Non-parametric characterization of gravitational-wave polarizations

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***Abstract*—Gravitational waves are polarized. Their polarization is essential to characterize the physical and dynamical properties of the source i.e., a coalescing binary of two compact objects such as black holes or neutron stars. Observations with two or more non coaligned detectors like Virgo and LIGO allow to reconstruct the two polarization components usually denoted by $h_+(t)$ and $h_\times(t)$. The amplitude and phase relationship between the two components is related to the source orientation with respect to the observer. Therefore the evolution of the polarization pattern provides evidence for changes in the orientation due to precession or nutation of the binary. Usually, some specific direct dynamical model is exploited to identify the physical parameters of such binaries. Recently, a new framework for the time-frequency analysis of bivariate signals based on a quaternion Fourier transform has been introduced in [1]. It permits to analyze the bivariate signal combining $h_+(t)$ and $h_\times(t)$ by defining its quaternion embedding as well as a set of non-parametric observables, namely Stokes parameters. These parameters are remarkably capable of measuring fine properties of the source, in particular by deciphering precession, without close bounds to a specific dynamical model.**

I. Introduction

A new kind of astronomy is born with the first advanced LIGO and advanced Virgo discoveries [2]–[6]. Those gravitational wave detectors allow the observation of astrophysical systems, such as binary black-holes, that have so far escaped conventional astronomy based on electromagnetic radiation. Gravitational waves carry information about the bulk motion of the emitting system relative to the observer. For instance, the wave frequency is related to the orbital or spinning period of the source mass distribution. Gravitational waves are polarized and the amplitude and phase relationship between the two polarizations components h_+ and $h_\times$ predicted by general relativity is related to the source orientation with respect to the observer. The evolution of the polarization pattern thus provides evidence for changes in the orientation due to precession or nutation of the system.

Precession of the binary orbital plane is an important information as it indicates that at least one binary component has a large spin, misaligned with the orbital angular momentum. In turn, this provides decisive hints on how the binary has formed. The presence of precession in the detected signal is classically tested by fitting the data with waveforms obtained from precessing binary physical models. This procedure does not test precession effects alone, but rather a full description of the binary orbital dynamics, which thus includes many other dynamical effects. In this contribution we propose a different approach.

Gravitation-wave detectors do not measure the two gravitational-wave polarizations independently but rather a linear mixture of them. However observations from two or more non-coaligned detectors allow to reconstruct the two gravitational-wave polarizations. We assume that h_+ and $h_\times$ from a binary merger are reconstructed from LIGO and Virgo observations, see e.g., [7]. Recently, a new framework for the frequency analysis of bivariate signals based on a quaternion Fourier transform has been introduced in [1]. It allows to analyze the bivariate signal combining $h_+(t)$ and $h_\times(t)$ by defining its quaternion embedding as well as a set of non-parametric observables, namely instantaneous Stokes parameters. Quaternion embedding and time-varying Stokes parameters are powerful and interpretable mathematical tools to describe the instantaneous polarization state of bivariate signals. We propose to infer the presence of precession by characterizing the evolution of the polarization state from the reconstructed signals $h_+(t)$ and $h_\times(t)$. Thanks to this approach we obtain new "non-parametric" observables capable of finely deciphering the geometrical configuration of the source, in particular precession, without close bounds to a specific dynamical model.

Section II introduces a standard gravitational wave model from precessing binaries. Section III describes instantaneous polarization features for bivariate signals. The fruitful combination of these two elements is presented first in Section IV and is further discussed in Section V with numerical examples. Section VI gathers concluding remarks.

