} &= x^k + d^k \end{aligned}$$

- Choose B_k (approximately) as $JR(x^k)^{-1}$
- Need to **damp** the last step to enforce convergence
- **Key:** penalty function for line search

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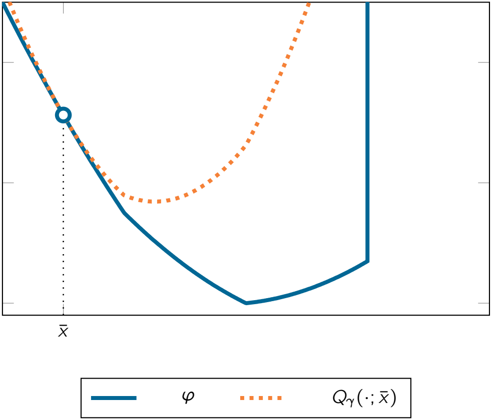
Forward-backward envelope

$$\varphi_\gamma(x) = \min_z \left\{ Q_\gamma(z;x) = f(x) + \langle z - x, \nabla f(x) \rangle + g(z) + \tfrac{1}{2\gamma}\|z - x\|^2 \right\}$$



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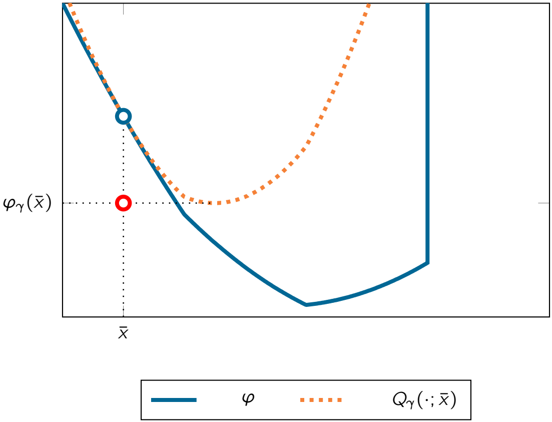

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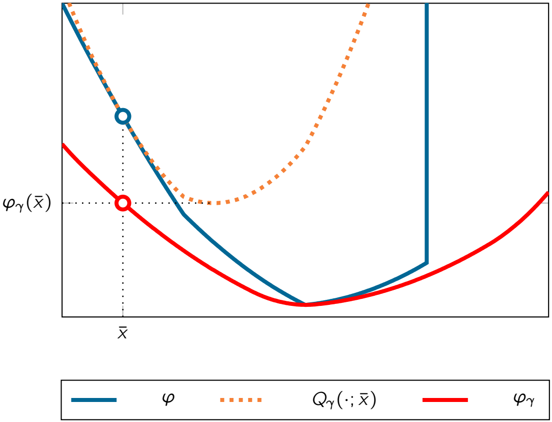

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Theorem

If g is convex (results can be extended to g being nonconvex)

1. φ_γ is strictly continuous
2. $\varphi_\gamma \le \varphi$ for any $\gamma > 0$
3. $\varphi(z) \le \varphi_\gamma(x) - \frac{1-\gamma L}{2\gamma}\|x - z\|^2$ where $z = \mathbf{prox}_{\gamma g}(x - \gamma \nabla f(x))$
4. $\varphi_\gamma(x) = \varphi(x)$ for any stationary point x
5. $\mathbf{inf}\, \varphi_\gamma = \mathbf{inf}\, \varphi$ and $\mathbf{argmin}\, \varphi_\gamma = \mathbf{argmin}\, \varphi$ for $\gamma \in (0, L^{-1})$

- **1.** implies that φ_γ is everywhere finite
- if $\gamma < L^{-1}$, **2.** and **3.** imply that z (strictly) decreases φ_γ
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Proximal averaged Newton-type method (PANOC)

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\begin{aligned}
z^k &= \mathbf{prox}_{\gamma g}(x^k - \gamma \nabla f(x^k)) && \\
d^k &= B_k(z^k - x^k) && B_k \in \mathbb{R}^{n\times n} \text{ nonsingular} \\
x^{k+1} &= (1-\tau_k)z^k + \tau_k(x^k + d^k) && \tau_k \in (0,1]
\end{aligned}
$$

From the theorem: φ_γ continuous and

$$\varphi_\gamma(z^k) \le \varphi_\gamma(x^k) - \tfrac{1-\gamma L}{2\gamma}\|x^k - z^k\|^2$$

Therefore: $\tau_k \in (0,1]$ exists such that

$$\varphi_\gamma(x^{k+1}) \le \varphi_\gamma(x^k) - \alpha\tfrac{1-\gamma L}{2\gamma}\|x^k - z^k\|^2 \qquad \alpha \in (0,1)$$

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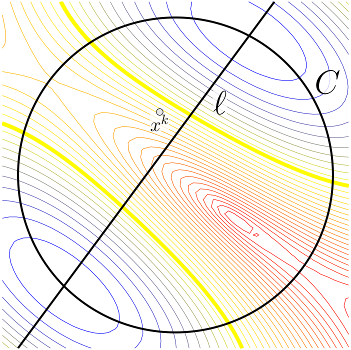
minimize $f(x) + g(x)$

$$f = \tfrac{1}{2}\,\mathbf{dist}^2(\cdot, \ell)$$

$$g = \delta_C$$



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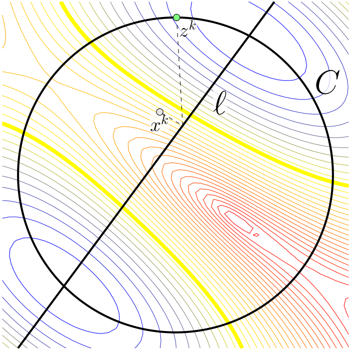

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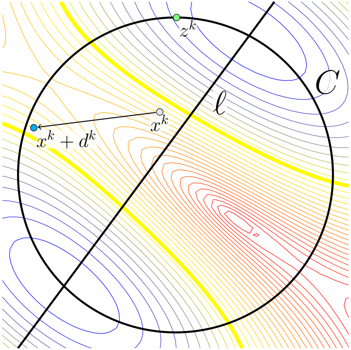

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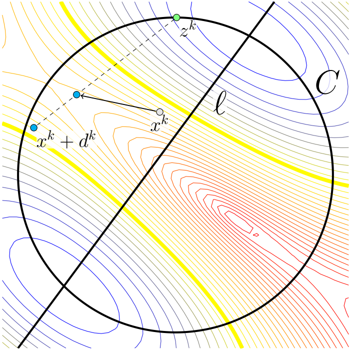

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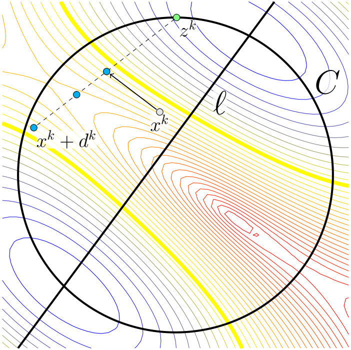

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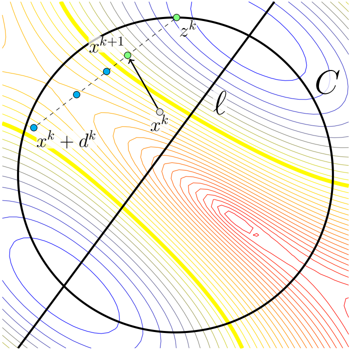

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z^k &= \mathbf{prox}_{\gamma g}(x^k - \gamma \nabla f(x^k)) & & \\
d^k &= B_k(z^k - x^k) & & B_k \in \mathbb{R}^{n\times n} \text{ nonsingular} \\
x^{k+1} &= (1-\tau_k) z^k + \tau_k (x^k + d^k) & & \tau_k \in (0,1]
\end{aligned}
$$

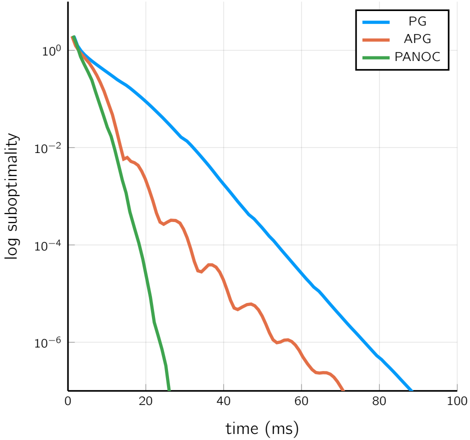
How to choose B_k? Quasi-Newton: start with nonsingular B_0, update it s.t.

$$
B_k y^k = s^k \quad \text{(inverse secant condition)} \qquad \begin{cases} s^k = x^k - x^{k-1} \\ y^k = R(x^k) - R(x^{k-1}) \end{cases}
$$

- (Modified) Broyden method yields superlinear convergence
- Limited-memory BFGS: works well in practice, **no need to store B^k**
- Products with B^k computed in $O(n)$ using inner products only

Example: lasso (or basis pursuit)

$$\underset{x}{\textbf{minimize}} \;\; \tfrac{1}{2}\|y - Ax\|^2 + \lambda\|x\|_1 \qquad A \in \mathbb{R}^{1000\times 2500}$$

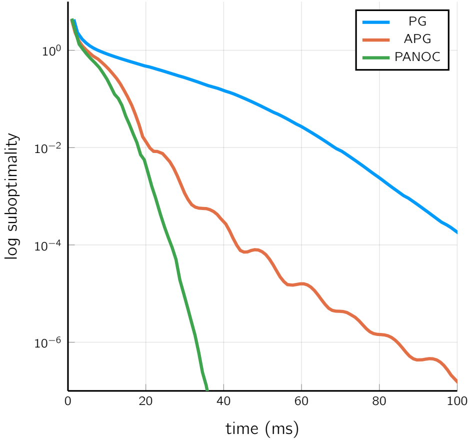


$$\lambda = 0.05\lambda_{\mathbf{max}}$$

$$nnz(x_\star) = 90$$

Example: lasso (or basis pursuit)

$$\underset{x}{\textbf{minimize}} \;\; \tfrac{1}{2}\|y - Ax\|^2 + \lambda\|x\|_1 \qquad A \in \mathbb{R}^{1000\times 2500}$$

