Estimation of Bandlimited Signals on Graphs From Single Bit Recordings of Noisy Samples

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***Abstract*—Recently, there has been interest in graph signal processing. We consider the problem of a bandlimited graph signal estimation (denoising) from single-bit samples obtained at each graph node. The samples before quantization are affected by zero-mean additive white Gaussian noise of known variance. Using Banach's contraction mapping theorem on complete metric spaces, we develop a recursive algorithm for bandlimited graph signal estimation. For our recursive algorithm, we show that the expected mean-squared error between the graph signal and its estimate is proportional to the bandwidth of the signal and inversely proportional to the size of the graph. We also consider the problem of choosing the nodes to sample based on the properties of graph Laplacian eigenvectors to minimize the mean-squared error of the estimate. Numerical tests with synthetic signals demonstrate the effectiveness of our estimation algorithm for Erdös-Rényi (ER) graphs, Barabäsi-Albert (BA) graphs, and Minnesota road-network graph.**

***Index Terms*—graph signal processing, quantization, estimation, sampling**

I. Introduction

Graph signal processing (GSP) studies signals defined on the nodes of a graph [1], [2]. The pairwise relationship between graph signal values at various nodes is captured by the edges of the graph, and the strength of the relationship is reflected in the edge weight. The growth of GSP has been motivated by the need to represent and analyze large data sets arising in various applications like social networks [3], transport networks [4], sensor networks [5], brain signals [6], power networks [7], and image processing [8]. Concepts of signal processing such as the Fourier transform [2] are extended to GSP via spectral graph theory [2], [9], [10].

It is observed that a large number of real world graph signals are approximately bandlimited [2]. The problem of sampling and estimation of a bandlimited graph signal has been recently studied [11], [12]. We consider the problem of a bandlimited graph signal estimation (denoising) based on its *single-bit* samples taken at each graph node. The samples before quantization are affected by zero-mean additive white Gaussian noise of known variance. In classical signal processing, the problem of continuous-time signal reconstruction from coarsely quantized samples is well known [13]–[18]. While working with single-bit signal samples, Masry obtained a mean square error of $O(1/K^{2/3})$, where K is the sampling rate. In Masry's method, the smoothness and not the bandlimitedness of the signal being sampled was used [13]. By exploiting the bandlimitedness of signal being sampled, Kumar and Prabhakaran obtained a maximum mean-squared error of $O(1/K)$ for bandlimited signals [19].

Main contributions: (i) Using Banach's contraction mapping theorem, we develop an iterative algorithm to estimate a bandlimited graph signal from its single-bit quantized noise-affected samples [20]. Let B be the bandwidth of a graph signal and N be the size of the graph. An expected mean-squared error of $O(B/N)$ is shown in this work, which is the counterpart of the result by Kumar and Prabhakaran [19]. (ii) A bandlimited signal can also be obtained by subsampling on the nodes. Based on the the lower order eigenvectors of the Laplacian matrix [21], we will present a sampling set selection strategy which is better than random sampling. Numerical simulations with synthetic signals verify the analytical results for Erdös-Rényi (ER) graphs, Barabási-Albert (BA) graphs, and Minnesota road-network graph [22].

Outline: In Sec. II, we have given an overview of graph signal processing pertinent to the sampling problem. Our problem setup is described in Sec. III. In Sec. IV, the proposed estimation algorithm is presented with abridged proofs. In Sec. V, the sampling set selection scheme is given. In Sec. VI, we have verified the results via simulations. Finally, we conclude in Sec. VII.

