Perturbation Analysis of Root-MUSIC-Type Methods for Blind Network-Assisted Diversity Multiple Access

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***Abstract*—We perform a first-order perturbation analysis of a root-MUSIC-type method for resolving collisions in the context of blind network-assisted diversity multiple access (BNDMA). Polynomial roots are computed as an intermediate step of the root-MUSIC algorithm for the purpose of blindly identifying the set of transmitters involved in a collision. We derive expressions for the individual and joint distributions of the noise-induced angular shifts of the computed roots. The expressions are analyzed in relation to the signal-to-noise ratio and the number of packet retransmissions made to resolve a collision. Results are verified numerically.**

Index — perturbation analysis, root-MUSIC, collision resolution, network-assisted diversity

I. Introduction

Network-assisted diversity multiple access (NDMA), first introduced in [1], is a cross-layer communication protocol that enables shared access to a communication channel and provides a solution for resolving packet collisions. In the case of simultaneous transmissions on the same frequency band, collided data is not discarded but rather stored by the receiver. The receiver relies on the medium access control (MAC) layer functionality and requests packet retransmissions from the involved transmitters. This creates the needed diversity which is exploited in the physical (PHY) layer using advanced signal processing in order to separate the colliding packets. In the blind version of NDMA (BNDMA), the set of active transmitters is unknown to the receiver beforehand. Each transmitter k multiplies its n^{th} transmission of packet $\vec{s}_k$ by $r_k^{(n-1)}$, where r_k is a complex exponential characteristic of transmitter k. This allows the receiver to detect the signatures of the transmitters in the received signal and blindly identify the active set. If each of K transmitters makes N transmissions, the received packets may be stacked in matrix Y_N given by [2]

$$Y_{N\times P} = W_{N\times K} \times S_{K\times P} + \Delta X \qquad (1)$$

$W_{N\times K}$ is a Vandermonde coefficient matrix whose k^{th} column is $[1, r_k, r_k^2, \ldots, r_k^{N-1}]^T$. $S_{K\times P}$ holds K user packets $\{\vec{s}_k\}_k$, each of which has P symbols. ΔX is the observation noise.

Equation (1) resembles the response of an antenna array of N sensor nodes that sample a mixture of K signals. The Vandermonde structure of $W_{N\times K}$, which acts as a virtual steering matrix, implies that the array has a uniform linear arrangement. Data from source k extends over the k^{th} row of S, and P snapshots are collected at each antenna output. The problem of blindly detecting the characteristic complex exponentials $\{r_k\}_k$ of the active transmitters is equivalent to the direction of arrival (DoA) estimation problem for K sources, where source k impinges on the antenna array at an angle $\omega_k = \angle r_k$. For the DoA problem, Estimation of Signal Parameters via Rotational Invariance Technique (ESPRIT), Multiple Signal Classification (MUSIC) and root-MUSIC are high resolution DoA estimation techniques. By making this analogy between BNDMA and DoA, an ESPRIT-type method and a root-MUSIC-type method for blind collision resolution are described in [2] and [3] respectively.

In this paper we do a perturbation analysis of the BNDMA method in [3]. Perturbation analysis refers to the effect of observation noise ΔX in (1) on the accuracy of detecting characteristic roots $\{r_k\}_k$ or equivalently $\{\omega_k\}_k$ from Y_N, which in turn affects the reconstruction of $W_{N\times K}$ and the decoding of packets S. Perturbation analysis for subspace decomposition in general is examined for instance in [4–7]. For the particular MUSIC-type subspace methods, results on perturbation analysis do exist in the literature in the context of DoA estimation. The dependence of performance on the signal-to-noise ratio (SNR), array size N, number of snapshots P, angular separation of the signals, etc. is studied numerically in [8–10]. Analytical results for the mean-squared errors (MSEs) of the DoA estimates are derived in [7], [11–13]. These results give insight on the effect of the array geometry, model error parameters, array size N, and the number of snapshots P on the performance of DoA estimation.

We analyze root-MUSIC in the context of collision resolution. This differs from DoA estimation in several respects. First, we focus on the noise averaging effect that is achieved by stacking a large number of packets N in Y_N, whereas in DoA estimation the number of sensors N cannot be flexibly varied. Second, we only consider packet transmissions of fixed symbol size P. In DoA estimation it is crucial to increase the time-averaging factor P to get better estimates of the spatial covariance matrix of the antenna array. Third, in [3] we decompose the measurement matrix Y_N itself. The observation error ΔX is assumed to follow a complex Gaussian distribution. By subspace-decomposing the sample covariance matrix computed from Y_N instead, as is typical in DoA estimation, the approximation error of the covariance matrix is modeled by a complex Wishart distribution [14], [15].