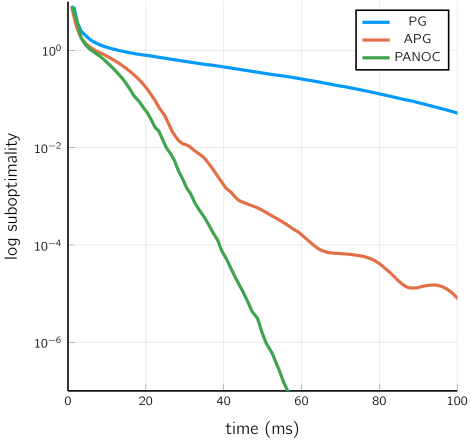


$$\lambda = 0.02\lambda_{\mathbf{max}}$$

$$nnz(x_\star) = 187$$

Example: lasso (or basis pursuit)

$$\underset{x}{\textbf{minimize}} \;\; \tfrac{1}{2}\|y - Ax\|^2 + \lambda\|x\|_1 \qquad A \in \mathbb{R}^{1000\times 2500}$$



$$\lambda = 0.01\lambda_{\mathbf{max}}$$

$$nnz(x_\star) = 404$$

Outline

Concluding remarks

Proximal gradient (PG) method:

- Extends classical gradient descent to composite problems
- Convergence rate guarantees in the convex case
- Convergence to local minima in the nonconvex case (under assumptions)
- Accelerated variants greatly improve convergence (and makes it practical)

Newton-type PG:

- Variable metric PG exist, which require solvng inner subproblem in general
- PANOC: Newton-type method for the composite optimality conditions
- **Same oracle** as PG: ∇f and $\mathbf{prox}_{\gamma g}$
- **Same global convergence** as PG
- Much faster local convergence using e.g. L-BFGS directions

References

Theory books:

- Bertsekas, "Convex Optimization Theory", 2009
- Rockafellar, Wets, "Variational Analysis", 2009
- Bauschke, Combettes, "Convex Analysis and Monotone Operator Theory in Hilbert Spaces", 2017

Algorithms books:

- Bertsekas, "Convex Optimization Algorithms", 2015
- Beck, "First-Order Methods in Optimization", 2017

(Accelerated) proximal gradient method:

- Beck, Teboulle, "A Fast Iterative Shrinkage-Thresholding Algorithm for Linear Inverse Problems", 2009
- Attouch et al, "Convergence of descent methods for semi-algebraic and tame problems: ...", 2011
- Nesterov, "Gradient methods for minimizing composite functions", 2013
- Li, Lin, "Accelerated Proximal Gradient Methods for Nonconvex Programming", 2015

Newton-type proximal gradient methods:

- Becker, Fadili, "A quasi-Newton proximal splitting method", 2012
- Lee et al, "Proximal Newton-Type Methods for Minimizing Composite Functions", 2014
- Chouzenoux et al, "Variable Metric Forward-Backward Algorithm for Minimizing the Sum of a Differentiable Function and a Convex Function", 2014
- Frankel et al, "Splitting Methods with Variable Metric for Kurdyka-Lojasiewicz Functions and General Convergence Rates", 2015
- Themelis et al, "Forward-backward envelope for the sum of two nonconvex functions: ...", 2016
- Stella et al, "A Simple and Efficient Algorithm for Nonlinear Model Predictive Control", 2017
- Stella et al, "Newton-type Alternating Minimization Algorithm for Convex Optimization", 2018

Proximal Gradient Algorithms: Applications in Signal Processing

Part III

Niccolò Antonello

niccolo.antonello@idiap.ch

EUSIPCO 2019

IDIAP Research Institute

StructuredOptimization.jl

Niccolò Antonello (Idiap Research Institute)

Outline

- Introduction to Julia
- StructuredOptimization.jl
    - AbstractOperators.jl
    - ProximalOperators.jl
    - ProximalAlgorithms.jl
- Demos

Introuction to

julia

The Julia language

- General-purpose programming language
- Designed specifically for *scientific computing*
- Young language
    - Born in 2012
    - 1.0 stable release Summer 2018

The Julia language features

- **Dynamic language** (Like Python, MATLAB)
- **Interoperability**
  - Easily call other languages (C, Fortran)
- **Designed to be *fast***
  - Approaches C, faster than Python
- **Open Source**

- Syntax very close to MATLAB

In [118]:

```
# solving a random linear system of equations (y = A*x)
A = randn(3,3)
x = randn(3)      # just a vector
y = A\x
```

Out[118]:

```
3-element Array{Float64,1}:
  2.0664785413005244
 -3.0015294370755936
 -1.5437559125502291
```

- ...but often very close to Python

In [119]:
```
a,b = (1,2) # Tuples
```

Out[119]:
```
(1, 2)
```

In [120]:
```
[i+j for i = 1:3, j =1:3] # Comprehensions
```

Out[120]:
```
3×3 Array{Int64,2}:
 2  3  4
 3  4  5
 4  5  6
```

- **Ahead-of-time compilation**

In [121]:

```
function foo(x)
    return sum(x)
end
x = randn(1000)
# first time you run a function code is compiled
@time foo(x)
# second time code is re-used
@time foo(x);
```

```
  0.024094 seconds (3.84 k allocations: 170.800 KiB)
  0.000005 seconds (5 allocations: 176 bytes)
```

Learning Julia

- Full documentation [docs.julialang.org (https://docs.julialang.org/en/v1/)](https://docs.julialang.org/en/v1/)
- Responsive and helpful community in [Discourse (https://discourse.julialang.org)](https://discourse.julialang.org)
- Help funtion

In [125]:
```
?cos # Interactive help through Read–Eval–Print Loop (REPL)
```

```
search: cos cosh cosd cosc Cos cospi cosine acos acosh acosd sincos const clos
e
```

Out[125]:
```
cos(x)
```

Compute cosine of `x`, where `x` is in radians.

```
cos(A::AbstractMatrix)
```

Compute the matrix cosine of a square matrix `A`.

If `A` is symmetric or Hermitian, its eigendecomposition (`eigen` (@ref)) is used to compute the cosine. Otherwise, the cosine is determined by calling `exp` (@ref).

Examples

```
jldoctest
julia> cos(fill(1.0, (2,2)))
2×2 Array{Float64,2}:
  0.291927  -0.708073
 -0.708073   0.291927
```

```
cos(x::AbstractExpression)
```

Cosine function:

$\cos(\mathbf{x})$

See documentation of `AbstractOperator.Cos`.

Integrated development environment (IDE)

- Juno (http://junolab.org) (Atom extension) $\leftarrow$ user friendly IDE
- Jupyter notebooks (https://jupyter.org) (such as this one) available also online (juliabox (https://juliabox.com))
- Other editors such as Vim, Spacemacs have dedicated extensions

Package manager

- Julia has built-in package manager

- Installing packages

```
julia> ] add AbstractOperators
```

Optimization in Julia

- [JuMP.jl (https://github.com/JuliaOpt/JuMP.jl)](https://github.com/JuliaOpt/JuMP.jl)
    - LP, MIP, SOCP, NLP
- [Convex.jl (https://github.com/JuliaOpt/Convex.jl)](https://github.com/JuliaOpt/Convex.jl)
    - Convex Optimization (like MATLAB's CVX)
- [Optim.jl (https://github.com/JuliaNLSolvers/Optim.jl)](https://github.com/JuliaNLSolvers/Optim.jl)
    - Smooth Nonlinear Programming

StructuredOptimization.jl

- **Large scale** and **nonsmooth** problems
- Convex & Nonconvex
- PG algorithms
- Modeling language with mathematical formulation

StructuredOptimization.jl: Package ecosystem

Joins 3 *independent* packages:

- **ProximalOperators.jl**
- **AbstractOperators.jl**
- **ProximalAlgorithms.jl**

ProximalOperators.jl

Proximal Operators

$$\mathbf{y} = \operatorname{prox}_{\gamma g}(\mathbf{x}) = \arg\min_{\mathbf{z}} \left\{ g(\mathbf{z}) + \frac{1}{2\gamma} \|\mathbf{z} - \mathbf{x}\|^2 \right\},$$

where $g : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}, \gamma > 0.$

- Generalization of projection
- Often has efficient closed form

Library of functions with efficient prox

- **Indicators of sets**
  - Norm balls e.g. $S = \left\{x : \sum_i |x_i| \leq r\right\}$
- **Norm and regularization functions**
  - Norms e.g. $\|x\|_1, \|x\|_\infty$
- **Penalties and other functions**
  - Least squares, Huber loss, Logistic loss

Example: ℓ_1-norm

In [126]:
```
using ProximalOperators # load a package
lambda = 3.5         # regularization parameter
f = NormL1(lambda) # one can create the L1-norm as follows
```

Out[126]:
```
description : weighted L1 norm
domain      : AbstractArray{Real}, AbstractArray{Complex}
expression  : x ↦ λ||x||_1
parameters  : λ = 3.5
```

In [128]:

```
# `prox` evaluates proximal operator at `x`
# optional positive stepsize `gamma`
gamma = 0.5
x = randn(10)
y, fy = prox(f, x, gamma)
# returning proximal point y and the value of the f(y)
```

Out[128]:

```
([0.0, 0.0, 0.0, -0.00310438, 0.0, 0.0, 0.0, 0.0, 0.0, 0.0], 0.010865314173881
369)
```

In [129]:

```
# `prox!` evaluates the proximal operator in-place
# (Note: by convention func. with ! are in-place)

y = similar(x); # pre-allocate y
fy = prox!(y, f, x, gamma)
```

Out[129]:

```
0.010865314173881369
```

ProximalOperators.jl calculus rules

Modify & combine functions

- Convex conjugate
- Functions combinations (Separable sum)
- Function regularization (Moreau Envelope)
- Pre and post composition

Example: Precomposition

Construct a least squares function with diagonal matrix:

$$f(\mathbf{x}) = \|\mathbf{D}\mathbf{x} - \mathbf{y}\|^2$$

In [130]:

```
d, y = randn(10), randn(10);
```

In [131]:
```
f_ls = SqrNormL2() # smooth function
```

Out[131]:
```
description : weighted squared Euclidean norm
domain      : AbstractArray{Real}, AbstractArray{Complex}
expression  : x ↦ (λ/2)||x||^2
parameters  : λ = 1.0
```

In [132]:
```
f = PrecomposeDiagonal(f_ls, d, y)
```

Out[132]:
```
description : Precomposition by affine diagonal mapping of weighted squared Eu
clidean norm
domain      : AbstractArray{Real}, AbstractArray{Complex}
expression  : x ↦ f(diag(a)*x + b)
parameters  : f(x) = x ↦ (λ/2)||x||^2, a = Array{Float64,1}, b = Array{Float6
4,1}
```

In [133]:

```
x = randn(10)
y, fy = prox(f,x)
```

Out[133]:

```
([1.73048, -0.384114, 0.298391, 0.885389, -0.567672, -0.166953, -0.103638, -0.
368458, -0.240603, 0.396272], 2.1379379614662475)
```

In [134]:

```
gradfx, fx = gradient(f,x)
```

Out[134]:

```
([-0.0572666, 2.11036, 2.10945, 0.358283, -1.87252, 0.0232842, 0.177813, -1.00
873, 1.43095, -0.21392], 5.933046424575226)
```

AbstractOperators

AbstractOperators.jl extends syntax typically used for matrices to mappings.