II. Review of Graph Signal Processing

Consider an unweighted graph $\mathcal{G}$ of N nodes with node set $\mathcal{N} = \{1, 2, \ldots, N\}$ and a set of undirected edges $\mathcal{E} = \{(i, j) : i \text{ is connected to } j\}$. The real edge weights form the symmetric adjacency matrix. More generally, define the graph shift operator (GSO) $\mathbf{S}$ as an $N \times N$ matrix having the same sparsity pattern as the graph $\mathcal{G}$. Defining a suitable and robust GSO is a crucial problem in GSP [23], [24]. In this work, the graph Laplacian is used as the GSO [1]. The graph Laplacian is symmetric for an undirected graph and thus has a set of real orthonormal eigenvectors. It can be decomposed as $\mathbf{S} = \mathbf{U}\mathbf{\Lambda}\mathbf{U}^{\mathrm{T}}$. Columns of the $\mathbf{U}$ matrix are the eigenvectors of $\mathbf{S}$. Let the eigenvector corresponding to the l^{th} smallest eigenvalue of $\mathbf{S}$ be denoted by $\vec{u}_l$. A *graph signal* is defined as a function from the set of nodes to the set of N dimensional reals, i.e., $\vec{g} : \mathcal{N} \to \mathbb{R}^N$ [1]. The graph signal can also be viewed as a vector in $\mathbb{R}^N$. The i^{th} entry of $\vec{g}$ is the signal on the i^{th} node of the graph.

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The *graph Fourier transform* of a graph signal $\vec{g}$ is defined as its expansion in terms of the eigenvectors of $\mathbf{S}$ [2]

$$\hat{\vec{g}}(l) = \sum_{i=1}^{N} \vec{g}(i)\vec{u}_l(i), \quad l \in \{1,2,..,N\}.$$

The inverse graph Fourier transform is then given by

$$\vec{g}(i) = \sum_{l=1}^{N} \hat{\vec{g}}(l)\vec{u}_l(i), \quad i \in \{1,2,..,N\}.$$

For $B \in \{1,2,..,N\}$, a graph signal is said to be *bandlimited* with bandwidth B [11], if its graph Fourier transform satisfies

$$\hat{\vec{g}}(l) = 0, \text{ for all } l > B.$$

III. Signal and Sampling Model

Random variables will be denoted by uppercase letters (e.g., $\vec{X}$) and its realization will be denoted by the corresponding lowercase letters (e.g., $\vec{x}$). We consider bounded and bandlimited signals on a general graph. The signal $\vec{g}(i)$ on node i lies in $[-1,1]$ and is corrupted with additive white Gaussian noise (AWGN) $\vec{W}$ of known variance σ^2. Additional AWGN $\vec{W}_d$ of variance σ_d^2 is added to dither the signal for reconstruction. Dithering and quantization is a classical topic [25]. In our sampling model illustrated in Fig. 1, single-bit samples of the signal $\vec{g}$ on each node of the graph is recorded, i.e.,

$$\vec{X}(i) = \mathbb{1}\Big(\vec{g}(i) + \vec{W}(i) + \vec{W}_d(i) > 0\Big) - \frac{1}{2}.$$

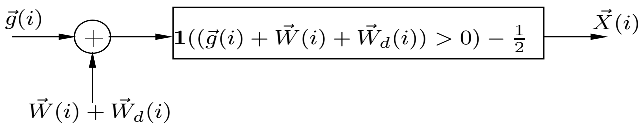

Fig. 1. The graph signal sampling scheme is illustrated.

The error metric is taken as the expected mean-square distortion between the original signal $\vec{g}$ and its estimate $\vec{G}_{\text{1-bit}}$.

$$\text{MSE} = \mathbb{E}\Big[\|\vec{g} - \vec{G}_{\text{1-bit}}\|_2^2\Big].$$

IV. Our algorithm for signal estimation

Using a linear transformation, we convert the observed single-bit samples $\vec{x}$ to a bandlimited estimate $\vec{y}$ of the 'companded' version of the signal $\vec{g}$. This bandlimited estimate $\vec{y}$ is then inverted to estimate the original $\vec{g}$. The inversion process is non-trivial, and is the core problem tackled by our algorithm. The companding function is the cumulative distribution function (CDF) of the noise. Dithering by $\vec{W}_d$ is necessary for the estimation of the signal $\vec{g}$, while ensuring stability against noise addition (see Sec. IV).