In the next section we derive first-order approximations of both the individual and the joint distributions of the angular displacements $\{\Delta\omega_k\}_k$ of $\{r_k\}_k$ in the complex plane. We prove these shifts are jointly Gaussian and fully characterize

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the means and covariances. While in [7] it is shown that the MSE of a DoA estimate monotonically decreases with the number of sensors N, in section III we argue that the MSE decays quadratically in the number of stacked packets N. Section IV presents numerical results and section V concludes the paper.

II. Noise Analysis

The signal component of Y_N in (1) can be expressed by SVD as

$$X = W_{N\times K} \times S = U_{\|}\Sigma_s V_s^H + U_{\perp}\Sigma_n V_n^H \tag{2}$$

where $U_{\|}$ and $U_{\perp}$ constitute an orthonormal basis for the left singular vectors of X, V_s and V_n form an orthonormal basis for the right singular vectors, Σ_s is a diagonal matrix holding the K non-zero singular values of X, and Σ_n is a matrix of $(N-K)\times(N-K)$ zeros.

The SVD of the noisy signal Y_N in (1) can be re-expressed as

$$Y_N = X + \Delta X = \hat{U}_{\|}\hat{\Sigma}_s\hat{V}_s^H + \hat{U}_{\perp}\hat{\Sigma}_n\hat{V}_n^H \tag{3}$$

where the perturbation ΔX in (1) leads to a perturbation of the singular vectors $U_{\|}$, $U_{\perp}$, V_s and V_n and the singular values $\text{diag}(\Sigma_s)$ and $\text{diag}(\Sigma_n)$. In particular, a perturbation of $U_{\perp}$ leads to a perturbation of the noise subspace projection matrix:

$$\hat{P}_{U_n} = \hat{U}_{\perp}\hat{U}_{\perp}^H = P_{U_n} + \Delta P_{U_n} = U_{\perp}U_{\perp}^H + \Delta P_{U_n} \tag{4}$$

This leads to a displacement of the roots $\{r_k\}_k$ generated by

$$\overrightarrow{w'}_N(z)^H \times P_{U_n} \times \overrightarrow{w'}_N(z) = 0 \tag{5}$$

for an arbitrary coding vector $\overrightarrow{w'}_N(z) = \left[1, z^1, \ldots, z^{N-1}\right]^T$, which impacts the identification of the set of active transmitters [3].

A. Perturbation of the noise projection matrix P_{U_n}

Referring to [4], for an arbitrary matrix X of SVD as in (2), a perturbation ΔX leads to a first-order perturbation $\Delta U_{\|}$ of the form

$$\Delta U_{\|} = U_{\|}R + U_{\perp}U_{\perp}^H\Delta X V_s\Sigma_s^{-1} \tag{6}$$

where $R = D\odot(U_{\|}^H\Delta X V_s\Sigma_s + \Sigma_s V_s^H\Delta X^H U_{\|})$ and $\odot$ is the Hadamard product. D is a $K\times K$ matrix whose first diagonal elements are zero while the off-diagonal elements have the form $D(k_1,k_2) = 1/(\sigma_{k_2}^2 - \sigma_{k_1}^2), 1 \le k_1 \ne k_2 \le K$. Values $\{\sigma_k\}_k$ correspond to the K non-zero singular values of X whose rank is K.

Define the signal subspace projection matrix as $P_{U_s} = U_{\|}U_{\|}^H$. By the orthonormality of the left singular vectors of X we have

$$P_{U_s} + P_{U_n} = I \tag{7}$$

Therefore,

$$\Delta P_{U_n} = -\Delta P_{U_s} = -\Delta U_{\|}U_{\|}^H - U_{\|}\Delta U_{\|}^H \tag{8}$$

