Blind Spectrum Sensing Based on Recurrence Quantification Analysis in the Context of Cognitive Radio

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***Abstract*—In Cognitive Radio, spectrum sensing methods can be classified in three categories: temporal, frequential and hybrid (temporal and frequential) methods. Temporal methods require a long observation period; frequential and hybrid methods have a high calculation cost and they are very sensitive to frequency resolution. In very low signal-to-noise ratio (SNR) and non-cooperative conditions, spectrum sensing methods present some limitations. To overcome these shortcomings, we propose a new blind strategy to detect the unoccupied spectral bands during a very short observation period. This new strategy is a temporal method based on Recurrence Quantification Analysis (RQA) of the received signal. Since the recurrence level in a communication signal is different from that of White Gaussian Noise, the detector can evaluate the recurrence level of the observed signal to detect the presence of a communication signal over a given spectral bandwidth. First, we estimate the three fundamental parameters of the recurrence matrix: the time delay parameter, the embedding dimension and the recurrence threshold. With these parameters, during a detection stage, the detector evaluates the recurrence level through the recurrence rate and compare it to a predetermined threshold estimated in absence of the signal of interest. The spectrum sensing based on RQA is very fast, free of frequency resolution issue and able to distinguish communication signal from a White Gaussian Noise. The results of our simulations prove the robustness of proposed RQA detector acting over limited number of samples and under very low SNR conditions.**

***Index Terms*—Cognitive Radio, Spectrum Sensing, Recurrence Quantification Analysis, Embedding parameters, Mutual Information, False Nearest Neighbours.**

I. Introduction

The need to make better use of the radio spectrum is leading to the development of new spectrum access strategies. Among these strategies, the opportunistic access proposed in Cognitive Radio allows a sharing of spectral bandwidth between two categories of users: Primary Users "PU" and Secondary Users "SU". PU is the one who holds the license of a bandwidth; and SU are all other opportunist transmitters. In this context, the main challenge for opportunistic users (SU) is the detection of unoccupied spectral bands. Many methods of spectrum sensing such as Energy Detection (ED), Waveform Detection (WFD), Cyclostationary Features Detection (CFD) have been developed [1]–[4]. However, most of these detectors suffer from certain limitations. We can mention among others a very high computational cost, an inefficiency in very low Signal Noise Ratio (SNR) conditions, a prior knowledge of PU's signal characteristics, etc. In this paper, we propose a new blind strategy to detect the state of PU's activity. This new strategy is based on the Recurrence Quantification Analysis (RQA) of the received signal. The recurrence is a fundamental characteristic of many dynamical systems. The quantification of this recurrence can be used to find out some intrinsic features of systems such as hidden periodicities, stationarity or non-stationarity, linearity or non-linearity properties. In radio communication, due to transmission technics, the transmitted signals contain hidden periodicities. From this idea, we propose to use Recurrence Quantification Analysis (RQA) tools to detect if the spectrum allocated to PU is free or occupied. Since PU's signal contains recurrent states, the SU can be able to evaluate the recurrence level contained in the observed signal by using RQA tools. And then, by comparing the recurrence level to a predetermined threshold, SU can make a decision about the presence or not of PU on a given spectrum. The starting point of RQA is the recurrence plot (RP). The RP is a graphical representation of the recurrence matrix [5]. The behavior of RP depends strongly on three fundamental parameters: the time delay τ, the phase space dimension m and the recurrence threshold ε. The time delay τ and the phase space dimension m are called embedding parameters [6]. In order to observe all intrinsic features contained in the signal, we must choose the optimal values of τ, m, ε. As, RP is a visual analysis tool, its analysis is not objective. For this raison, RQA tools are used to obtain objective analysis [5,17]. In our work, Recurrence Rate (RR) is used as a RQA tool.

The rest of this paper is organized in five sections. In section II, we present the concepts of recurrence analysis. After that, we detail the RP construction principle in section III. Sections IV and V deal respectively with our proposed detection model and its performance analysis. The last section concerns the conclusion and perspective for future works.