Let $\vec{l} := \big(F(\vec{g}) - \frac{1}{2}\big)$. Then by the definition of $\vec{X}$

$$\mathbb{E}[\vec{X}] = \vec{l}.$$

Consider the matrix $\mathbf{P}$ that projects a vector to the subspace spanned by the B eigenvectors of $\mathbf{S}$ corresponding to the B smallest eigenvalues. Let $\mathbf{U_B} = [\vec{u}_1, \vec{u}_2, ..., \vec{u}_B]$. Then $\mathbf{P}$ is given by $\mathbf{U_B}\mathbf{U_B}^{\mathrm{T}}$. The projection matrix $\mathbf{P}$ is the ideal low pass graph filter with bandwidth B [2]. Let $\vec{y} = \mathbf{P}\vec{x}$. Then

$$\mathbb{E}[\vec{Y}] = \mathbf{P}\mathbb{E}[\vec{X}] = \mathbf{P}\vec{l}$$

Proposition 1. *The average variance of elements in $\vec{Y}$ is $O(B/N)$.*

Proof. By definition $\vec{x}(i) \in \{-\frac{1}{2}, \frac{1}{2}\}$. For any $1 \le i \le N$, $\text{var}(\vec{X}(i)) \le \frac{1}{4}$. From $\vec{y} = \mathbf{P}\vec{x}$, we have,

$$\vec{y}(n) = \sum_{j=1}^{N}\Big(\sum_{i=1}^{B} \vec{u}_i(n)\vec{u}_i(j)\Big)\vec{x}(j).$$

Since $\vec{W}$ and $\vec{W}_d$ are (respectively) i.i.d., the elements of $\vec{X}$ are independent. So,

$$\begin{aligned}
\text{var}(\vec{Y}(n)) &\le \frac{1}{4}\sum_{j=1}^{N}\Big(\sum_{i=1}^{B} \vec{u}_i(n)\vec{u}_i(j)\Big)^2 \\
&= \frac{1}{4}\sum_{j=1}^{N}\sum_{i=1}^{B}\sum_{k=1}^{B} \vec{u}_i(n)\vec{u}_i(j)\vec{u}_k(n)\vec{u}_k(j) \\
&= \frac{1}{4}\sum_{i=1}^{B}\sum_{k=1}^{B} \vec{u}_i(n)\vec{u}_k(n)\delta[i-k] \\
&\le \frac{1}{4}\sum_{i=1}^{B} \vec{u}_i^2(n).
\end{aligned}$$

Averaging over n, we get

$$\begin{aligned}
\frac{1}{N}\sum_{n=1}^{N}\text{var}(\vec{Y}(n)) &\le \frac{1}{4N}\sum_{n=1}^{N}\sum_{i=1}^{B} \vec{u}_i(n)^2 \\
&= \frac{1}{4N}\sum_{n=1}^{B}\sum_{i=1}^{N} \vec{u}_i(n)^2 = \frac{B}{4N},
\end{aligned}$$

as the vectors $\vec{u}_i$ are unit norm. □

From the unbiased estimate $\vec{Y}$ of $\mathbf{P}\vec{l} = \mathbf{P}\big(F(\vec{g}) - \frac{1}{2}\big)$, with an average variance of $O(B/N)$, an estimate for $\vec{g}$ will be obtained. The underlying technique uses the Banach's fixed point theorem along with contraction mapping [20]. This approach is inspired from the work of Kumar and Prabhakaran [19].

Consider the set of bounded graph signals

$$S_B = \{\vec{m} : \|\vec{m}\|_\infty \le 1\}.$$

Now we define a contraction mapping $T : S_B \to S_B$ which will be used to obtain an estimate of $\vec{g}$ from $\vec{y}$. The following function is used in the contraction map to restrict the dynamic range of the argument to $[-1,1]$ as $\|\vec{g}\|_\infty \le 1$ is assumed:

$$\text{clip}[z] := \begin{cases} z, & \text{if } \mid z \mid \le 1 \\ \text{sgn}(z), & \text{otherwise} \end{cases}$$