Substituting (6) in (8) and noting that $R^H = -R$ we have

$$\Delta P_{U_n} = -P_{U_n}\Delta X X^+ - {X^+}^H \Delta X^H P_{U_n} \tag{9}$$

where

$$X^+ = V_s\Sigma_s^{-1}U_{\|}^H \tag{10}$$

B. Angular displacements of the characteristic complex exponentials $\{r_k\}_k$

Denote by $d_k = 1$ and ω_k the respective magnitude and angle of characteristic complex exponential r_k of transmitter k, i.e. $r_k = d_k\exp(j\omega_k)$. A coding vector $\overrightarrow{w}_k$ of transmitter k is defined as

$$\overrightarrow{w}_k = \overrightarrow{w'}_N(r_k) = [1, \exp(j\omega_k), \ldots, \exp\left(j(N-1)w_k\right)]^T \tag{11}$$

where $\overrightarrow{w'}_N(z)$ is an arbitrary coding vector as in (5). By definition of $U_{\perp}$ in (2) we have

$$\overrightarrow{w}_k^H P_{U_n}\overrightarrow{w}_k = 0,\ 1 \le k \le K \tag{12}$$

Because of ΔX, perturbed roots $\{\hat{r}_k\}_k$ are generated by (5) as approximations for $\{r_k\}_k$. Define

$$\begin{aligned}\overrightarrow{w}_k^{(1)} &= \frac{d\overrightarrow{w'}}{dz}(r_k) \\ &= [0, \exp(j\omega_k), \ldots, (N-1)\cdot\exp\left(j(N-1)\omega_k\right)]^T\end{aligned} \tag{13}$$

Referring to [11], (5) can be approximated as a first-order perturbed version of (12) as follows:

$$\begin{aligned}&(\overrightarrow{w}_k^H - j{\overrightarrow{w}_k^{(1)}}^H\Delta\omega_k - {\overrightarrow{w}_k^{(1)}}^H\Delta r_k + \text{h.o.t}) \times (P_{U_n} + \Delta P_{U_n}) \\ &\times(\overrightarrow{w}_k + j\overrightarrow{w}_k^{(1)}\Delta\omega_k + \overrightarrow{w}_k^{(1)}\Delta r_k + \text{h.o.t}) = 0\end{aligned} \tag{14}$$

where h.o.t refers to higher order terms to be neglected in a first-order analysis. (14) implies perturbation ΔP_{U_n} of noise projection matrix U_n expectedly shifts root r_k to new position $(1+\Delta r_k)\exp(j(\omega_k + \Delta\omega_k))$ in the complex plane. The real and imaginary parts in the left-hand side of (14) should be equated to zero. Unless higher-order terms are considered, $\Delta r_k = 0$. Moreover, the identification of the active set of users depends on the angles of the detected roots. A first-order approximation of the angular shift $\Delta\omega_k$ is given by

$$\Delta\omega_k = \frac{{\overrightarrow{w}_k^{(1)}}^H P_{U_n}\Delta X X^+\overrightarrow{w}_k - \overrightarrow{w}_k^H {X^+}^H\Delta X^H P_{U_n}\overrightarrow{w}_k^{(1)}}{2j{\overrightarrow{w}_k^{(1)}}^H P_{U_n}\overrightarrow{w}_k^{(1)}} \tag{15}$$

Note that $\Delta\omega_k$ in (15) is real-valued.

C. Individual distributions of angular shifts $\{\Delta\omega_k\}_k$

Recall that ΔX has dimensions $N\times P$. We assume noise is independent for the different packet symbols, so the columns of ΔX are independent. We assume noise is also independent over the different slot durations, so the entries of each column of ΔX are independent. We finally assume each entry $\Delta X_{n,p}$ of ΔX is circularly symmetric complex normal of mean zero $E[\Delta X_{n,p}] = 0$, variance $E[\Delta X_{n,p}^H\Delta X_{n,p}] = \sigma^2$ and relation

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$E[\Delta X_{n,p}\Delta X_{n,p}] = 0$. Therefore, we may associate a complex matrix normal distribution to ΔX:

$$\Delta X \sim \mathcal{CN}(0_{N\times P}, \sigma^2 I_{N\times N}, I_{P\times P}) \tag{16}$$