II. Gravitational waves from precessing systems

Following [8], we assume quasi-circular orbits and introduce a set of two frames to model the sensing of gravitational waves. The modelling of the GW signal from the precessing

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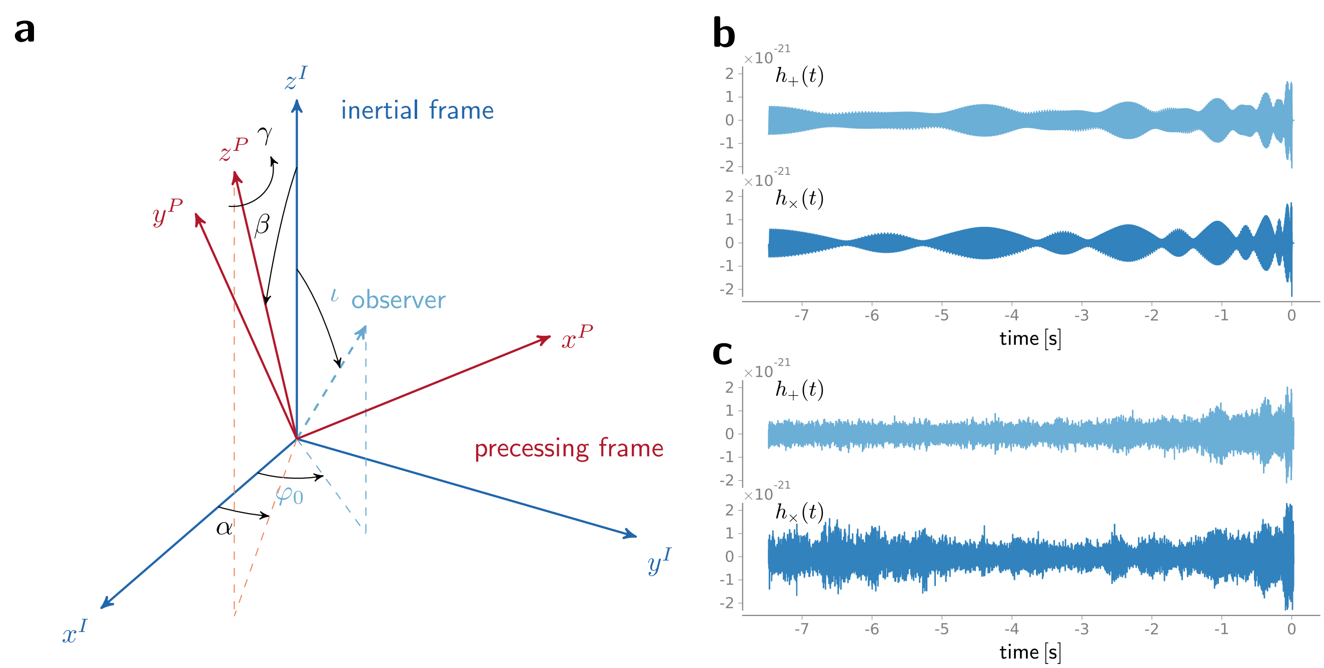

Fig. 1. (a) The relation between precessing and inertial frames is parameterized by Euler angles α, β, γ in the zyz-convention. Spherical coordinates (ι, φ_0) denote the position of the observer in the inertial frame. (b) Waveforms of the two polarizations h_+ and $h_\times$ for a strongly precessing binary system formed by a neutron-star and a 10-solar mass black-hole with misaligned spin $\mathbf{s}_1 = (0.7, 0.7, 0)$. The binary is face-on and located at a distance of 10 Mpc. (c) Reconstructed polarizations from simulated LIGO and Virgo observations of the same binary system, see [7] .

binaries is usually done in two steps. First, the computation of the GW modes is done in the frame ^{P} instantaneously co-precessing with the binary orbital plane. Those modes are the result of the decomposition of the signal in the spin -2 weighted spherical harmonics. In the second step, the modes are rotated to the inertial frame ^{I} associated with the binary configuration at some fiducial time (which is usually associated with the time when the signal enters the observational band of the detector). This inertial frame is then associated with the position and orientation of the GW detectors (LIGO, Virgo).