Definition 1. *Define the map T as:*

$$T(\vec{m}) := \mathit{clip}\left\{\gamma\mathbf{P}\vec{l} + \mathbf{P}\left(\vec{m} - \gamma\left(F(\vec{m}) - \frac{1}{2}\right)\right)\right\}.$$

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Observe that $\vec{g}$ is a solution (fixed point) of the equation $T(\vec{m}) = \vec{m}$. With $\vec{g}_0 = \vec{0}$ the following recursion is defined:

$$\vec{g}_{k+1} = T(\vec{g}_k) := \text{clip}\{\gamma \mathbf{P}\vec{l} + \mathbf{P}\big(\vec{g}_k - \gamma\big(F(\vec{g}_k) - \frac{1}{2}\big)\big)\}$$

Proposition 2. *The mapping T is a contraction on the set S_B with l^2 distance as the metric.*

Proof. Note that T is closed on S_B as the clip function ensures that the l^∞ norm of the signal $\vec{g}_k$ is restricted to $[0, 1]$. Define $\vec{r}_k = \gamma \mathbf{P}\vec{l} + \mathbf{P}\Big[\vec{g}_k - \gamma\big(F(\vec{g}_k) - \frac{1}{2}\big)\Big]$. Then,

$$\|\vec{r}_1 - \vec{r}_2\|_2 = \|\mathbf{P}[\vec{g}_1 - \vec{g}_2 - \gamma\big(F(\vec{g}_1) - F(\vec{g}_2)\big)]\|_2. \qquad (1)$$

Since $\|\mathbf{P}\|_2 = 1$, so

$$\|\vec{r}_1 - \vec{r}_2\|_2 \le \left\|(\vec{g}_1 - \vec{g}_2)[1 - \gamma\Big(\frac{F(\vec{g}_1) - F(\vec{g}_2)}{(\vec{g}_1 - \vec{g}_2)}\Big)]\right\|_2$$
$$\text{or } \|\vec{r}_1 - \vec{r}_2\|_2 \le \|1 - \gamma f\|_\infty \|(\vec{g}_1 - \vec{g}_2)\|_2$$

where $f(x)$ is the derivative of $F(x)$, i.e., $f(x)$ is the probability density of the noise. The clip function reduces maximum distance [19], so

$$\|\text{clip}(\vec{r}_1) - \text{clip}(\vec{r}_2)\|_2 \le \|\vec{r}_1 - \vec{r}_2\|_2.$$

Next, note that

$$\begin{aligned}\|T(\vec{r}_1) - T(\vec{r}_2)\|_2 &= \|\text{clip}(\vec{r}_1) - \text{clip}(\vec{r}_2)\|_2 \\ &\le \|1 - \gamma f\|_\infty \|(\vec{g}_1 - \vec{g}_2)\|_2\end{aligned}$$

Define $\alpha = \|1 - \gamma f\|_\infty$, where $f(0) \ge f(x) \ge f(\pm 1)$. For T to be a contraction, we require $0 < \alpha < 1$. This is ensured by restricting γ to $(0, \frac{2}{f(1)})$. It is also required that $\frac{2f(0)}{f(1)} < 1$, which is ensured by a sufficiently large variance in $\vec{W}_d$. □

It can be seen by substitution that $\vec{g}$ is a fixed point of T. By the uniqueness of fixed point in a contraction mapping in a complete metric space, $\vec{g}$ is the only fixed point.

In the above analysis, we took perfect samples $\mathbf{P}\vec{l}$. Next, estimation error in $\vec{g}$ will be upper bounded, if an estimate of $\mathbf{P}\vec{l}$, i.e., $\vec{Y}$ is used. Consider the recursion:

$$\vec{G}_{k+1} = \text{clip}\Big\{\gamma \vec{Y} + \mathbf{P}\big(\vec{G}_k - \gamma\big(F(\vec{G}_k) - \frac{1}{2}\big)\big)\Big\}$$