In [135]:
```
using AbstractOperators
A = DCT(3,4) # create a 2-D Discrete Cosine Transform operator
```

Out[135]:
```
ℱc  ℝ^(3, 4) -> ℝ^(3, 4)
```

In [136]:
```
x = randn(3,4) # notice that x is not restricted to a vector!
y = A*x        # apply the linear operator
```

Out[136]:
```
3×4 Array{Float64,2}:
  0.383866  -1.40719   -0.727502  -0.452948
 -0.468083   0.740363  -0.237086  -0.49221
  0.492823  -1.47733    0.313025  -2.24597
```

Fast (Matrix free) operators library

- **Basic operators** (Eye, DiagOp)
- **DSP**
    - Transformations (e.g DFT, DCT)
    - Filtering (e.g Conv, Xcorr)
- **Nonlinear functions** (Cos, Sin)

Matrix free?

Use fast operators, avoid building matrices.

In [138]:

```
# Fourier transform
N = 2^9
x = randn(Complex{Float64},N)
A = [exp(-im*2*pi*k*n/(N)) for k =0:N-1, n=0:N-1]; #Fourier Matrix
```

In [139]:

```
A_mf = DFT(Complex{Float64},(2^9,)) # (matrix free)
```

Out[139]:

```
ℱ  ℂ^512 -> ℂ^512
```

In [140]:
```
# not good for memory
println("Size Fourier Matrix:     ", sizeof(A))
println("Size Abstract Operator: ", sizeof(A_mf))
```

```
Size Fourier Matrix:    4194304
Size Abstract Operator: 24
```

In [141]:
```
# ...and neither for speed!
print("Fourier Matrix:")
@time A*x
print("Abstract Operators:")
@time A_mf*x;
```

```
Fourier Matrix:  0.000896 seconds (5 allocations: 8.281 KiB)
Abstract Operators:  0.000017 seconds (5 allocations: 8.281 KiB)
```

AbstractOperators.jl calculus rules

- **Concatenation** HCAT, VCAT, DCAT
- **Composition**
    - Linear and Nonlinear
- **Transformations**
    - Scale, Affine addition
    - **Adjoint** and **Jacobian**

Automatic differentiation

$$f(\mathbf{x}) = \tilde{f}(AB\mathbf{x}),$$

where A and B linear operators

$$\nabla f(\mathbf{x}) = B^* A^* \nabla \tilde{f}(AB\mathbf{x})$$

A^* and B^* adjoint operators with fast transformation

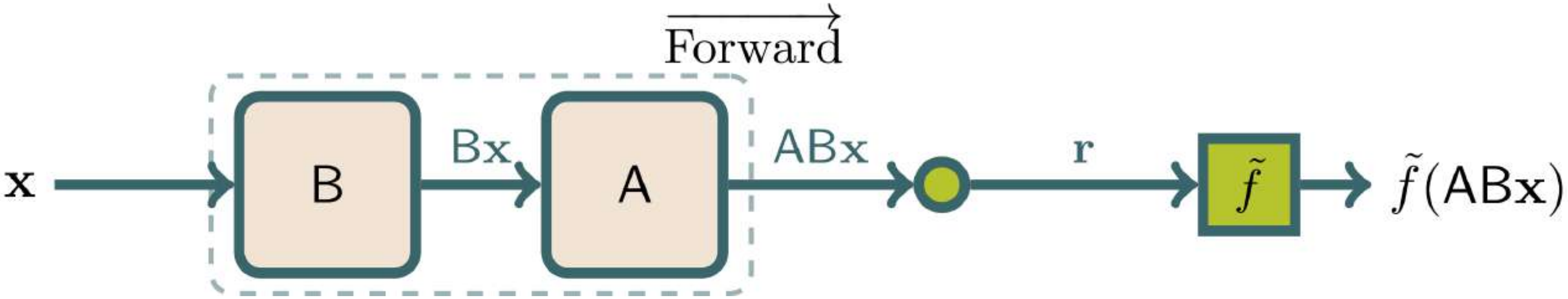
In [143]:
```
# define operators
B = IDCT(5)              # inverse DCT transform
A = FiniteDiff((5,))  # finite difference operator
B, A
```

Out[143]:
```
(ℱc⁻¹  ℝ^5 -> ℝ^5 , δx  ℝ^5 -> ℝ^4 )
```

In [144]:
```
C = A*B # can combine operators
```

Out[144]:
```
δx*ℱc⁻¹  ℝ^5 -> ℝ^4
```

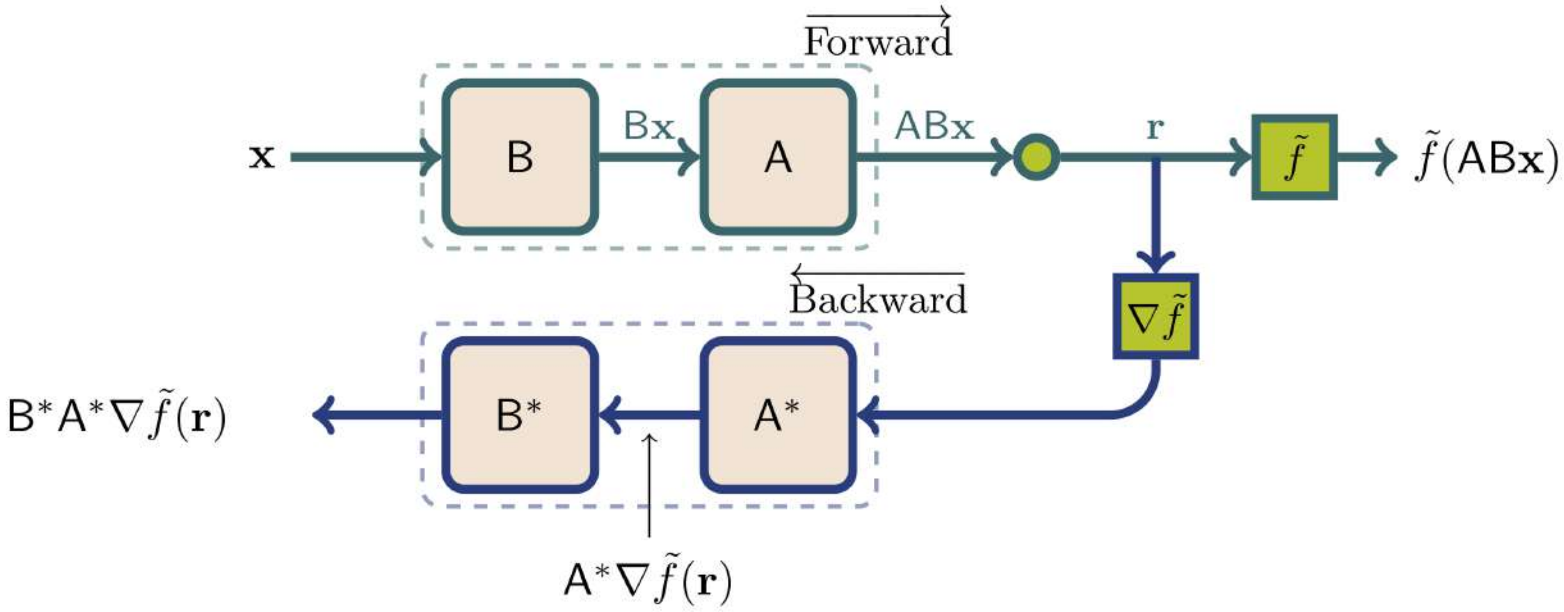
$$f(\mathbf{x}) = \tilde{f}(A B\mathbf{x})$$



In [145]:
```
x = randn(5)            # random point
r = C*x                 # r = A*B*x (Forward pass)
f_t = SqrNormL2()       # least squares cost function
f = f_t(r)              # evaluate f(x) = g(A*B*x)
```

Out[145]: 4.5266875469669605

$$\nabla f(\mathbf{x}) = B^* A^* \nabla \tilde{f}(AB\mathbf{x})$$



In [146]:
```
∇f_t, f_tx = gradient(f_t,r)
∇f = C'*∇f_t;  # get gradient: adjoint operator C' (Backpropagation)
```

In [147]:

```
# gradient using finite differences
using LinearAlgebra
x_eps = zero(x)
∇f_FD = zero(x)
for i = 1:length(x_eps)
    x_eps .= 0
    x_eps[i] = sqrt(eps())
    ∇f_FD[i] = (f_t(C*(x.+x_eps))-f)./sqrt(eps())
end
norm( ∇f_FD - ∇f ) # testing gradient using
```

Out[147]:

```
2.067756216756867e-7
```

ProximalAlgorithms.jl

Proximal algorithms for nonsmooth optimization in Julia.