In the precessing frame attached to the binary, the complex gravitational wave strain $h^P = h^P_+ - ih^P_\times$ can be decomposed into spherical harmonics $h^P_{\ell m}(t)$ such that

$$h^P(t;\Omega) = \sum_{\ell=2}^{\infty} \sum_{m=-\ell}^{\ell} h^P_{\ell,m}(t) {}_{-2}Y_{\ell,m}(\Omega) \tag{1}$$

where Ω is the (time-varying) angle of the observer in the precessing frame and ${}_{-2}Y_{\ell,m}$ are the -2-spin weighted spherical harmonics.

The key idea of [8] is that the gravitational wave modes in the precessing frame resemble that of a non-precessing binary. The dominant modes correspond to $(\ell = 2, m = \pm 2)$ and they can be approximated as

$$h^P_{2,\pm 2}(t) = a_0(t) e^{\mp i\Phi(t)}. \tag{2}$$

While [8] derives explicit expressions for the instantaneous amplitude $a_0(t)$ and phase $\Phi(t)$ by resolving the binary orbital motion, we do not assume here any specific evolution for $a_0(t)$ and $\Phi(t)$.

The modes are rotated from the precessing frame ^{P} to the inertial frame labelled with ^{I} using the (time-dependent) Euler angles α, β and γ, see Fig. 1a. This change of frame involves the following correspondence between spherical harmonics coefficients of the gravitational waves expressed in each frame:

$$h^I_{\ell,m} = \sum_{m'=-\ell}^{\ell} h^P_{\ell,m'} \, D^{\ell,*}_{m'm}(-\gamma, -\beta, -\alpha) \tag{3}$$

where $D^\ell_{m'm}$ are the Wigner-D functions [9]. When the binary does not precess, the frames ^{P} and ^{I} coincide and $\alpha = \beta = \gamma = 0$. Since there are only $\ell = 2$ modes in the frame ^{P}, only $\ell = 2$ modes will contribute in the frame ^{I}. However, all m modes contribute to the observed signal, which therefore reads

$$h^I(t;\iota,\varphi_0) = \sum_{m=-2}^{2} h^I_{2,m}(t) {}_{-2}Y_{2,m}(\iota,\varphi_0) \tag{4}$$

where (ι, φ_0) are the spherical coordinates of the observer [1].

By combining Eqs (1–4), we can express the two polarizations h_+ and $h_\times$ of the incident gravitational wave in the observation frame as a generic function of the binary orbital

[1] In the geocentric frame, ι therefore corresponds to the inclination of the binary orbital plane with the line of sight.

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dynamics and the orientation of the binary. For a strongly precessing binary system composed of a neutron star and a 10 solar-mass black-hole with misaligned spin $\mathbf{s}_1 = (0.7, 0.7, 0)$, this results in the waveforms shown in Fig. 1b. Precession causes changes in the orientation of the binary's orbital plane with respect to the line of sight, that leads to the characteristic amplitude modulations clearly seen on the waveform envelop. It also leads to less obvious interrelationships between the "+" and "×" phases that we intend to discriminate with the mathematical tools introduced in the next section.

III. Instantaneous polarization features

Recently, a new framework based on a tailored Quaternion Fourier Transform (QFT) has been introduced for the time-frequency analysis of bivariate signals [1]. It allows to extend familiar concepts such as Fourier transform and spectral analysis to the case of bivariate signals. While it is usual to embed real signals in the space of complex signals, this approach proposes to consider bivariate signals as complex valued signals which are embedded in the higher dimensional space of quaternions. We recall here the concept of quaternion embedding, which is the bivariate counterpart of the well-known analytic signal originally introduced in [10].

We form a bivariate signal by combining the two observed polarizations in a complex-valued signal $h(t) = h_+(t) - \boldsymbol{i}h_\times(t)$. The QFT used in [1] takes values in the 4-dimensional space of quaternions $\mathbb{H}$, *i.e.* it has a scalar (real) part and a vectorial part made of 3 imaginary parts carried by the standard quaternion square roots of -1: $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$; see for example [11] for a review on quaternions.