Let the limit of this recursive mapping be $\vec{G}_{\text{1-bit}}$. We will now derive a bound on $\frac{1}{N}\mathbb{E}[\|\vec{G}_{\text{1-bit}} - \vec{g}\|_2^2]$. Consider two recursions, one using $\vec{Y}$ having $\vec{G}_{\text{1-bit}}$ as its limit, and the other using $\mathbf{P}\vec{l}$ having $\vec{g}$ as its limit:

$$\vec{R}_k = \gamma \vec{Y} + \mathbf{P}[\vec{G}_{k-1} - \gamma\big(F(\vec{G}_{k-1}) - \frac{1}{2}\big)]$$
$$\vec{r}_k = \gamma \mathbf{P}\vec{l} + \mathbf{P}[\vec{g}_{k-1} - \gamma\big(F(\vec{g}_{k-1}) - \frac{1}{2}\big)].$$

Being interested in the distortion, we consider

$$\vec{R}_k - \vec{r}_k = \gamma(\vec{Y} - \mathbf{P}\vec{l}) + \mathbf{P}\left[(\vec{G}_{k-1} - \vec{g}_{k-1})\left(1 - \gamma\left(\frac{F(\vec{G}_{k-1}) - F(\vec{g}_{k-1})}{(\vec{G}_{k-1} - \vec{g}_{k-1})}\right)\right)\right]$$

Using the *Minkowski inequality* for the l_2 norm [20],

$$\begin{aligned}\|\vec{R}_k - \vec{r}_k\|_2 &\le \gamma\|\vec{Y} - \mathbf{P}\vec{l}\|_2 + \alpha\|\mathbf{P}\|_2\|\vec{G}_{k-1} - \vec{g}_{k-1}\|_2 \\ &\le \gamma\|\vec{Y} - \mathbf{P}\vec{l}\|_2 + \alpha\|\vec{G}_{k-1} - \vec{g}_{k-1}\|_2\end{aligned}$$

Squaring both sides and using *Cauchy Schwartz inequality*

$$\|\vec{R}_k - \vec{r}_k\|_2^2 \le 2\gamma^2\|\vec{Y} - \mathbf{P}\vec{l}\|_2^2 + 2\alpha^2\|\vec{G}_{k-1} - \vec{g}_{k-1}\|_2^2.$$

Now consider the following mean square difference,

$$\begin{aligned}&\|T(\vec{G}_{k-1}) - T(\vec{g}_{k-1})\|_2^2 \\ &= \|[\text{clip}(\vec{R}_k)] - [\text{clip}(\vec{r}_k)]\|_2^2 \\ &\le \|\vec{R}_k - \vec{r}_k\|_2^2 \\ &\le 2\gamma^2\|\vec{Y} - \mathbf{P}\vec{l}\|_2^2 + 2\alpha^2\|\vec{G}_{k-1} - \vec{g}_{k-1}\|_2^2\end{aligned}$$

Taking expectation on both sides in the above equation,

$$\mathbb{E}[\|T(\vec{G}_{k-1}) - T(\vec{g}_{k-1})\|_2^2] \le 2\gamma^2\mathbb{E}[\|\vec{Y} - \mathbf{P}\vec{l}\|_2^2] + 2\alpha^2\mathbb{E}[\|\vec{G}_{k-1} - \vec{g}_{k-1}\|_2^2]$$

Since $T(\vec{G}_{k-1}) - T(\vec{g}_{k-1}) = \vec{G}_k - \vec{g}_k$, so

$$\mathbb{E}[\|\tilde{\vec{g}}_k - \vec{g}_k\|_2^2] \le 2\gamma^2\mathbb{E}[\|\vec{Y} - \mathbf{P}\vec{l}\|_2^2] + 2\alpha^2\mathbb{E}[\|\vec{G}_{k-1} - \vec{g}_{k-1}\|_2^2]$$