ISBN 978-90-827970-1-5

II. Concept of Recurrence Analysis

Recurrence analysis come from the fact that during its evolution, some states of a dynamical system can be reproduced several times. The different states of the system form the *phase space* of the system. Each state is called the *state vector* and is defined by state variables. The number of state variables required to define a state vector is the *dimension* of the system's phase space. The temporal evolution of the system is defined by the evolution equation which allows to determine the state of system at any time. Modelling this evolution equation is a very complex task. In practice, we don't observe a phase space object but time series, as a sequence of scalar measurements [5,7]. From this time series, we should reconstruct the phase space. The standard strategy for the phase space reconstruction is delay-coordinate embedding [6,7]. Specifically, we construct $m-dimensional$ state vectors X_n from m time-delayed samples of the measurement y_n:

$$X_n = \left[y_n, y_{n-\tau}, y_{n-2\tau}, ..., y_{n-(m-1)\tau}\right] \quad (1)$$

where τ and m are respectively the time-delay parameter and the embedding dimension (dimension of the phase space). In order to detect the maximum recurrence in the time series, one must use the optimal values of τ and m. Many algorithms are proposed to estimate the optimal values of τ and m [7]–[11]. Here, we use Mutual Information (MI) to estimate the time delay τ and False Nearest Neighbours (FNN) algorithm [10] to estimate m.

A. Estimation of time delay parameter τ

Experiment noise can generate statistical dependence among the subsequent vectors X_n. Hence, the time delay τ has to be chosen in order to reduce this statistical dependence [5]. We distinguish three methods to determine the optimal τ: autocorrelation function, MI and geometrical approach. In this work, we use MI to determine the optimal time delay τ because it can measure the general dependence between two random variables. Therefore, it provides a better criterion for the optimal time delay τ. The optimal τ corresponds to τ value which produces the first local minimum of MI [12]. Let's $Y_1, ..., Y_K$ be K random vectors with a joint probability distribution $f_{Y_1,...,Y_K}(y_1, ..., y_K)$ and marginal probability density functions $f_{Y_1}(y_1), ..., f_{Y_K}(y_K)$, mutual information between these vectors is defined as the Kullback-Leiber divergence between two probability distributions $\Pi_{k=1}^{K} f_{Y_k}(y_k)$ and $f_{Y_1,...,Y_K}(y_1, ..., y_K)$:

$$I(Y_1, ..., Y_K) = -\mathbb{E}\left[log\left(\frac{f_{Y_1}(y_1)...f_{Y_K}(y_K)}{f_{Y_1,...,Y_K}(y_1, ..., y_K)}\right)\right] \quad (2)$$

$I(Y_1, ..., Y_K)$ is non-negative, and it becomes zero if and only if random vectors are independents [13]. MI can be defined with the entropy H [13]–[15]:

$$I(Y_1, ..., Y_K) = \sum_{k=1}^{K} H(Y_k) - H(Y_1, ..., Y_K) \quad (3)$$

where: $H(Y) = -\sum_{i=1}^{K} f_Y(y_i) \log\Big[f_Y(y_i)\Big]$.

We can distinguish two widespread methods to estimate MI. The first is based on a partionning of space defined by the two systems of interest X and Y. The second used the k-nearest neighbor statistics. In this paper, we use the k-nearest neighbor statistics approach proposed by Kraskov et *al.* [15]. In our case, we should determine for which value of the time delay τ, a noisy communication signal $y(n)$ and its delay version $y(n-\tau)$ become independent. At first, we estimate the mutual information $I\left[y(n), y(n-\tau)\right]$ for $\tau \in [0, N]$ with N being the number of samples contained in $y(n)$. Then, we choose τ as the first minimum value of $I\left[y(n), y(n-\tau)\right]$.

B. Estimation of embedding dimension m

From the literature, we notice three basic approaches to estimate the optimal embedding dimension m [9]–[11,16]. The most used is the False Nearest Neighbours (FNN) method. In this paper, we use the FNN algorithm developped by Cao in [11], because of its simplicity and its low calculation cost. The principle is the following: if m is qualified as the optimal embedding dimension by the embedding theorem [6], then any two points which stay close in the $m-dimensional$ reconstructed phase space will be closed in $(m+1)-dimensional$ reconstructed phase space. Such a pair of points are called true neighbors, otherwise, they are called false neighbors. Perfect embedding means that no false neighbors exist [11,16]. The first parameter which helps to determine the FNN is defined by:

$$a(i, m) = \frac{\left\|X_{i_{m+1}} - X_{j_{m+1}}\right\|_\infty}{\left\|X_{i_m} - X_{j_m}\right\|_\infty} \quad (4)$$

where $i, j = 1, 2, ..., N - m\tau$. $X_{i_{m+1}}$ is the reconstructed state vector with embedding dimension $(m+1)$: $X_{i_{m+1}} = [y_i, y_{i-\tau}, y_{i-2\tau}, ..., y_{i-m\tau}]$; X_{j_m} is the nearest neighbor of X_{i_m} in the $m-dimensional$ reconstructed phase space and $\|.\|_\infty$ denotes the maximum norm, i.e.,

$$\|X_{i_m} - X_{j_m}\|_\infty = \max_{0\le k\le m-1} |y_{i+k\tau} - y_{j+k\tau}|; \quad (5)$$

$a(i, m)$ can be viewed as a neighborhood criterion. The major drawback here is that $a(i, m)$ changes with the considered state vector X_i. To overcome this matter, the mean value of all $a(i, m)$ is used:

$$\mathbb{E}(m) = \frac{1}{N - m\tau} \sum_{i=1}^{N-m\tau} a(i, m) \quad (6)$$

$\mathbb{E}(m)$ depends only on the dimension m and the lag τ. To investigate its variation from m to $m+1$, the ratio $R(m)$ is calculated:

$$R(m) = \frac{\mathbb{E}(m+1)}{\mathbb{E}(m)} \quad (7)$$

By plotting $R(m)$, we notice that $R(m)$ becomes constant when m becomes greater than a defined value m_0. In this case, (m_0+1) becomes the minimum embedding

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dimension [11]. The embedding parameters m and τ are necessary to construct the Recurrence Plot (RP).

III. Recurrence Plot

Recurrence plot (RP) is a visual tool showing the behaviour of recurrences contained in a signal. It is defined by the recurrence matrix [5,17]:

$$\mathbf{R}_{i,j}^{(\varepsilon,m)} = \Theta\left\{\varepsilon - \left\| \ X_i - X_j \ \right\|\right\} \tag{8}$$

where $i, j = 1, ..., M$ and $M = N - (m-1)\tau$ is the number of reconstructed state vectors X_i, N denotes the number of samples contained in the observed signal; ε is the recurrence threshold and $\Theta(.)$ represents the Heaviside step function and $\|.\|$ is a norm. We use here $L_2 - norm$. On the plot, recurrence is represented by a black dot. The main challenge to define recurrence among the different state vectors is the choice of the adequate recurrence threshold ε. If ε is chosen too small, there may be almost no recurrence points and we cannot learn anything about the recurrence structure contained in the signal. On the other hand, if ε is chosen too large, almost every point is a neighbour of every point leading to a lot of false recurrences. Consequently, we have to find a compromise about ε value. There are many approaches to estimate ε [17,18]. In our detection model, the signal of interest is corrupted by the noise. In order to obtain similar result as for noise-free situation, the value $\varepsilon = 5\sigma_b$ is proposed in the literature. σ_b is the standard deviation of the White Gaussian Noise (WGN). However, this value is not adequate for very low SNRs ($SNR \leqslant -10dB$) nor very high SNRs ($SNR \geqslant 20dB$) conditions. Another drawback with this relation is that the detector should know the noise variance σ_b^2, before defining recurrent states. To overcome these issues, we chose empirically $\varepsilon = 3\sigma_y$, where σ_y denotes the standard deviation of the received signal $y(n)$. Visual analysis of RP is not objective because differents observers can see things differently. For this raison, Zbilut and Webber introduced some definitions and procedures to quantify RP structures [19]. In the literature, we have five classical tools for RQA [5]. In our detection model, we use the reccurrence rate (RR) as the detection criterion. RR is the measure of the relative density of recurrence points in the sparce recurrence matrix and is defined by [5]:

$$RR(\varepsilon, M) = \frac{1}{M^2}\sum_{i=1}^{M}\sum_{j=1}^{M}\mathbf{R}_{i,j}^{(\varepsilon,m)} \tag{9}$$

where $M = N - (m-1)\tau$ and N the number of samples contained in the observation $y(n)$.