The quaternion embedding of a signal $h(t)$, denoted $h_\mathbb{H}(t)$, is defined by:

$$h_\mathbb{H}(t) = h(t) + \mathcal{H}\left[h(t)\right]\boldsymbol{j} \in \mathbb{H} \tag{5}$$

where $\mathcal{H}[\cdot]$ stands for Hilbert transform. The restriction of the quaternion embedding signal $h_\mathbb{H}(t)$ to its real and $\boldsymbol{i}$-imaginary parts, belonging to span$\{1, \boldsymbol{i}\}$, is the original signal $h(t)$. Note that $\mathcal{H}\left[h(t)\right] \in \text{span}\{1, \boldsymbol{i}\}$ is a complex valued signal as well. Most importantly, $h_\mathbb{H}(t)$ and $h(t)$ share the same spectral content except that the QFT of $h_\mathbb{H}(t)$ is null at negative frequencies.

The quaternion nature of $h_\mathbb{H}(t)$ yields a convenient polar form expression [1], [12]:

$$h_\mathbb{H}(t) = a(t)e^{\boldsymbol{i}\theta(t)}e^{-\boldsymbol{k}\chi(t)}e^{\boldsymbol{j}\varphi(t)}, \tag{6}$$

which looks much alike the usual amplitude-phase decomposition of real signals, with 2 additional factors $(\theta(t), \chi(t))$ accounting for polarization properties. It can be related to *polarized AM-FM* signals by restricting the Cartesian form of $h_\mathbb{H}(t)$ to its span$\{1, \boldsymbol{i}\}$ part, leading to:

$$h(t) = a(t)e^{\boldsymbol{i}\theta(t)}\left(\cos\chi(t)\cos\varphi(t) + \boldsymbol{i}\sin\chi(t)\sin\varphi(t)\right). \tag{7}$$

Polarized AM-FM signals are the complex generalization of the standard AM-FM signals. Expressions (6) and (7) explicitely exhibit instantaneous waveform attributes $a(t)$ and $\varphi(t)$ as well as instantaneous geometric attributes $\theta(t)$ and $\chi(t)$.

Figure 2 depicts the way those parameters describe the instantaneous shape of the bivariate signal $h(t)$. The parameters $a(t)$, $\theta(t)$ and $\chi(t)$ represent the intensity, the orientation and the ellipticity of the instantaneous polarization ellipse. The phase $\varphi(t)$ encodes the local oscillatory evolution along this ellipse. The interpretation of any bivariate signal using the *polarized AM-FM* form is subject to conditions similar to the Bedrosian theorem in the univariate case. Details can be found in [1] and can be summarized by saying that geometric instantaneous parameters $a(t)$, $\theta(t)$ and $\chi(t)$ must have a much slower rate of change than $\varphi(t)$ for the interpretation to be valid.

An alternate and powerful way to parametrize the quaternion embedding signal $h_\mathbb{H}(t)$ is to consider its associated Stokes parameters defined as [13]:

$$S_0(t) = a^2(t) \tag{8}$$
$$S_1(t) = a^2(t)\cos 2\theta(t)\cos 2\chi(t) \tag{9}$$
$$S_2(t) = a^2(t)\sin 2\theta(t)\sin 2\chi(t) \tag{10}$$
$$S_3(t) = a^2(t)\sin 2\chi(t) \tag{11}$$

in terms of the ellipse parameters $(a(t), \theta(t), \chi(t))$ used to parametrize the quaternion embedding in (6). More precisely, the Stokes parameters, that are very popular in optics, carry the *polarization information* parametrized by $\theta(t)$ and $\chi(t)$. The first Stokes parameter $S_0(t)$ is purely energetic and gives the instantaneous power. Normalized Stokes parameters S_1/S_0, S_2/S_0 and S_3/S_0 describe the instantaneous polarization state only. Stokes parameters are second-order quantities, making their estimation numerically less sensitive than ellipse parameters θ and χ.