The first argument in $\mathcal{CN}(\cdot,\cdot,\cdot)$ is the mean, the second argument describes the dependencies among the entries of a single column (covariance matrix), and the third argument describes dependencies among the different columns. This is equivalent to

$$\mathrm{vec}(\Delta X) \sim \mathcal{CN}(0_{NP}, \sigma^2 I_{P\times P} \otimes I_{N\times N}) \tag{17}$$

where $\mathcal{CN}(\mu, \Gamma)$ is the complex multivariate normal distribution of mean μ and covariance matrix Γ,

$$\begin{aligned}\mathrm{vec}(\Delta X) = [&\Delta X_{1,1}, \ldots, \Delta X_{N,1}, \Delta X_{1,2},\\ &\ldots, \Delta X_{N,2}, \ldots, \Delta X_{1,P}, \ldots, \Delta X_{N,P}]^T\end{aligned} \tag{18}$$

and $A \otimes B$ is the Kronecker product of two arbitrary matrices $A \in \mathcal{C}^{m\times n}$ and $B \in \mathcal{C}^{p\times q}$. Define

$$C_k^H = \frac{\overrightarrow{w}_k^{(1)H} P_{U_n}}{\overrightarrow{w}_k^{(1)H} P_{U_n} \overrightarrow{w}_k^{(1)}} \tag{19}$$

$$D_k = X^+ \overrightarrow{w}_k \tag{20}$$

Using (16), $C_k^H \Delta X D_k$ is distributed as

$$C_k^H \Delta X D_k \sim \mathcal{CN}(0, \sigma^2 C_k^H I_{N\times N} C_k, D_k^H I_{P\times P} D_k) \tag{21}$$

By a transformation as in (17), $C_k^H \Delta X D_k$ is a simple complex random variable of distribution

$$C_k^H \Delta X D_k \sim \mathcal{CN}(0, \sigma^2 D_k^H D_k C_k^H C_k) \tag{22}$$

Using the fact that $P_{U_n} P_{U_n}^H = P_{U_n}$, we have

$$C_k^H C_k = 1/\left(\overrightarrow{w}_k^{(1)H} P_{U_n} \overrightarrow{w}_k^{(1)}\right) \tag{23}$$

Moreover, using (10) and $V_s^H V_S = I$ we have

$$D_k^H D_k = \overrightarrow{w}_k^H U_\parallel \Sigma_s^{-1} \Sigma_s^{-1} U_\parallel^H \overrightarrow{w}_k \tag{24}$$

Both $C_k^H C_k$ and $D_k^H D_k$ are real numbers. $\Delta\omega_k$ in (15) can be expressed as

$$\Delta\omega_k = \frac{\left(C_k^H \Delta X D_k\right) - \left(C_k^H \Delta X D_k\right)^H}{2j} = \mathrm{Im}\left(C_k^H \Delta X D_k\right) \tag{25}$$

where $\mathrm{Im}(z)$ is the imaginary part of complex variable z. As a first-order approximation, (22) and (25) imply that $\Delta\omega_k$ is a real Gaussian scalar distributed as

$$\Delta\omega_k \sim \mathcal{N}\left(0, \frac{\sigma^2}{2}(D_k^H D_k)(C_k^H C_k)\right) \tag{26}$$

D. Joint distribution of angular shifts $\{\Delta\omega_k\}_k$

We now prove that angular shifts $\{\Delta\omega_k\}_k$ are jointly Gaussian for a first-order analysis. Denote by $D_{k,p}$ the p^{th} element of vector D_k and by ΔX_p the p^{th} column of matrix ΔX, $1 \le p \le P$. Define K arbitrary real coefficients $\{\alpha_k\}_{k=1}^K$. Using (25), the weighted sum $\sum_k \alpha_k \Delta\omega_k$ can be expressed as

$$\begin{aligned}\sum_k \alpha_k \Delta\omega_k &= \sum_k \alpha_k \,\mathrm{Im}\left(\sum_p D_{k,p} C_k^H \Delta X_p\right)\\ &= \sum_p \mathrm{Im}\left(\left(\sum_k \alpha_k D_{k,p} C_k^H\right)\Delta X_p\right)\\ &= \sum_p \mathrm{Im}\left(\beta_p^H \Delta X_p\right)\end{aligned} \tag{27}$$

The second equality in (27) follows from the linearity of the imaginary operator. While row vector β_p^H is N-dimensional, it is a weighted sum of only K vectors $D_{k,p} C_k^H$. Moreover, there are P such vectors $\{\beta_p^H\}_p$. Therefore, the problem of selecting K coefficients $\{\alpha_k\}_k$ so that all vectors $\{\beta_p^H\}_p$ are zero vectors admits NP equations. It is thus overdetermined and admits no non-trivial solutions for $\{\alpha_k\}_k$ almost surely. Since the entries of vectors $\{\Delta X_p\}_p$ are complex Gaussian, and assuming set $\{\alpha_k\}_k$ is non-trivial, $\sum_k \alpha_k \Delta\omega_k$ in (27) is a weighted sum of real Gaussians and is thus Gaussian-distributed. Therefore, angular shifts $\{\Delta\omega_k\}_k$ in (26) are jointly Gaussian.