- (Accelerated) **Proximal Gradient** (aka Forward-backward)
- **PANOC**

Many others:

- Asymmetric forward-backward-adjoint algorithm (AFBA)
- Chambolle-Pock primal dual algorithm
- Davis-Yin splitting algorithm
- Douglas-Rachford splitting algorithm
- Vũ-Condat primal-dual algorithm

`PANOC`, `ZeroFPR`, `ForwardBackward`

Solve problem:

$$\underset{\mathbf{x}}{\operatorname{argmin}}\, f(A\mathbf{x}) + g(\mathbf{x})$$

- f smooth function
- A linear operator
- g nonsmooth function

Example: sparse deconvolution

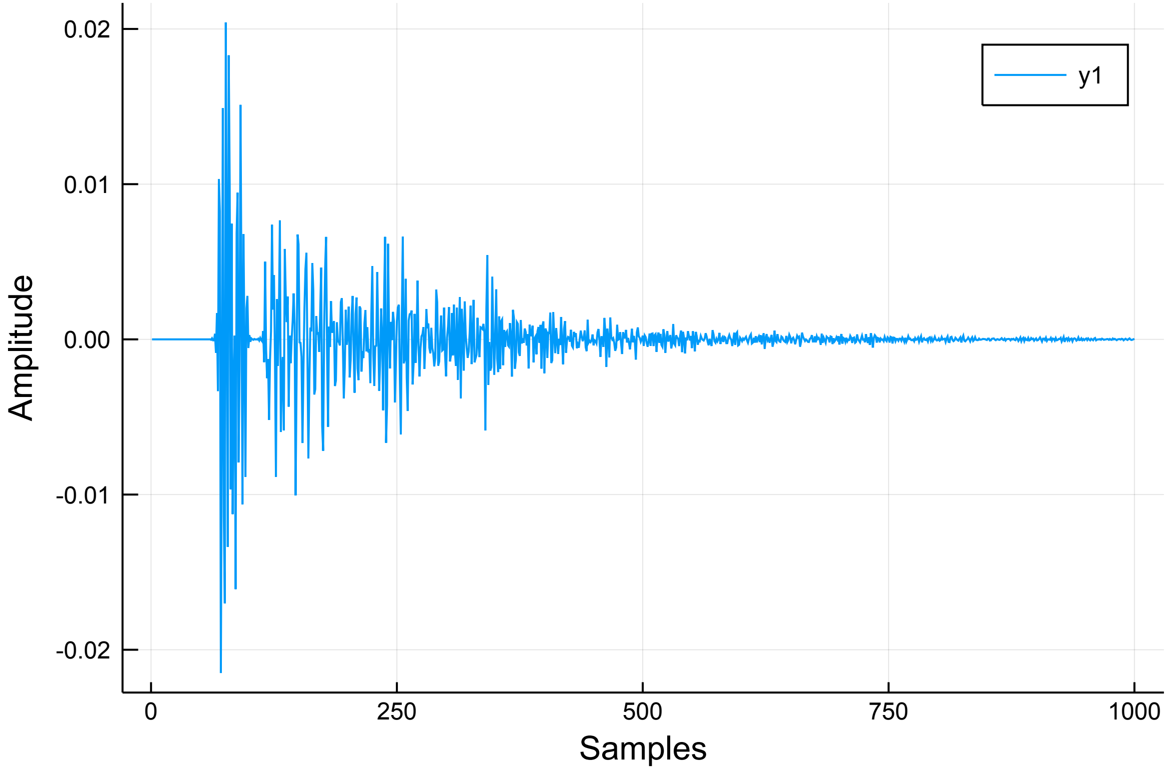
$$\mathbf{x}^\star = \underset{\mathbf{x}}{\operatorname{argmin}} \frac{1}{2} \|\mathbf{h} * \mathbf{x} - \mathbf{y}\|^2 + \lambda \|\mathbf{x}\|_1$$

- $\mathbf{h}$ impulse response (FIR)
- $\mathbf{y}$ noisy measurement
- $\mathbf{x}$ unkown source clean signal (sparse)

In [148]:

```
using DelimitedFiles, Plots
h = readdlm("data/h.txt")[:] #load impulse response
plot(h; xlabel="Samples", ylabel="Amplitude")
```

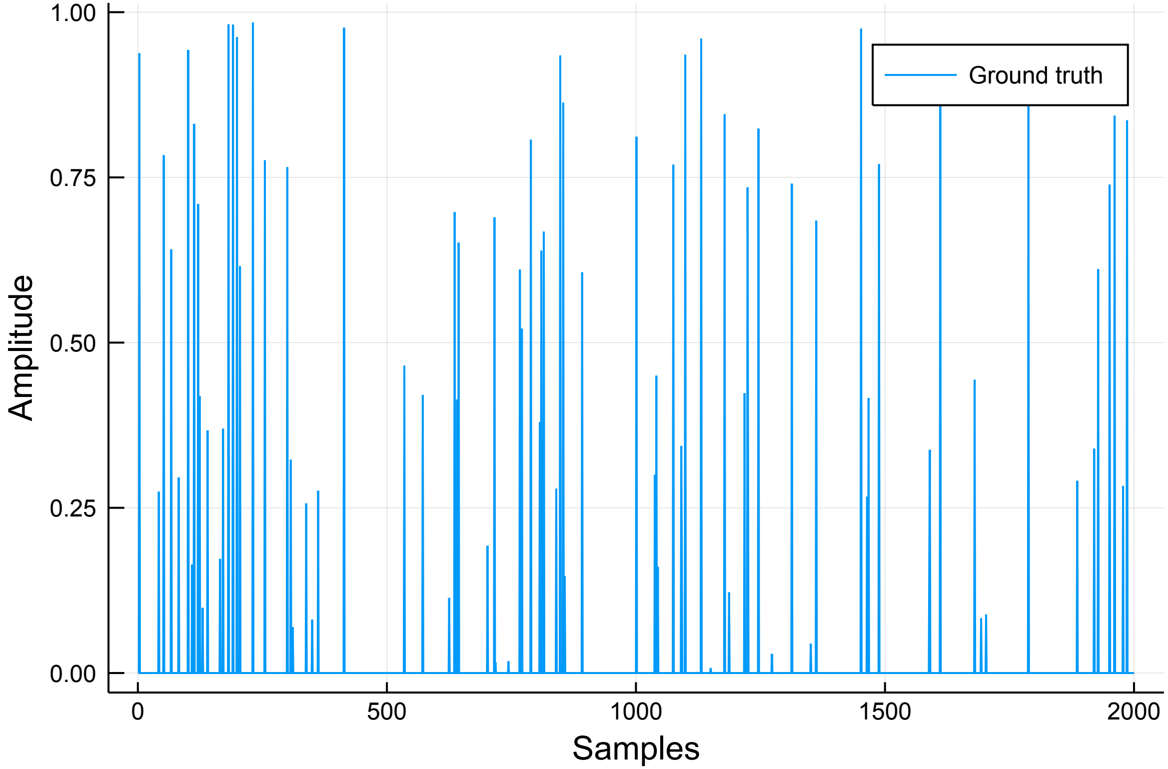
Out[148]:



In [149]:

```julia
using SparseArrays, Random; Random.seed!(123)
x_gt = Array(sprand(2000,0.05))     # random sparse vector
plot(x_gt; xlabel="Samples", ylabel="Amplitude", label="Ground truth")
```

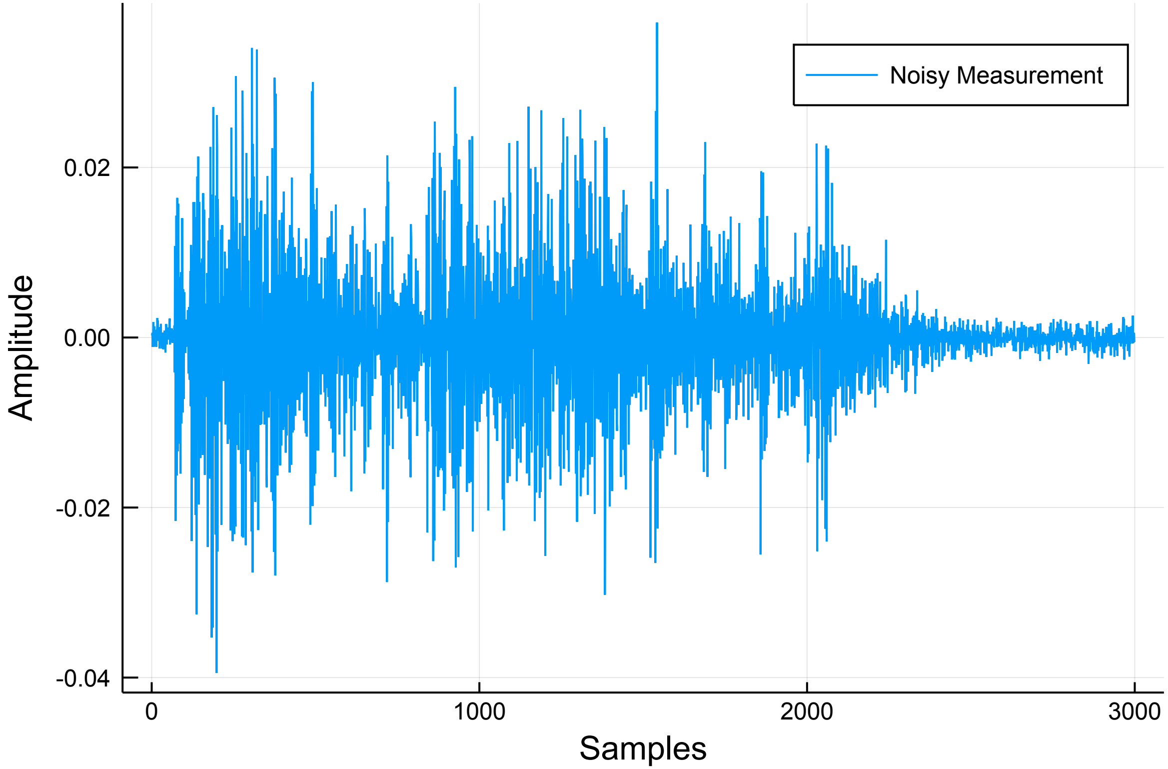
Out[149]:



In [150]:

```
using DSP
y = conv(h,x_gt) + 1e-3 .* randn(length(h)+length(x_gt)-1)
plot(y; xlabel="Samples", ylabel="Amplitude", label="Noisy Measurement")
```