IV. Detection Model

The aim of our work is to detect the presence or absence of communication signal on a given spectrum. Let y(n) be the observed signal containing N samples:

$$y(n) = \eta s(n) + b(n) \tag{10}$$

where $s(n)$ denotes the communication signal and $b(n)$ an AWGN. η is a binary variable ($\eta = 1$ if there is presence of a communication signal else $\eta = 0$).
Let us consider the following hypotheses H_0 and H_1:

- H_0 : absence of communication signal; $\eta = 0$
- H_1 : presence of communication signal; $\eta = 1$

The detection principle is to compare the RR of the observed signal to a predetermined threshold λ_{RR} to make a decision. During the spectrum sensing stage, the detector evaluates the $RR(\varepsilon, N)$ and applies the following criterion:

$$RR(\varepsilon, N) \underset{H_0}{\overset{H_1}{\gtrless}} \lambda_{RR} \tag{11}$$

λ_{RR} is the Recurrence Rate estimated when there is no communication signal on the spectral band. By performing several Monte Carlo simulations, we noticed that λ_{RR} can be modelized by a random variable depending on SNR. The following lines present the analyical expressions of the distance matrix D components $d_{i,j}$ (equation (12)), recurrence condition in WGN (equation (14)) and detection threshold λ_{RR} (equation (15)). From (10) and (1), we can write:

$$\begin{aligned} d_{i,j}^2 &= \|X_i - X_j\|^2 \\ &= m(\hat{\sigma}_i^2 + \hat{\sigma}_j^2) + m(\hat{\mu}_i^2 + \hat{\mu}_j^2) - 2m\mathbb{E}\left[X_i X_j^H\right] \end{aligned} \tag{12}$$

where $\hat{\sigma}_i^2$, $\hat{\sigma}_j^2$, $\hat{\mu}_i$ and $\hat{\mu}_j$ are random variables and represent respectively the estimation of variance and expectation of the different state vectors X_i and X_j. Hence, for a received signal $y(n)$, the components $d_{i,j}$ of the distance matrix D can be modelled as a random variable. X_j^H denotes the Hermitian vector of X_j. In the case where $y(n) = b(n)$, if we suppose that $b(n)$ is a centered WGN ($\mathbb{E}[b(n)] = 0 \implies \mathbb{E}[X_i] = \mathbb{E}[X_j] = 0$) equation (12) becomes:

$$d_{i,j}^2 = m(\hat{\sigma}_i^2 + \hat{\sigma}_j^2) \tag{13}$$

Consequently, recurrence condition in WGN becomes:

$$\mathbf{R}_{i,j}^{(\varepsilon,m)} = 1 \iff \hat{\sigma}_i^2 + \hat{\sigma}_j^2 < \frac{\varepsilon^2}{m} \tag{14}$$

As the detection threshold λ_{RR} corresponds the RR when there is no communication signal on the spectrum, we can write that:

$$\lambda_{RR} = P\left[\hat{\sigma}_i^2 + \hat{\sigma}_j^2 < \frac{\varepsilon^2}{m}\right] \tag{15}$$

Equation (15) shows that the detection threshold is independant of modulation technics. λ_{RR} depepends on the state vectors energy. Let $z_{ij} = \hat{\sigma}_i^2 + \hat{\sigma}_j^2$ be a realization of the random variable Z. The next step is to determine the Probability density function (Pdf) of Z. Let us consider the random variable Ψ such as $\psi = N\hat{\sigma}_i^2 = \sum_{k=0}^{m-1} b_{i-k\tau}^2$

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and $f_\Psi(\psi)$ the Pdf of Ψ. $f_\Psi(\psi)$ can be approximated by a χ^2 Pdf with m degree of freedom:

$$f_\Psi(\psi) = \frac{1}{(\sqrt{2})^m \Gamma\left(\frac{m}{2}\right)} \psi^{\frac{m}{2}-1} \exp\left[-\frac{\psi}{2}\right] \quad (16)$$

where $\Gamma(x)$ is Euler's Gamma function. Because $\hat{\sigma}_i^2$ and $\hat{\sigma}_j^2$ are independant, we can write:

$$\begin{aligned} f_Z(z) &= (f_\Psi * f_\Psi)(z) \\ &= \frac{e^{-\frac{z}{2}}}{2^m \Gamma^2\left(\frac{m}{2}\right)} \int_0^{+\infty} [u(z-u)]^{\frac{m}{2}-1} du \end{aligned} \quad (17)$$