Given that variables $\{\Delta\omega_k\}_k$ are jointly Gaussian, the joint distribution is fully characterized by the mean vector and covariance matrix. All angular shifts $\{\Delta\omega_k\}_k$ have zero mean as in (26). For the covariance matrix, we evaluate $E[\Delta\omega_k \Delta\omega_l]$ using (25):

$$\begin{aligned}4E[\Delta\omega_k \Delta\omega_l] &= E[(C_k^H \Delta X D_k)(C_l^H \Delta X D_l)^H]\\ &+ E[(C_k^H \Delta X D_k)^H (C_l^H \Delta X D_l)]\\ &- E[(C_k^H \Delta X D_k)^H (C_l^H \Delta X D_l)^H]\\ &- E[(C_k^H \Delta X D_k)(C_l^H \Delta X D_l)]\end{aligned} \tag{28}$$

$C_k^H \Delta X D_k$ is a weighted sum of the entries of ΔX. These entries are independent, have zero mean and are circularly symmetric. Thus, the last two expectations in (28) are zero. By linearity of the expectation we have

$$\begin{aligned}4E[\Delta\omega_k \Delta\omega_l] &= C_k^H E[\Delta X D_k D_l^H \Delta X^H] C_l\\ &+ D_k^H E[\Delta X^H C_k C_l^H \Delta X] D_l\end{aligned} \tag{29}$$

Note the following:

$$\begin{aligned}E[\Delta X D_k D_l^H \Delta X^H] &= E[(\sum_p D_{k,p}\Delta X_p)(\sum_p D_{l,p}^* \Delta X_p^H)]\\ &= \sum_p \sum_{p'} D_{k,p} D_{l,p'}^* E\left[\Delta X_p \Delta X_{p'}^H\right]\\ &= \sum_p D_{k,p} D_{l,p}^* \sigma^2 I_{N\times N}\\ &= \sigma^2 \left(D_l^H D_k\right) I_{N\times N}\end{aligned} \tag{30}$$

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where the second equality in (30) is implied by the linearity of expectation, while the third equality is obtained by utilizing the distribution of ΔX in (16). Similarly,

$$E[\Delta X^H C_k C_l^H \Delta X] = \sigma^2 \left(C_l^H C_k\right) I_{P\times P} \tag{31}$$

Plugging (30) and (31) in (29) we have

$$\begin{aligned} E[\Delta\omega_k \Delta\omega_l] &= \frac{\sigma^2}{4}\left[\left(D_l^H D_k\right)\left(C_k^H C_l\right) + \left(D_k^H D_l\right)\left(C_l^H C_k\right)\right] \\ &= \frac{\sigma^2}{2}\,\mathrm{Re}\left(\left(D_l^H D_k\right)\left(C_k^H C_l\right)\right) \end{aligned} \tag{32}$$

where $\mathrm{Re}(z)$ is the real part of complex variable z. For the case $k = l$, note that (32) becomes the variance of $\Delta\omega_k$ as in (26). The matrix holding $E[\Delta\omega_k\Delta\omega_l], 1 \le k, l \le K$ in (32) defines the first-order approximation of the covariance matrix of the joint distribution of $\{\Delta\omega_k\}_k$.

III. Noise averaging

We derive an upper bound on the variance of $\Delta\omega_k$ in (26). Using (24),

$$D_k^H D_k = \left|\Sigma_s^{-1} U_\parallel^H \overrightarrow{w}_k\right|_2^2 = \sum_{k'=1}^{K} \frac{1}{\sigma_{k'}^2}\left|U_{\parallel k'}^H \overrightarrow{w}_k\right|_2^2 \tag{33}$$

where $|v|_2$ is the $\mathcal{L}_2$ norm of vector (or scalar) v, set $\{\sigma_k\}_k$ is the set of K non-zero singular values of X along the diagonals of Σ_s, and $U_{\parallel k}$ is the k^{th} column of $U_\parallel$. By Cauchy-Schwarz inequality, we obtain

$$D_k^H D_k \le \sum_{k'=1}^{K} \frac{1}{\sigma_{k'}^2}\left|U_{\parallel k'}\right|_2^2 \left|\overrightarrow{w}_k\right|_2^2 = N\left(\sum_{k'=1}^{K}\frac{1}{\sigma_{k'}^2}\right) \tag{34}$$

where the columns of $U_\parallel$ have unit norm, and $|\overrightarrow{w}_k|_2^2 = N$ using (11).