Out[150]:



Construct $f(\mathbf{x}) = \frac{1}{2}\|\mathbf{h} * \mathbf{x} - \mathbf{y}\|^2$ and $g(\mathbf{x}) = \lambda\|\mathbf{x}\|_1$

In [151]:
```
using AbstractOperators
linop = Conv(size(x_gt), h) # convolution operator
```

Out[151]:
```
★  ℝ^2000 -> ℝ^2999
```

In [152]:
```
using ProximalOperators
smooth = PrecomposeDiagonal( SqrNormL2(), 1.0, -y )
nonsmooth = NormL1(1e-3);
```

Create `PANOC` solver

In [153]:
```
using ProximalAlgorithms: PANOC
solver = PANOC(verbose=true); # set options
```

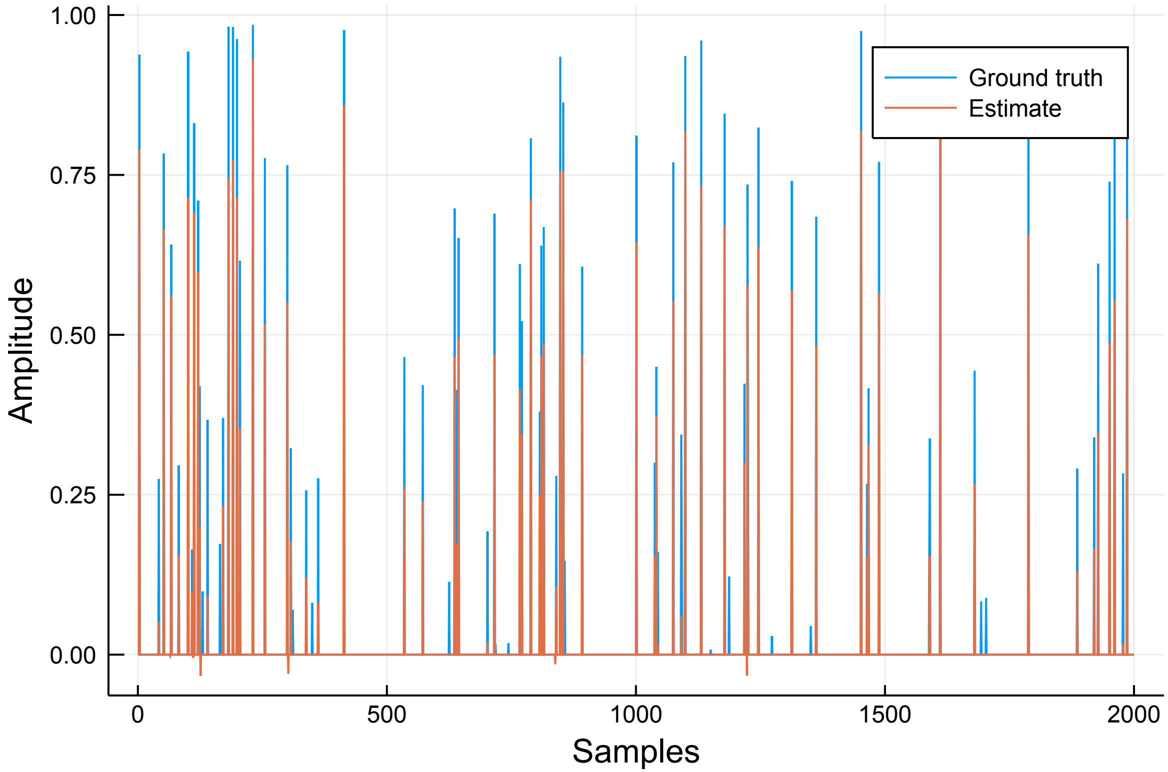
In [154]:
```
x0 = zeros(length(x_gt)) # initial estimate
println("  it  |      γ      |     res      |      τ   ")
x0, its = solver(x0; f=smooth, A=linop, g=nonsmooth);
```

```
   it  |      γ      |     res      |      τ
    10 | 2.200e+01 | 7.836e-04 | 1.000e+00
    20 | 2.200e+01 | 1.418e-04 | 1.000e+00
    30 | 2.200e+01 | 1.363e-06 | 1.000e+00
    40 | 2.200e+01 | 8.510e-08 | 1.000e+00
    47 | 2.200e+01 | 6.195e-09 | 1.000e+00
```

In [155]:

```
plot(x_gt; xlabel="Samples", ylabel="Amplitude", label="Ground truth")
plot!(x0, label="Estimate")
```

Out[155]:



StructuredOptimization.jl

Structured optimization problem:

$$\underset{\mathbf{x}}{\text{minimize}} f_1(A_1\mathbf{x}) + f_2(A_2\mathbf{x}) + \cdots + f_N(A_N\mathbf{x})$$

- Cost function composed of different terms
- f_i loss functions
- A_i linear operators
- Constraints: indicator functions

Structured optimization problem:

$$\underset{\mathbf{x}}{\text{minimize}} f_1(A_1\mathbf{x}) + f_2(A_2\mathbf{x}) + \cdots + f_N(A_N\mathbf{x})$$

StructuredOptimization.jl converts it to the PG general problem:

$$\underset{\mathbf{x}}{\text{minimize}} f(A\mathbf{x}) + g(\mathbf{x})$$

- f smooth (differentiable)
    - **automatic differentiation** $\rightarrow$ AbstractOperators.jl
- g nonsmooth (including constraints)
    - **efficient proximal mappings** $\rightarrow$ ProximalOperators.jl

Example: Sparse Deconvolution

$$\mathbf{x}^\star = \underset{\mathbf{x}}{\mathrm{argmin}} \frac{1}{2}\|\mathbf{h} * \mathbf{x} - \mathbf{y}\|^2 + \lambda\|\mathbf{x}\|_1$$

In [156]:

```
using StructuredOptimization
x = Variable(length(x_gt)) # define optimization variable
```

Out[156]:

```
Variable(Float64, (2000,))
```

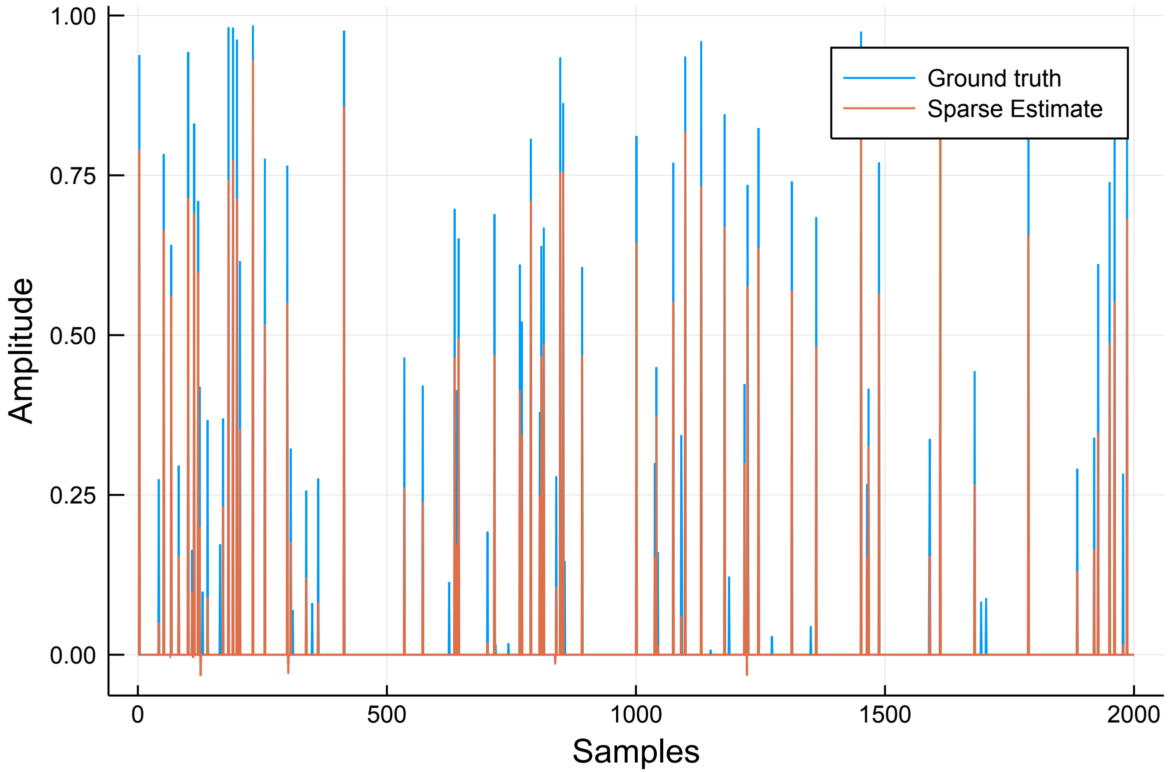
In [157]:

```
# (ls short hand for 0.5*norm(...)^2 )
@minimize ls( conv(x,h) - y ) + 1e-3*norm(x, 1); # solve problem
```

In [158]:

```
plot(x_gt; xlabel="Samples", ylabel="Amplitude", label="Ground truth")
plot!(~x, label="Sparse Estimate")
# ~x to access the solution
```

Out[158]:



Matrix free optimization

In [159]:
```
~x .= 0
_, its = @time @minimize ls( conv(x,h) - y ) + 1e-3*norm(x, 1)
x_mf = copy(~x);
```

```
0.185090 seconds (17.13 k allocations: 6.785 MiB, 6.34% gc time)
```

In [160]:
```
~x .= 0; Nx = length(x_gt)
H = hcat([[zeros(i);h;zeros(Nx-1-i)] for i = 0:Nx-1]...)
_, its_mf = @time @minimize ls( H*x - y ) + 1e-3*norm(x, 1);
its_mf == its
```

```
0.342836 seconds (17.44 k allocations: 6.720 MiB)
```

Out[160]:
```
true
```

Example: constraint optimization

Refine the LASSO solution using:

$$\begin{aligned} &\underset{\mathbf{z}}{\text{minimize}}\ \tfrac{1}{2}\|(\mathbf{E}\mathbf{z}) * \mathbf{h} - \mathbf{y}\|^2 \\ &\text{subject to } \mathbf{z} \geq 0 \end{aligned}$$