Considering the cumulative density function F_Z of f_Z, we obtain:

$$\lambda_{RR} = F_Z\left(\frac{\varepsilon^2}{m}\right) \quad (18)$$

Hence:

$$f_{\lambda_{RR}}(r) = f_{RR/H_0}(r) = f_Z(r) \quad (19)$$

Simulations show that the probabilty density functions $f_{RR/H_0}(r)$ and $f_{RR/H_1}(r)$ of RR under H_0 and H_1 overlap (Figure 1). The main difficulty is to determine an optimal value of λ_{RR} in order to minimize detection errors according to the Neymann-Pearson detection theory [2,20]. Because detection errors are mainly

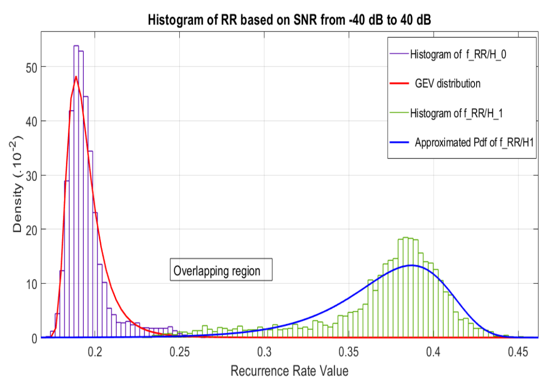

Fig. 1: Histogram of RR under hypothesis H_0 and H_1. The probability density functions $f_{RR/H_0(r)}$ and $f_{RR/H_1}(r)$ of RR under H_0 and H_1 overlap. λ_{RR} must be chosen to maximize detection probability and minimise probability of false alarm.

from Probability of False Alarm (P_{fa}), we estimate $f_{RR/H_0}(r)$. And then, based on P_{fa}, we calculate the corresponding optimal value of detection threshold λ_{RR}. Equation (17) gives the analytical expression of $f_{RR/H_0}(r)$. From Kolmogorov-Smirnov test, we can also reliably approximated $f_{RR/H_0}(r)$ by the Generalized Extreme Values distribution

$$f_{RR/H_0}(r;\mu,\sigma,\xi) = \frac{1}{\sigma}\left[1+\xi\left(\frac{r-\mu}{\sigma}\right)\right]^{-\frac{1}{\xi}-1} \times \exp\left\{-\left[1+\xi\left(\frac{r-\mu}{\sigma}\right)\right]^{-\frac{1}{\xi}}\right\} \quad (20)$$

where ξ is the shape parameter, σ stands for the scale parameter and μ represents the location parameter. Using the minimizing quadratic error principle, we have estimated these parameters and obtained the following values: $\xi = -0.17$, $\sigma = 0.0056$ and $\mu = 0.085$. With the equation (21), we have calculated the threshold detection optimal λ_{RR} based on P_{fa} value by using the inverse of cumulative density function F_{RR/H_0}.

$$P_{fa} = \int_{\lambda_{RR}}^{+\infty} f_{RR/H_0}(r)dr, \quad (21)$$

Using the obtained threshold, we will analyze the performance of RQA detector in the following section.

V. Simulations Results

To evaluate robustness of the proposed algorithm, we have tested our detection model on several kind of signals (BPSK,16-QAM, 4-ASK) in low SNR conditions ($SNR \leq 0dB$). In this paper, we present results about 16-QAM signals. For simulations we use parameters from the table I closed to wideband wireless communication systems. Performing Monte-Carlo simulations,

Parameters	Value
Sampling frequency	30.72 GHz
Data frequency	200 KHz
Carrier frequency	2.6 GHz
Number of samples	1000
embedding dimension m	8
Time delay τ	6
Recurrence threshold ε	$3\sigma_y$