Moreover, (23) implies

$$(C_k^H C_k)^{-1} = \left|U_\perp^H \overrightarrow{w}_k^{(1)}\right|_2^2 = \left|U_\perp^H \overrightarrow{u}_{\overrightarrow{w}_k^{(1)}}\right|_2^2 \left|\overrightarrow{w}_k^{(1)}\right|_2^2 \tag{35}$$

where $\overrightarrow{u}_{\overrightarrow{w}_k^{(1)}}$ is a unit vector in the direction of $\overrightarrow{w}_k^{(1)}$. Using (13),

$$\begin{aligned} \left|\overrightarrow{w}_k^{(1)}\right|_2^2 &= 1 + 2^2 + \cdots + (N-1)^2 \\ &= \frac{(N-1)^3}{3} + \frac{(N-1)^2}{2} + \frac{(N-1)}{6} \end{aligned} \tag{36}$$

(35) becomes

$$(C_k^H C_k)^{-1} \ge \left|U_\perp^H \overrightarrow{u}_{\overrightarrow{w}_k^{(1)}}\right|_2^2 \frac{(N-1)^3}{3} = p_{N,k}\frac{(N-1)^3}{3} \tag{37}$$

We now prove by contradiction that $p_{N,k}$ is strictly positive. Assume $p_{N,k} = 0$. By definition of $p_{N,k}$ and since spaces $U_\perp$ and $U_\parallel$ are orthogonal, $\overrightarrow{w}_k^{(1)}$ has to be spanned by the columns of $U_\parallel$. Let vector $\overrightarrow{v}$ hold the first K entries of $\overrightarrow{w}_k^{(1)}$ and matrix M hold the first K rows of $U_\parallel$. M is a $K \times K$ Vandermonde matrix and is full rank. We therefore have

$$U_\parallel \times \left(M^{-1}\overrightarrow{v}\right) = \overrightarrow{w}_k^{(1)} \tag{38}$$

Denote by $\overrightarrow{u}_{\parallel,n}^H$ the n^{th} row of $U_\parallel$ and $\overrightarrow{w}_k^{(1)}(N)$ the N^{th} entry of $\overrightarrow{w}_k^{(1)}$. From (38),

$$\left|\overrightarrow{w}_k^{(1)}(N)\right|_2 = \left|\overrightarrow{u}_{\parallel,N}^H \times \left(M^{-1}\overrightarrow{v}\right)\right|_2 \le K \times \left|M^{-1}\overrightarrow{v}\right|_2 \tag{39}$$

The inequality in (39) follows from Cauchy-Schwarz relation and the fact that all entries of $U_\parallel$ have norm less than unity since the columns of $U_\parallel$ are orthonormal. From (13), the left hand side of (39) is unbounded as N grows, while the right hand side of (39) is independent of N. This is a contradiction. Therefore, $p_{N,k}$ in (37) is strictly positive. For a fixed number of transmitters K we lower-bound $p_{N,k}$ by a positive constant. Combining (26), (34) and (37), we obtain an upper limit on the variance of angular shift $\Delta\omega_k$ that drops for higher signal powers $\{\sigma_{k'}\}_{k'}$ relative to the noise power σ^2. It also decays quadratically in the number of observed packets N.

IV. Numerical Experiments

In this section we verify the theoretical results for the individual and joint distributions of the angular perturbations of the roots, which are computed for the purpose of identifying the set of active transmitters as discussed in [3]. We also verify the noise-averaging effect of section III. Consider two simultaneously active transmitters 1 and 2 of respective characteristic complex exponentials r_1 and r_2 where $\omega_1 = \angle r_1 = \pi/4$ and $\omega_2 = \angle r_2 = 3\pi/4$. Each transmitter sends a packet consisting of a real random sequence of ± 1s and of length $P = 1000$. Because of collision, retransmissions are necessary. The receiver stacks the N observed packets in matrix Y_N and computes estimates of angles ω_1 and ω_2 as described in [3]. The SNR is defined as $\mathrm{SNR} = 10\log_{10}(1/\sigma^2)$, where σ^2 is the noise power as in (16). Each numerical experiment is repeated 1000 times and mean-statistics are computed.