For the recursion to converge to a fixed point, we require that

$$\alpha^2 < \frac{1}{2} \qquad (2)$$

and then

$$\lim_{k\to\infty}\mathbb{E}[\|\vec{G}_k - \vec{g}_k\|_2^2] \le \frac{2\gamma^2\mathbb{E}[\|\vec{Y} - \mathbf{P}\vec{l}\|_2^2]}{(1 - 2\alpha^2)}$$
$$\text{or } \mathbb{E}[\|\vec{G}_{\text{1-bit}} - \vec{g}\|_2^2] \le \frac{2\gamma^2\mathbb{E}[\|\vec{Y} - \mathbf{P}\vec{l}\|_2^2]}{(1 - 2\alpha^2)}.$$

Averaging over all nodes and using the result from Proposition 1, we get,

$$\frac{1}{N}\sum_{n=1}^{N}\mathbb{E}[\|\vec{G}_{\text{1-bit}} - \vec{g}\|_2^2] \le \frac{2\gamma^2}{(1 - 2\alpha^2)}\frac{B}{4N}.$$

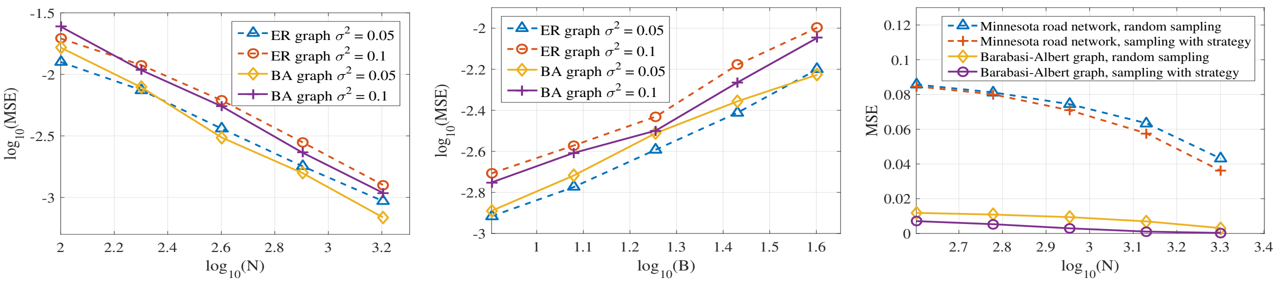
The choice of parameter γ is done such that the error is minimum while adhering to the condition in (2). As every component of the signal $\vec{g}_k$ is restricted to $[-1, 1]$, (2) translates to the following condition on the parameter γ:

$$\frac{-1}{\sqrt{2}} < \alpha = \|1 - \gamma f\|_\infty < \frac{1}{\sqrt{2}}, \text{ or}$$
$$\Big(1 - \frac{1}{\sqrt{2}}\Big)\frac{1}{f(a)} < \gamma < \Big(1 + \frac{1}{\sqrt{2}}\Big)\frac{1}{f(a)}, \forall a \in [-1, 1].$$

In this interval, $f(a)$ attains maxima at 0 and minima at ± 1,

$$\Big(1 - \frac{1}{\sqrt{2}}\Big)\frac{1}{f(1)} < \gamma < \Big(1 + \frac{1}{\sqrt{2}}\Big)\frac{1}{f(0)}.$$

For larger noise variance, the noise PDF $f(x)$ has a smaller dynamic range, for x in $[-1, 1]$. Such a γ is guaranteed to exist if sufficient amount of dithering is done. This justifies the role of dithering. The constant γ is set according to the noise variance and does not depend on N. Thus, we have established that the mean square error of estimation depends as $O(B/N)$. Numerical tests in Sec. VI demonstrate this result.



(a) (b) (c)

Fig. 2. (a) $\log_{10}$ of mean square error v/s $\log_{10}$ of number of nodes for two graphs: BA and ER and different noise variances. (b) $\log_{10}$ of mean square error v/s $\log_{10}$ of bandwidth of signal for two graphs: BA and ER and different noise variances. (c) Mean square error v/s log of number of sampled nodes for two graphs: BA and Minnesota road network; comparison between random sampling and sampling with the proposed strategy.