- $\mathbf{z}$ is a vector of length $\|\mathbf{x}\|_0$
- $\mathbf{E}$ is a matrix that expands $\mathbf{z}$ to the support of $\mathbf{x}^\star$

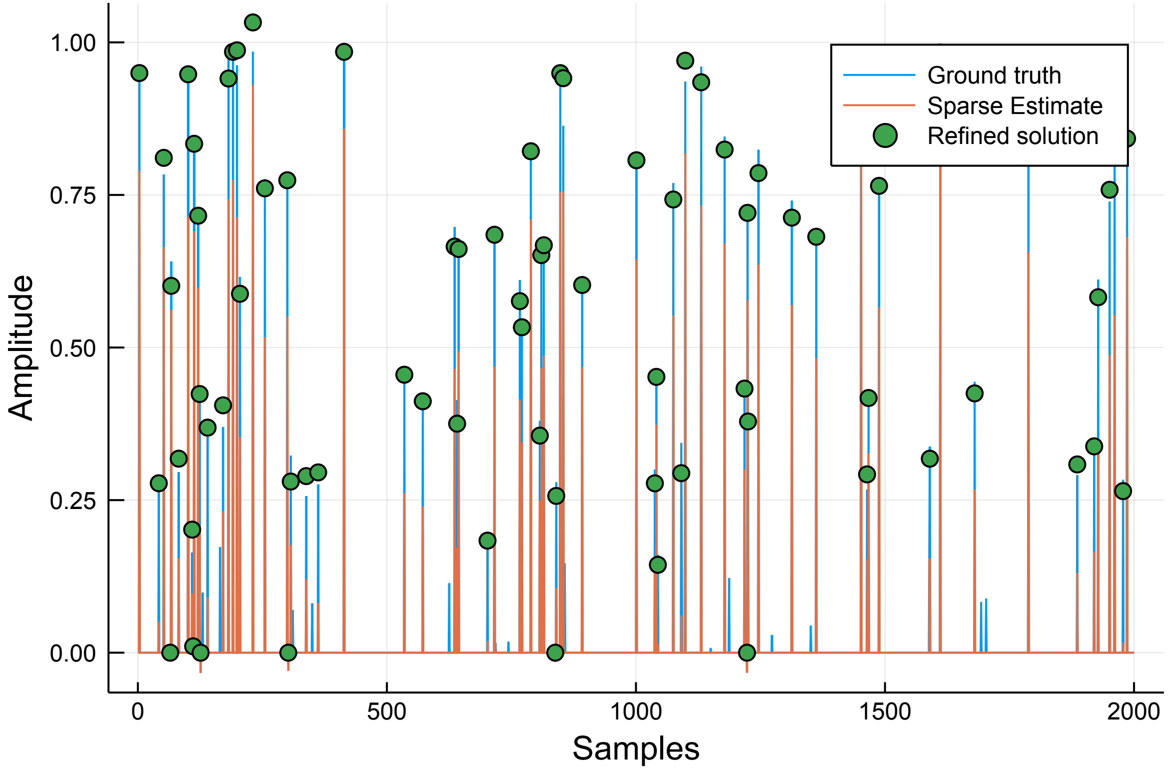
In [162]:

```julia
using LinearAlgebra
idx = findall(.!( (~x) .≈ 0.0 ))       # indices nonzero elements
E = Diagonal(ones(length(~x)))[:,idx] # create expansion matrix

z = Variable((~x)[idx]) # initialize var. with nonzero elements
@minimize ls( conv(E*z,h) - y ) st z >= 0.0; # solve problem
```

In [163]:

```
plot(x_gt; xlabel="Samples", ylabel="Amplitude", label="Ground truth")
plot!(~x, label="Sparse Estimate")
scatter!(idx, ~z, label="Refined solution" )
```

Out[163]:



Limitations

- Only PG algorithms supported
- g must be **efficiently computable proximal mappings**

Nonsmooth function $g(\boldsymbol{B}\mathbf{x})$ must satisfy:

1. $\boldsymbol{B}$ is a **tight frame**
   - $\boldsymbol{B}\boldsymbol{B}^* = \mu \boldsymbol{I}$, where $\mu \geq 0$
2. g is a *separable sum*: $g(\boldsymbol{B}\mathbf{x}) = \sum_j h_j(\boldsymbol{C}_j \mathbf{x}_j)$
   - $\mathbf{x}_j$ non-overlapping slices of $\mathbf{x}$
   - $\boldsymbol{C}_j$ tight frames

In [164]:

```
n = 10
A,b = randn(2*n,n),randn(2*n)
x = Variable(n)
@minimize ls(A*x-b)+norm(dct(x),1);
# nonsmooth fun composed with orthogonal operator (rule 1)
```

In [165]:

```
#@minimize ls(A*x-b)+norm(A*x,1)
# rule 1 not satisfied!
```

In [166]:

```
#@minimize ls( A*x - b ) + maximum(x) st x >= 2.0
# rule 2 not satisfied!
```

In [167]:

```
@minimize ls( A*x - b ) + maximum(x[1:5]) st x[6:10] >= 2.0;
# accepted: optimization variables partitioned in nonoverlapping groups
```

Demos

- Line Spectral estimation
- Video background removal
- Audio Declipping

Line Spectral Estimation

Goal:

- recover frequencies & amplitudes of signal $\mathbf{y}$

Assumption:

- $\mathbf{y}$ sparse mixture of N sinusoids.

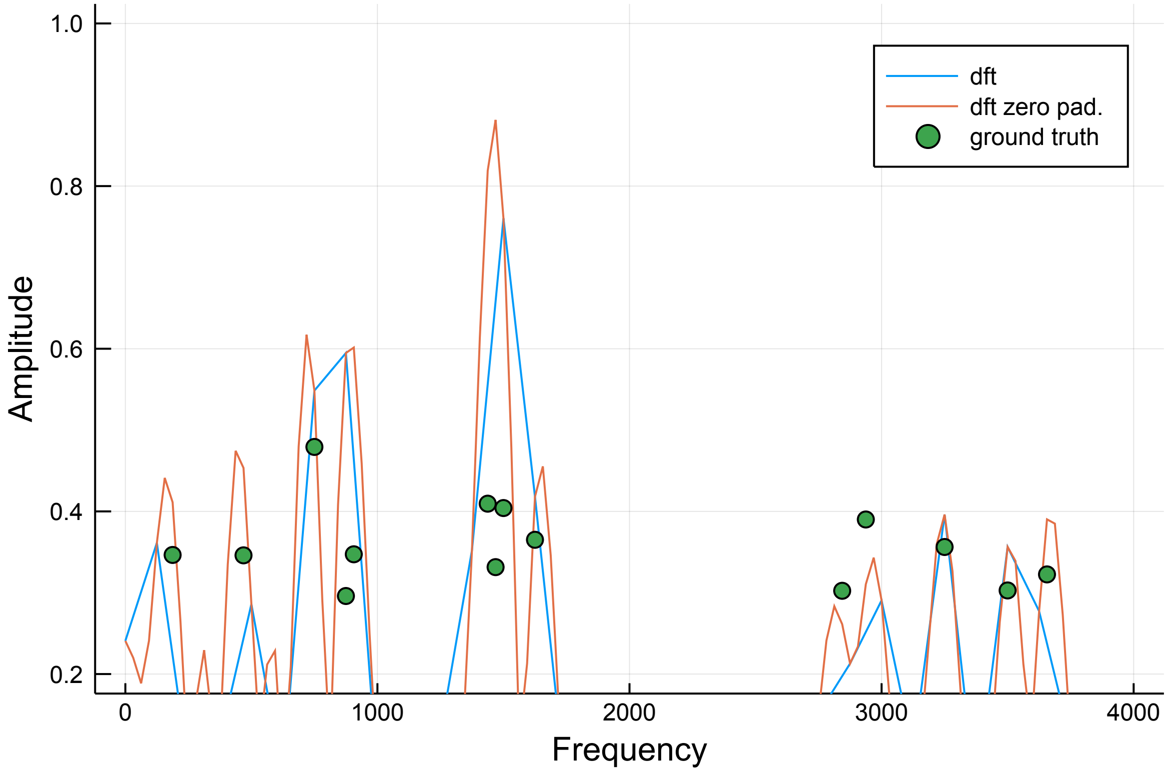
Simple solutions:

- DFT
- zero-padded DFT of $\mathbf{y}$ with s super-resolution factor.

In [18]:

```
plot(f,    abs.(fft(y)./Nt )[1:div(Nt,2)+1]; label = "dft")
plot!(f_s, abs.(xzp./Nt )[1:div(s*Nt,2)+1];  label = "dft zero pad.", ylim=[0.2;1
], xlim=[0;4e3], xlabel="Frequency", ylabel="Amplitude")
scatter!(fk, abs.(A) ./2; label = "ground truth")
```

Out[18]:



Spectral leakage:

- frequencies merge
- amplitude not estimated correctly

Lasso formulation:

$$\mathbf{x}_1^\star = \underset{\mathbf{x}}{\operatorname{argmin}} \frac{1}{2} \| S F^{-1} \mathbf{x} - \mathbf{y} \|^2 + \lambda \| \mathbf{x} \|_1,$$

- F^{-1}: Inverse Fourier transform
- S: selection mapping takes first N_t samples