As explained in [1], Stokes parameters associated to a bivariate signal $h(t)$ can be directly computed from $h_\mathbb{H}(t)$ by $|h_\mathbb{H}(t)|^2 = S_0(t)$ and $h_\mathbb{H}(t)\boldsymbol{j}h_\mathbb{H}^\star(t) = \boldsymbol{i}S_3(t) + \boldsymbol{j}S_1(t) + \boldsymbol{k}S_2(t)$, where '$\star$' stands for quaternion conjugation. This shows that Stokes parameters are related to quadratic quantities in the quaternion embedding. They also have invariance properties with respect to the QFT [1]. In the sequel, Stokes parameters associated to a bivariate signal will be used to extract some of its time-varying properties. Their direct identification to the imaginary parts of a quadratic quantity derived from the quaternion embedding makes them good candidates for non-parametric analysis and estimation as proposed in the following Sections.

IV. From Stokes to astrophysical parameters

Instantaneous polarization attributes obtained thanks to the quaternion embedding method provide a straightforward characterization of polarization evolution for precessing binaries. As a first example, let us consider the observed signal (4) obtained for face-on $(\iota, \varphi_0) = (0, 0)$ binaries, that is when

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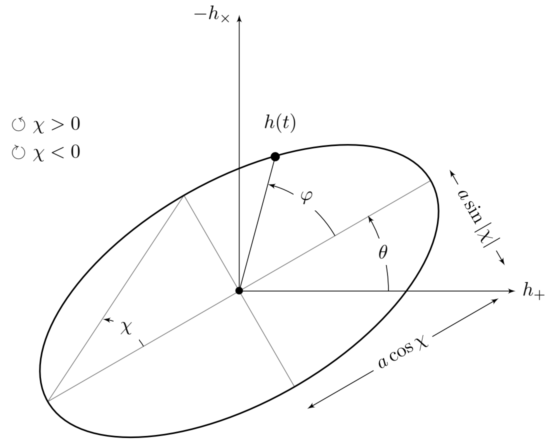

Fig. 2. Instantaneous ellipse tangent to the trajectory of the bivariate signal $h(t) = h_+(t) - \boldsymbol{i}h_\times(t)$. This ellipse is characterized by the polarization parameters $(a(t), \theta(t), \chi(t))$.

the observer is in direction z^I. From (4), one gets the simple expression (time-dependence omitted for conciseness):

$$h^I = ka_0 e^{-2\boldsymbol{i}\alpha} \left\{ (1 + \cos^2 \beta) \cos \Phi_\gamma - 2\boldsymbol{i} \cos \beta \sin \Phi_\gamma \right\} \tag{12}$$

where $k > 0$ is a constant and $\Phi_\gamma = \Phi + 2\gamma$. For most cases of astrophysical relevance the orbital dynamics can be described by osculating orbits where the precession timescale is much longer than the orbital timescales. This means that Euler angles $[\alpha(t), \beta(t), \gamma(t)]$ vary much slower than the phase $\Phi(t)$. It thus allows a direct identification of instantaneous polarization parameters by comparing (12) with (7):

$$a(t) = ka_0(t) \left(1 + \cos^2 \beta(t))^2 + 4 \cos^2 \beta(t) \right)^{1/2} \tag{13}$$

$$\theta(t) = -2\alpha(t) \tag{14}$$

$$\chi(t) = -\arctan \frac{2 \cos \beta(t)}{1 + \cos^2 \beta(t)} \tag{15}$$

$$\varphi(t) = \Phi(t) + 2\gamma(t) \tag{16}$$

Eqs. (13)–(16) highlights the direct relation between standard descriptors of bivariate signals and GW parameters. In particular, Eqs. (14) and (15) explicitly show how precessing binaries generate polarization modulation effects on the observed signal $h^I(t)$. For this face-on case, one observes a nice decoupling between orientation $\theta(t)$ depending only on $\alpha(t)$ and ellipticity $\chi(t)$ depending only on $\beta(t)$. Note that $\gamma(t)$ only affects the phase $\varphi(t)$ and does not produce polarization modulation effects.