V. Subsampler Design

Consider the scenario where we can sample only a part of the node set and have to estimate the signal on all nodes. This problem has been studied extensively, e.g., in [11], [12]. The usual setting has simplifying assumptions such as bandlimited or piece-wise constant or globally smooth signals [11]. Here we obtain single bit recordings. The sampled signal $\vec{x}$ has no such property; it is a non-linear function of noise affected underlying signal $\vec{g}$. We propose a sampling strategy aimed at minimizing the mean square error in $\vec{y}$ as an estimate of $\mathbf{P}\vec{l}$. Recall that $\vec{y} = \mathbf{P}\vec{x}$. Thus, we are interested only in the component of $\vec{x}$ that lies in the subspace spanned by the eigenvectors of $\mathbf{U}$ having the B smallest eigenvalues, i.e., the low-pass component of $\vec{x}$ with bandwidth B.

It is observed in [21], [11] that for large complex graphs, the lower order eigenvectors are highly localized. The authors of [21] have used the *inverse participation ratio (IPR)* to quantify the localization and have studied it on various datasets. The *energy concentration ratio (ECR)* is defined in [11] as the smallest fraction of nodes that account for 95% of the signal's energy. *ECR*, which lies in $(0, 1]$, is small if the signal is energy-concentrated. It is reported in [11] that for bandlimited signals on complex networks, the *ECR* is usually small.

To minimize the mean square error in $\vec{y}$, we sample the nodes that account for most of its energy. The graph Fourier transform of $\vec{y}$, i.e., $\hat{\vec{y}}$ can be given by the following:

$$\hat{\vec{y}}(b) := \begin{cases} \sum_{j=1}^{N} \vec{u}_b(j)\vec{x}(j), & \text{if } b \leq B \\ 0, & \text{otherwise} \end{cases}$$

Definition 2. *We define the strength of a node (NS) for the case of single bit recordings as a function of bandwidth B as:*

$$NS(j, B) = \sum_{b=1}^{B} \vec{u}_b^2(j)$$

Let $\phi = \{i : \text{ node i is sampled}\}$ be the set of sampled nodes. Since $\vec{x}(j) \in \{-\frac{1}{2}, \frac{1}{2}\}$, we assume the elements of $\vec{x}$ corresponding to nodes that are not sampled as the unbiased value i.e., zero. Thus, we get the following sampled signal:

$$\vec{x}_s(j) := \begin{cases} \vec{x}(j) & j \in \phi \\ 0, & j \notin \phi \end{cases}$$

Define $\vec{y}_s = \mathbf{P}\vec{x}_s$. Its graph Fourier transform is:

$$\hat{\vec{y}}_s(b) := \begin{cases} \sum_{\{j:j\in\phi\}} \vec{u}_b(j)\vec{x}(j), & \text{if } b \leq B \\ 0, & \text{otherwise} \end{cases}$$

As the graph Fourier transform preserves energy, the mean square error due to subsampling can be given as:

$$\begin{aligned} \frac{1}{N}\|(\vec{y} - \vec{y}_s)\|_2^2 &= \frac{1}{N}\|(\hat{\vec{y}} - \hat{\vec{y}}_s)\|_2^2 \\ &= \frac{1}{N}\sum_{b=1}^{B}\Big(\sum_{j:j\notin\phi} \vec{u}_b(j)\vec{x}(j)\Big)^2 \end{aligned}$$

Since $\vec{x}(j) \in \{-\frac{1}{2}, \frac{1}{2}\}$, using the *Cauchy-Schwartz inequality*

$$\begin{aligned} \frac{1}{N}\|(\vec{y} - \vec{y}_s)\|_2^2 &\leq \Big(\frac{N - |\phi|}{4N}\Big)\sum_{b=1}^{B}\sum_{\{j:j\notin\phi\}} \vec{u}_b^2(j) \\ &= \Big(\frac{N - |\phi|}{4N}\Big)\sum_{\{j:j\notin\phi\}}\sum_{b=1}^{B} \vec{u}_b^2(j) \\ &= \Big(\frac{N - |\phi|}{4N}\Big)\sum_{\{j:j\notin\phi\}} NS(j, B) \end{aligned}$$

To minimize the error, we choose the node subset ϕ such that $\big(\sum_{j:j\notin\phi} NS(j, B)\big)$ is minimum. In other words, we sample the nodes with the highest node strength for bandwidth B.