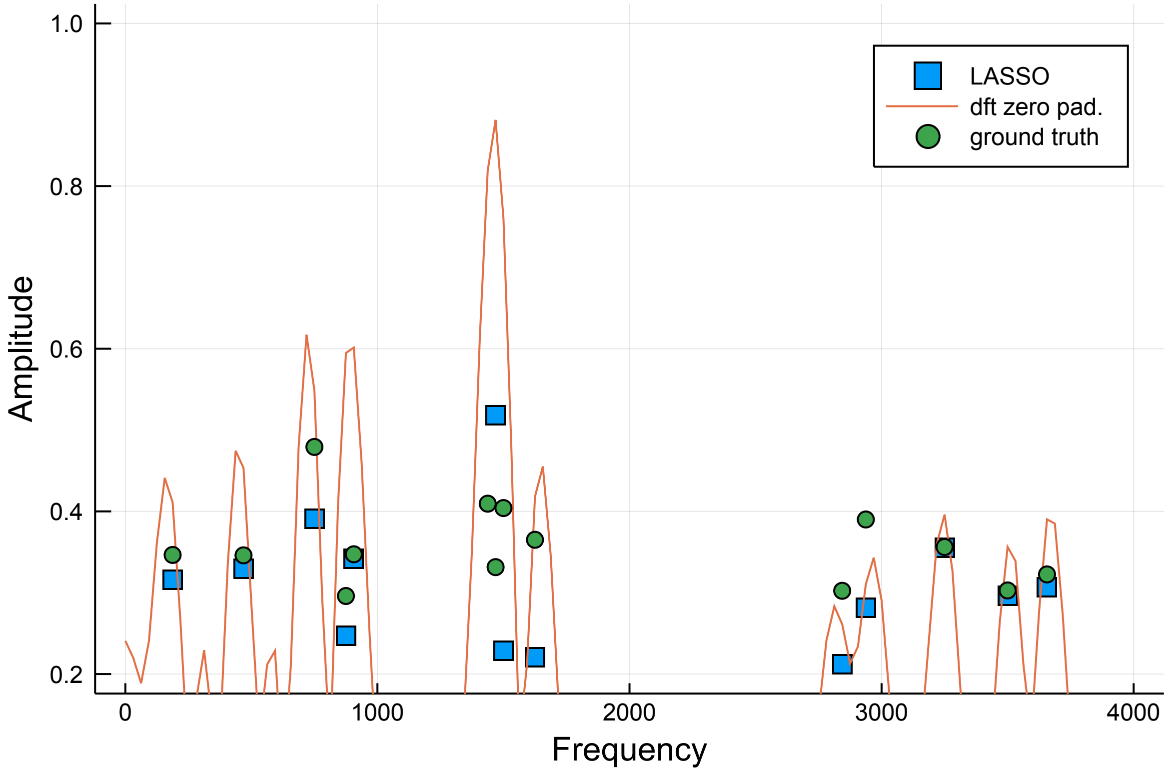
In [19]:

```
using StructuredOptimization

x = Variable(Complex{Float64}, s*Nt) # define complex-valued variable
lambda = 1e-3*norm(xzp./(s*Nt),Inf)  # set lambda

@minimize ls(ifft(x)[1:Nt]-complex(y))+lambda*norm(x,1) with PANOC(tol = 1e-8)
x1 = copy(~x); # copy solution
```

In [20]:

```
scatter(f_s, abs.(x1[1:div(s*Nt,2)+1]./(s*Nt) ); label = "LASSO", m=:square)
plot!(f_s, abs.(xzp./Nt )[1:div(s*Nt,2)+1];  label = "dft zero pad.", ylim=[0.2;1
], xlim=[0;4e3], xlabel="Frequency", ylabel="Amplitude")
scatter!(fk, abs.(A) ./2; label = "ground truth", ylim=[0.2;1], xlim=[0;4e3])
```

Out[20]:



Lasso results

- $\mathbf{x}_1^\star$ estimates improve!
- Amplitude usually underestimated

Non-convex problem

$$\mathbf{x}_0^\star = \underset{\mathbf{x}}{\operatorname{argmin}} \frac{1}{2} \| SF^{-1}\mathbf{x} - \mathbf{y} \|^2 \text{ s.t. } \|\mathbf{x}\|_0 \leq 2N.$$

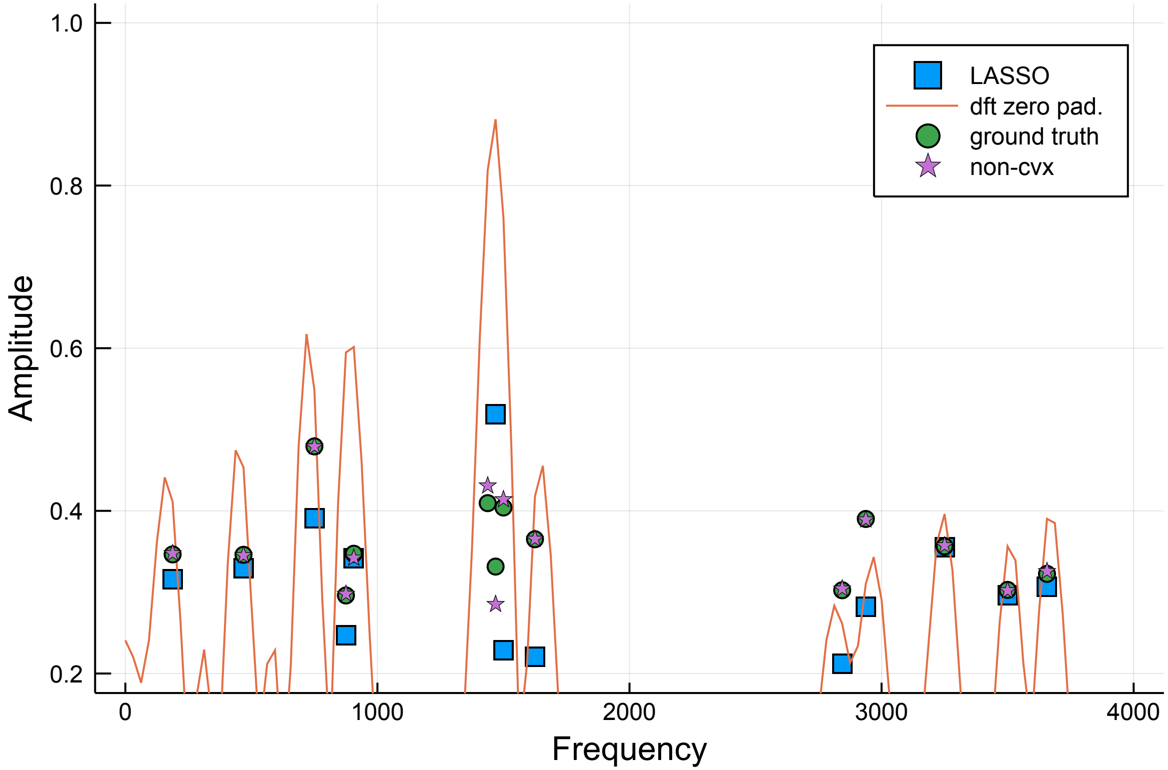
In [22]:

```
# notice that following problem is warm-started by previous solution
@minimize ls(ifft(x)[1:Nt]-complex(y)) st norm(x,0) <= 2*N  with PANOC(tol = 1e-8
);
x0 = copy(~x);
```

In [23]:

```
scatter!(f_s, abs.(x0[1:div(s*Nt,2)+1]./(s*Nt) ); label = "non-cvx", m=:star)
```

Out[23]:



Demo: Video Background removal

Video

- Static background
- Moving foreground

Goal

- Separate foreground from static background

In [5]:

```
using Images
include("utils/load_video.jl")
n, m, l = size(Y)
Gray.([Y[:,:,1] Y[:,:,2] Y[:,:,3]])
```

Out[5]:

Low rank approximation

$$\begin{aligned}&\underset{\mathbf{L},\mathbf{S}}{\text{minimize}}\ \frac{1}{2}\|\mathbf{L}+\mathbf{S}-\mathbf{Y}\|^2+\lambda\|\text{vec}(\mathbf{S})\|_1\\&\text{subject to } \text{rank}(\mathbf{L})\leq 1\end{aligned}$$

- $\mathbf{Y}$: l-th column has l-th frame
- $\mathbf{L}$: background (low-rank)
- $\mathbf{S}$: foreground (sparse)

$$\underset{\mathbf{L},\mathbf{S}}{\text{minimize}}\ \tfrac{1}{2}\|\mathbf{L}+\mathbf{S}-\mathbf{Y}\|^2 + \lambda\|\text{vec}(\mathbf{S})\|_1$$

$$\text{subject to}\ \text{rank}(\mathbf{L}) \leq 1$$

In [6]:

```
using StructuredOptimization

Y = reshape(Y,n*m,l) # reshape video
L = Variable(n*m,l)  # define variables
S = Variable(n*m,l)

@minimize ls(L+S-Y) + 3e-2*norm(S,1) st rank(L) <= 1 with PANOC(tol = 1e-4);
```

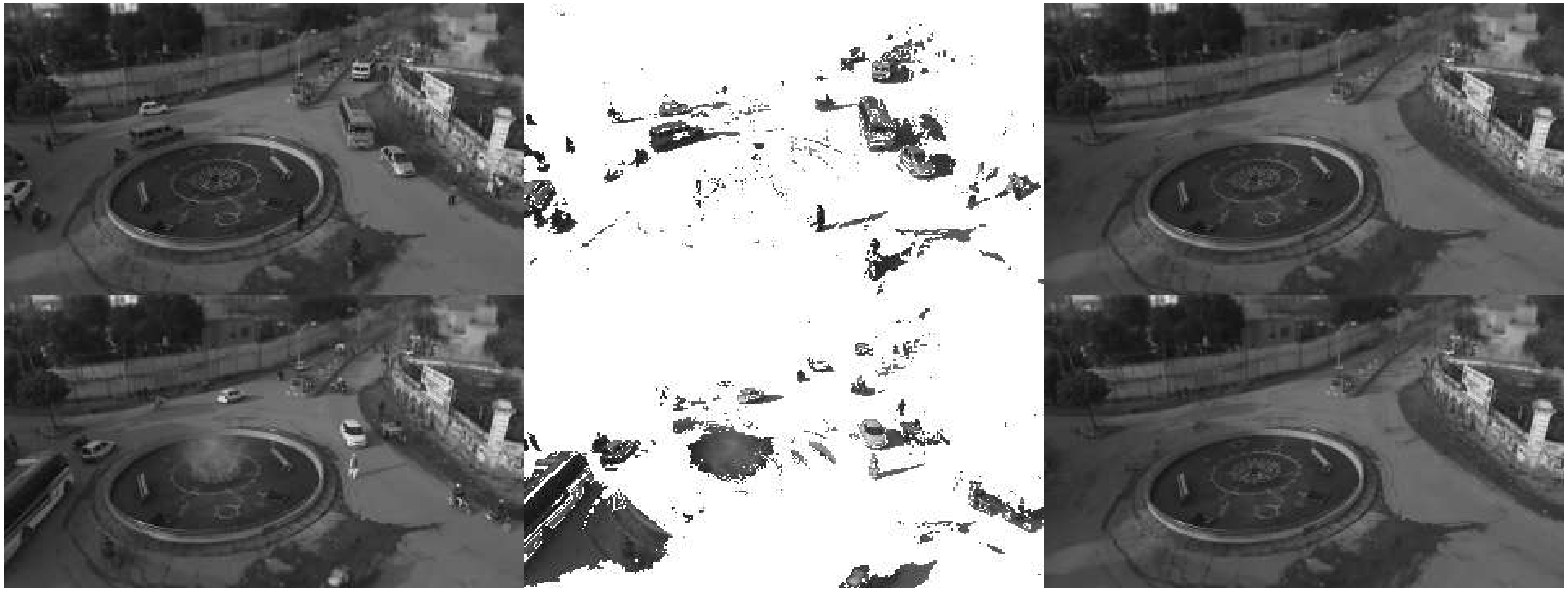
In [7]:

```
L, S = ~L, ~S # extract vectors from variables
S[ S .!= 0 ] .= S[ S .!= 0 ] .+L[ S .!= 0 ]
# add background to foreground changes in nonzero elements
S[S.== 0] .= 1.0
# put white in null pixels
Y, S, L = reshape(Y,n,m,l), reshape(S,n,m,l), reshape(L,n,m,l);
```