In the general case, *i.e.* for an arbitrary observer position $(\iota, \varphi_0) \neq (0, 0)$, this approach is no longer so direct and practical due to the complexity of the expression of $h^I(t)$. Rather, precession and polarization modulation effects in $h^I(t)$ can be easily characterized using instantaneous Stokes parameters. In particular, normalized Stokes parameters $S_1/S_0, S_2/S_0, S_3/S_0$ provide a convenient description of the instantaneous polarization state of $h^I(t)$. Their explicit expressions can still be obtained from the quaternion embedding of the generic model (4) but they become far too voluminous to be reproduced here. They provide a direct connection between precession parameters, Euler angles α, β, and the instantaneous polarization state of $h^I(t)$. Note that γ only affects the instantaneous phase of $h^I(t)$, as for the special case of face-on binaries.

V. Applications and discussions

We illustrate our findings on simulated gravitational waveforms from precessing binaries. Simulations are carried out using the generic SEOBNRv3 model of a (strongly) precessing black-hole/neutron star binary [14]. This precessing case is somehow extreme and is not favoured by current binary formation models. However it is not excluded and remains physically possible. Above all, waveforms presented in Fig. 1b serve our illustrative purposes. Fig. 1b depicts the two polarizations $h_+(t)$ and $h_\times(t)$ of a gravitational wave emitted by this binary system.

Stokes parameters provide a straightforward diagnosis of precession. The theoretical relation between normalized Stokes parameters $S_1(t)/S_0(t), S_2(t)/S_0(t), S_3(t)/S_0(t)$ and Euler angles can be explicitly derived, see Sec. IV eqs. (14)-(15) for face-on binaries; this holds in general with more elaborated formulas when $(\iota, \varphi_o) \neq (0, 0)$. For non-precessing binaries, Euler angles $\alpha(t), \beta(t), \gamma(t)$ are identically zero. In this case, the instantaneous polarization state of $h^I(t)$ is constant since θ and χ remain constant. As a result, normalized Stokes parameters remain constant as well, see (8)-(11). Since they are readily computed from the quaternion embedding of the observed signal $h^I(t)$, they provide a useful and sensitive tool for the analysis of precession effects.

Fig. 3 shows the instantaneous normalized Stokes parameters obtained from the quaternion embedding $h_{\mathbb{H}}(t)$ of the bivariate signal $h^I(t) = h_+(t) - \boldsymbol{i}h_\times(t)$ with waveforms presented in Fig. 1b. These gravitational waveforms correspond to a strong precessing binary observed face-on $(\iota, \varphi_0) = (0, 0)$. The non-parametric estimates of Stokes parameters (thin white lines) from simulated $h^I(t)$ close to perfectly match the values expected from the explicit physical model involving Euler angles (thick blue lines): the 2 curves are superposed. The presence of oscillations indicates that the instantaneous polarization state of $h^I(t)$ is modulated. This polarization modulation is directly explained by the precession dynamics. In particular for face-on binaries $S_3(t)/S_0(t)$ is a function of the precession angle $\beta(t)$ only, see Eqs (11) & (15).

Fig. 4 presents the normalized Stokes parameters obtained for the bivariate signal $h^I(t) = h_+(t) - \boldsymbol{i}h_\times(t)$ using reconstructed polarizations depicted in Fig. 1c. The polarization reconstruction from LIGO/Virgo observations require solving an inverse problem. Here, this is performed using sparsity-promoting regularization techniques (LASSO) presented in [7]. The reconstructed polarizations are obtained from observations of the black-hole/neutron-star binary system considered in Fig. 1c in simulated LIGO/Virgo noise using sensitivity curves comparable to that of the last O2 science run.