TABLE I: Simulation parameters

we generate Receiver Operating Characteristic (ROC) curves. Fig. 2 shows the capability of our proposed model to detect communication signal buried in noise for different values of SNR. One can note through these ROC curves that the proposed detector can reliably detect communication signals in Gaussian channels where $SNR \geqslant -4dB$. For example, when $SNR = -4dB$, our detector can detect the communication signal with $P_d = 0.97$ against $P_{fa} = 0.1$. When $SNR = -6dB$, this detector detects the communication signal with $P_d = 0.8$ against $P_{fa} = 0.10$. Hereon, we have confronted RQA Detection (RQD) model to Energy Detection (ED) and Cyclostationary Features Detection (CFD) algorithms. ED is the most used detection method because of its low implementation complexity. Recently works have proposed a method to overcome the issue of the optimal detection threshold for ED design [21,22]. The biggest disadvantage with ED is the inability to differentiate a communication signal from a white noise with high energy. The CFD is the best trade-off between complexity and accuracy in detection strategy. However, because of the cyclic spectrum estimation, it requires a very high computational cost for high frequency resolutions [2,23]. Fig. 3 shows the P_d of these three detectors in different SNR values. Among these three detection algorithms, the CFD is the most efficient. However, it requires a very high computational cost and is very sensitive to the frequency resolution. ED seems equally powerful, but it is unable to differentiate the communication signal from

ISBN 978-90-827970-1-5 © EURASIP 2018

a high energy noise. Unlike these two detectors, the RQD has a low calculation cost and is able to distinguish a communication signal from noise even with a low SNR. Also, we notice that when $SNR \geqslant -5dB$, RQD has the same performance that ED and CFD. The performance of RQD can be signicatively improve by combining RR with others RQA tools.

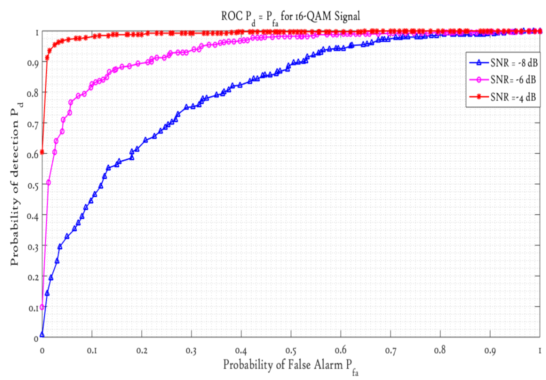

Fig. 2: Receiver Operating Characteristics (ROC). These results concern 16-QAM signal buried in White Gaussian Noise.

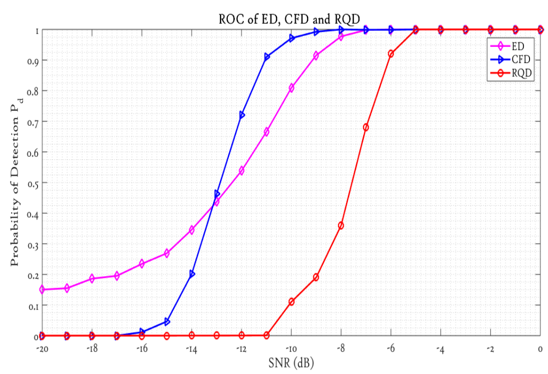

Fig. 3: Receiver Operating Characteristics of ED, CFD and RQD for $SNR \in [-20; 0]$ dB. When $SNR \geq -5$ dB, RQD has the same performance like ED and CFD.

VI. Conclusion

This paper deals with the problematic of blind spectrum sensing in very low SNR conditions in Cognitive Radio. We have proposed a detection model based on Recurrence Quantification Analysis (RQA). RQA can reveal and quantify the hidden recurrences in communication signals . During spectrum sensing stage, Secondary User evaluates the recurrence level contained in observed signal and compare it to a predetermined threshold in order to make a decision. Our simulations show that RQA is a powerfull method to detect communication signal buried in noise. Using RQA in spectrum sensing is a new and promising approach. Our current simulations using recurrence rate allow to detect PU's for $SNR \geqslant -6dB$ with a low probability of false alarm. In future works, we propose to combine several RQA tools to force back this limit to $SNR \leqslant -15dB$ to corroborate the efficiency of RQA detection model.

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