VI. Numerical Tests

Experiment 1. In this test, we verify the $O(B/N)$ dependence of the mean square error. We consider two types of graphs: Erdös-Rényi (ER) and Barabási-Albert (BA) graphs. For ER graphs, the edge-presence probability p is taken as 0.1. In the BA graphs, a new node attaches to 4 existing nodes. The network starts to evolve from a 10-node clique. *Part (a)* describes dependence of MSE on N for fixed B and *part (b)* describes its dependence on B for fixed N. In *part (a)*, synthetic signals of bandwidth $B = 10$ are considered. Five graphs of sizes $N = 100, 200, 400, 800, 1600$ are generated from both ER and BA models. In *part (b)*, the size of the graph is taken as $N = 1000$ and 5 signals of bandwidth $B = 8, 12, 18, 27, 40$ are considered. The signals are generated such that the non-zero graph frequency components are i.i.d. zero mean Gaussian random variables. Further, the signals are scaled to a have a dynamic range within $[-1, 1]$ as required by

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the signal model. Noise is AWGN of variance $\sigma^2 = 0.05, 0.1$ in two different cases in both parts of the experiment. The reported results correspond to average over 100 instances.

In Fig. 2.(a), the variation of log_{10}(MSE) with log_{10}(N) is plotted. The slopes of the best-fit lines of the four curves are: ER graph ($\sigma^2 = 0.05$) : -0.96; ER graph ($\sigma^2 = 0.1$) : -1.00, BA graph ($\sigma^2 = 0.05$) : -1.15; BA graph ($\sigma^2 = 0.1$) : -1.12.

In Fig. 2.(b), the variation of log_{10}(MSE) with log_{10}(B) is plotted. The slopes of the best-fit lines of the four curves are: ER graph ($\sigma^2 = 0.05$) : 1.02; ER graph ($\sigma^2 = 0.1$) : 1.04, BA graph ($\sigma^2 = 0.05$) : 0.97; BA graph ($\sigma^2 = 0.1$) : 1.00. This confirms the results of section IV.

Experiment 2. This test illustrates the merit of the subsampler design proposed in section V. We consider the Minnesota road network ($N = 2645$) [22] and a BA graph ($N = 2645$). A synthetic signal of bandwidth $B = 20$ and dynamic range within $[-1, 1]$ is generated similar to the previous experiment. For both graphs, we consider 5 cases in which a sampling set of sizes, $|\phi| = 400, 600, 900, 1350, 2000$ is taken. Noise is AWGN of variance $\sigma^2 = 0.01$. Mean square error corresponding to 2 schemes is reported: (a) random sampling, (b) sampling with given strategy. The results reported in Fig. 2.(c) correspond to the error averaged over 100 instances. The proposed sampling strategy works better than random sampling for all 5 values of $|\phi|$ considered for both graphs. It is interesting to note that the error due to subsampling is smaller in the BA graph as compared to the road map graph due to the presence of *hubs* [26] and thus, greater localization of the lower order eigenvectors. Thus, it is sufficient to sample only a few *prominent* nodes with high node strength, $NS(j, B)$.

VII. Conclusions

In this paper, we have given an algorithm to estimate a bandlimited graph signal using single bit recordings from graph nodes. The signal $\vec{x}$ consisting of the single bit recordings is converted to an estimate $\vec{y}$ of a low-pass filtered version of a companded form of the signal to be estimated, $\vec{g}$. The estimate $\vec{y}$ is inverted, using a contraction mapping T, to get an estimate $\vec{g}_{\text{1-bit}}$ of $\vec{g}$. The expected value of the mean square error is shown to be of $O(B/N)$. The result is validated by simulations on synthetic signals. We have also considered the case of subsampling. Leveraging on the observation that the lower order eigenvectors of the Laplacian are highly localized, a notion of strength of a node is defined. A subsampler design is given to minimize the mean square error in the estimate.

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