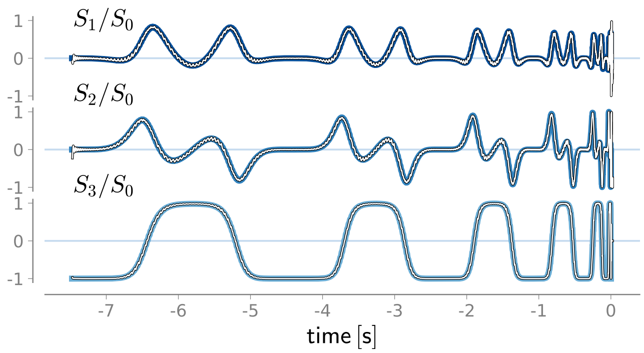

Fig. 3. Non-parametric estimates of normalized instantaneous Stokes parameters S_1/S_0, S_2/S_0 and S_3/S_0 (thin white lines) computed from computed $h^I(t)$ depicted in Fig. 1b. They close to perfectly match the values expected from the explicit physical model involving Euler angles (thick blue lines). Oscillations characterize the dynamics of precessing binary.

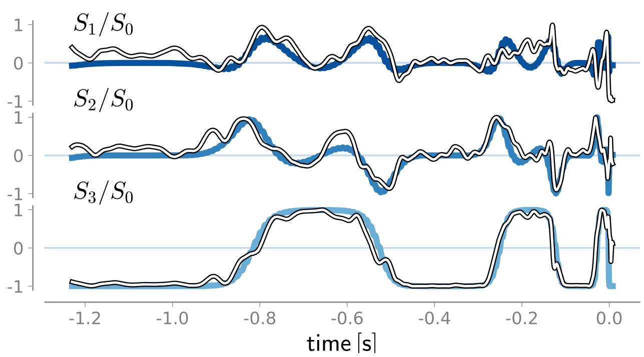

Fig. 4. Non-parametric estimates of normalized instantaneous Stokes parameters S_1/S_0, S_2/S_0 and S_3/S_0 (thin white lines) obtained from the reconstructed polarizations shown in Fig. 1c. Expected values from the explicit physical model involving Euler angles (thick blue lines) are depicted for comparison.

Stokes parameters are extracted from the ridge of a quaternion continuous wavelet transform [1], in order to overcome the remaining noise in reconstructed polarizations in Fig. 1c. This noise hinders the direct use of the quaternion embedding method. On the ridge, one approximately recovers the quaternion embedding of the noiseless signal $h^I(t)$, see [1] for details. The extracted ridge corresponds to the end of the inspiral (-1.2 s $\leq t \leq 0$) since SNR increases and becomes large enough as the binary comes close to the merger. The good agreement between reconstructed normalized Stokes parameters and their explicit physical model involving Euler angles (thick blue lines) demonstrate the relevance of use of Stokes parameters to diagnosis and characterize precession.

VI. Conclusion

We have shown that Stokes parameters estimated from the observed gravitational wave directly connect waveform features to dynamical properties of the source. When applied to the case of coalescing compact binaries, they permit to test the presence of precession of the orbital plane prior to the merger (when the binary collapses). Most importantly these new observables are non parametric and bring robust information to provide a support to more conventional waveform fitting procedures based on a comprehensive and detailed model of the binary dynamics. In some sense, Stokes parameters are a reparametrization that directly encodes orbital properties of the source which are very difficult to obtain individually. In the case of the observation of a simulated simple face-on binary, with a dominating quadrupolar mode, our results show a remarkable agreement between theoretical predictions and numerical estimations. They can be extended to arbitrary binary orientations and higher-order modes. This approach could also yield the detailed physical parameters from the Stokes observables by reverting a system of non-linear equations. Together with the reconstruction of polarizations described in [7], it provides a complete procedure to analyze polarization-related effects in experimental data from LIGO and Virgo detectors. It has the potential of revealing any dynamical effect that affects the gravitational-wave polarization pattern, i.e, not only precession but also e.g., orbital eccentricity.

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