In [8]:

```
idx = [1;3]
img = Gray.(vcat([ [Y[:,:,i] S[:,:,i] L[:,:,i]] for i in idx]...))
```

Out[8]:



Demo: Audio de-clipping

Audio recoding of loud source can saturate

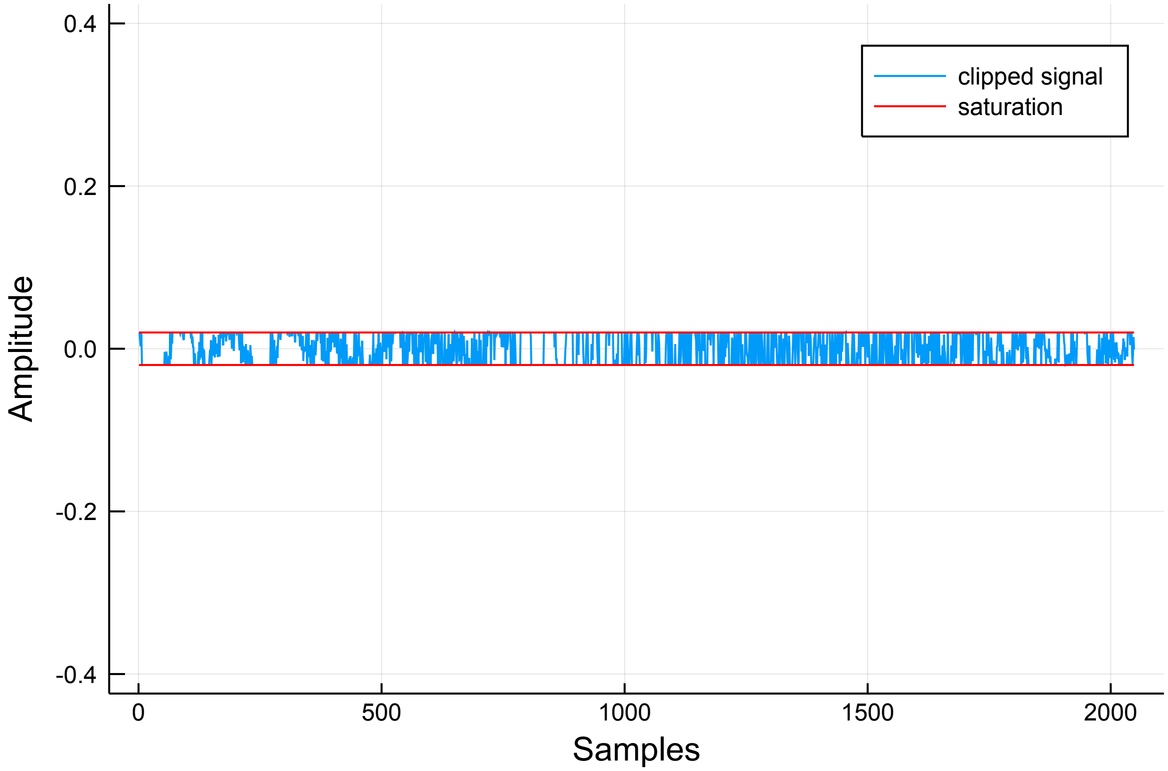
In [1]:

```
using WAV, Plots
# load wav file
yt, Fs = wavread("data/clipped.wav"); yt = yt[:,1][:]
C = maximum(abs.(yt))      # clipping level
# plotting a frame of the audio signal
idxs = 2^11+1:2^12;
```

In [2]:

```
plot(yt[idxs]; label = "clipped signal", xlabel="Samples", ylabel="Amplitude", ylim=[-0.4; 0.4])
plot!([1;length(idxs)],  [C.*ones(2), -C.*ones(2)]; color=[:red :red], label = ["saturation" ""])
```

Out[2]:



$$
\begin{aligned}
\underset{\mathbf{x},\mathbf{y}}{\text{minimize}} \quad & \frac{1}{2}\|F_{i,c}\mathbf{x} - \mathbf{y}\|^2, \\
\text{subject to} \quad & \|M\mathbf{y} - M\tilde{\mathbf{y}}\| \leq \epsilon \\
& M_{+}\mathbf{y} \geq C \\
& M_{-}\mathbf{y} \leq -C \\
& \|\mathbf{x}\|_0 \leq N
\end{aligned}
$$

Input:

- $\tilde{\mathbf{y}}$ frame of clipped signal

Optimization variables:

- $\mathbf{x}$ DCT transform declipped frame ($F_{i,c}$ brings to time domain)
- $\mathbf{y}$ time domain declipped frame

Constraints on $\mathbf{y}$:

- M selection matrix of uncorrupted samples
- $M_{\pm}$ selection matrix of saturated samples

Constraints on $\mathbf{x}$:

- $\mathbf{x}$ is sparse ℓ_0-ball constraint (sparsity DCT domain) (*nonconvex*)

$$
\begin{aligned}
\underset{\mathbf{x},\mathbf{y}}{\text{minimize}} \quad & \frac{1}{2}\|F_{i,c}\mathbf{x} - \mathbf{y}\|^2, \\
\text{subject to} \quad & \|M\mathbf{y} - M\tilde{\mathbf{y}}\| \leq \epsilon \\
& M_+\mathbf{y} \geq C \\
& M_-\mathbf{y} \leq -C \\
& \|\mathbf{x}\|_0 \leq N
\end{aligned}
$$

Nonconvex problem: refine solution by increasing N

In [3]:

```
using StructuredOptimization, DSP
Nl = 2^10                          # time window length
Nt = length(yt)                    # signal length
yd = zeros(Nt)                     # allocate declipped output

x, y = Variable(Nl), Variable(Nl) # optimization variables
f = ls( idct(x) - y )              # cost function
yw = zeros(Nl)                     # allocate weighted clipped frame

# wieight window options
win = sqrt.(hanning(Nl+1)[1:Nl])
overlap = div(Nl,2);
```

In [ ]:

```
z, ϵ = 0, sqrt(1e-5) #weighted Overlap-Add
while z+Nl < Nt
    fill!(~x,0.); fill!(~y,0.) # initialize variables
    Ip = sort(findall(     yt[z+1:z+Nl]  .>=  C) ) #pos clip idxs
    In = sort(findall(     yt[z+1:z+Nl]  .<= -C) ) #neg clip idxs
    I  = sort(findall(abs.(yt[z+1:z+Nl]) .<   C))  #uncor idxs

    yw .= yt[z+1:z+Nl].*win # weighted frame
    for N = 30:30:30*div(Nl,30)   # increase active components DCT
        cstr = (norm(x,0) <= N,
                norm(y[I]-yw[I]) <= ϵ,
                y[Ip] >=  C.*win[Ip],
                y[In] <= -C.*win[In]   )
        @minimize f st cstr with PANOC(tol = 1e-4, verbose = false)
        if norm(idct(~x) - ~y) <= ϵ break end
    end
    yd[z+1:z+Nl] .+= (~y).*win # store declipped signal
    z += Nl-overlap            # update index
end
```

In [ ]:

```
plot(yd[idxs], label = "declipped signal", xlabel="Time (samples)", ylabel="Amplit
ude", ylim=[-0.4; 0.4])
plot!(yt[idxs]; label = "clipped signal")
plot!([1;length(idxs)],  [C.*ones(2), -C.*ones(2)]; color=[:red :red], label = ["s
aturation" ""])
```

In [ ]:

```
using LinearAlgebra
wavwrite( 0.9 .* normalize(yd[:],Inf), "data/declipped.wav"; Fs = Fs, nbits = 16,
compression=WAVE_FORMAT_PCM) # save wav file
```

Clipped audio:

0:00 / 0:15

Declipped audio:

0:00 / 0:15

Conclusions

- Proximal gradient (PG) methods apply to wide variety of signal processing tasks
- PG framework applies to large-scale inverse problems with non-smooth terms
- PG framework applies to both convex and nonconvex problems
- Accelerated and Newton-type extensions of PG enjoy much faster convergence
- Julia software toolbox offers modeling language with mathematical notation
- More signal processing demos & examples available @ `https://github.com/kul-forbes/StructuredOptimization.jl`

Conclusions

Additional resources

- N. Antonello, L. Stella, P. Patrinos and T. van Waterschoot, "*Proximal gradient algorithms: applications in signal processing*", arXiv:1803.01621, Mar. 2018. `https://arxiv.org/abs/1803.01621`
- Software packages:
  - `https://github.com/kul-forbes/ProximalOperators.jl`
  - `https://github.com/kul-forbes/AbstractOperators.jl`
  - `https://github.com/kul-forbes/ProximalAlgorithms.jl`
  - `https://github.com/kul-forbes/StructuredOptimization.jl`

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