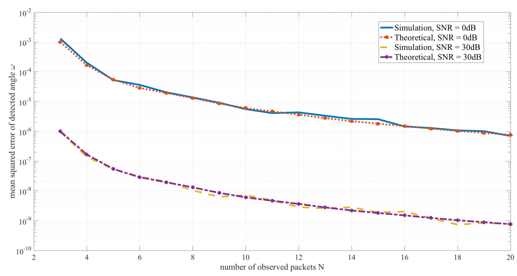

Fig. 1: Theoretical and numerical results of the mean squared error $E[\Delta\omega_1^2]$ versus the number of received packets N for two SNR conditions: SNR = 0dB and SNR = 30dB. $K = 2$, $\omega_1 = \pi/4$, $\omega_2 = 3\pi/4$.

In figure 1 we plot the MSE for estimating ω_1 versus N for two SNR conditions. As expected, the MSE decays for larger values of N (noise-averaging effect) and higher SNR. Moreover, we compute the theoretical curves using (26). The matching between the theoretical and simulation curves indicates that the first-order perturbation analysis is accurate for predicting the statistics of $\Delta\omega_k$.

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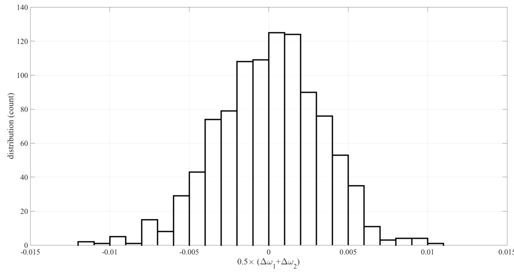

Fig. 2: Distribution of $(\Delta\omega_1 + \Delta\omega_2)/2$ for $K = 2$, $N = 7$, SNR = 0dB, $\omega_1 = \pi/4$, $\omega_2 = 3\pi/4$.

Figure 2 shows a histogram of the average of the two angular shifts $\Delta\omega_1$ and $\Delta\omega_2$. The plot fairly has a Gaussian bell shape. This is expected for a weighted sum of jointly Gaussian random variables as we derived in section II-D. Moreover, for the plot in figure 2, the simulated MSE is 1.0679e-05. By computing the covariance matrix of $\Delta\omega_1$ and $\Delta\omega_2$ using (32), the theoretical MSE is given by

$$E\left[\left(\frac{\Delta\omega_1 + \Delta\omega_2}{2}\right)^2\right] = \begin{pmatrix}0.5 & 0.5\end{pmatrix} \times \text{Cov}\begin{pmatrix}\Delta\omega_1 \\ \Delta\omega_2\end{pmatrix} \times \begin{pmatrix}0.5 \\ 0.5\end{pmatrix}$$
$$= 1.1871\text{e-}05 \quad (40)$$

which is of the same order as the simulated MSE.

Finally, figure 3 shows reduced symbol error rate (SER) upon decoding the collided packets for a larger number of stacked packets N and higher SNR values.

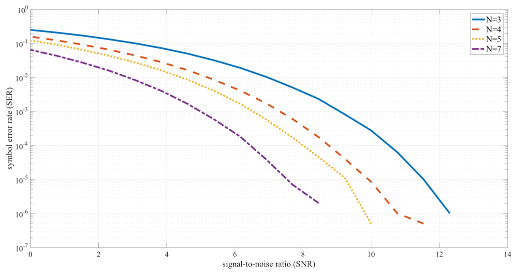

Fig. 3: Symbol error rate versus SNR for $K = 2$, $\omega_1 = \pi/4$, $\omega_2 = 3\pi/4$, and $N \in \{3, 4, 5, 7\}$.

V. Conclusion

Distributions for the perturbations of the roots computed to identify the active user set are derived and verified in simulations. The algorithm of [3] becomes more robust for higher SNR values and upon noise-averaging. A future research direction is to integrate the derived distributions into an optimal method for blind user identification. Another research direction is to do throughput analysis due to the additional retransmissions incurred by noise-averaging.

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