Approximate Recovery of Initial Point-like and Instantaneous Sources from Coarsely Sampled Thermal Fields via Infinite-Dimensional Compressed Sensing

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***Abstract*—We propose a method for resolving the positions of the initial source of heat propagation fields. The method relies on the recent theory of compressed sensing off the grid, i.e. TV-minimization. Based on the so-called soft recovery framework, we are able to derive rigorous theoretical guarantees of approximate recovery of the positions. Numerical experiments show a satisfactory performance of our method.**

***Index Terms*—Infinite-dimensional compressed sensing, Super-resolution, Thermal field, High-coherent dictionary.**

I. Introduction

The problem of resolving locations of thermal sources has application in a wide variety of fields. Examples include agriculture [1], climate change studies [2], thermal monitoring of CPUs [3], and protecting the environment [4]. In the literature, many strategies for attacking the problem of the thermal field and/or source estimation can be found. Let us summarize a few of the more recent ones:

In [5], [6], the authors have shown that the diffusion equation, which is the governing equation for thermal propagation phenomena, acts like a low-pass filter. This fact can be used to bound the aliasing error of the thermal reconstruction as a function of time samples and the number of sensors provided that the thermal field generated by initial sources.

The proposed mathematical spatio-temporal trade-off in [5] between the number of sensors and time samples has been further investigated in [7] as well as under a theoretical frameworks called “dynamic sampling” [8]–[10] and more recently in [11].

In order to address the problem of initial sources in [6] and extend it to time-varying sources, the authors have been extended their previous contribution in [12] by assuming the time-varying emissions rates lie in two specific low-dimensional subspaces [13, Page 134]. A similar extension of [6] has been done in [14] by assuming that the sources can induce in a time different than zero. Their method exploits a different mathematical framework called Prony’s method and can estimate the locations, time of inductions and also amplitudes of the instantaneous sources, under a separation condition on the induction times. Several extensions of their theory have been made such as the extension to other types of sources, i.e. straight-line and polygonal sources [15] or other governing equations such as wave and Poisson equation [16]. Their method requires relatively many samples and has not exploited the inherent sparsity of the problem.

This is instead done in [17] and then extended to two-dimensional sources in [18]. In these works the authors exploit the spatiotemporal correlation of the samples governed by diffusion equation as a side information into the recovery method, combined with a compressed sensing approach, utilizing the sparsity of thermal fields in the wavelet domain.

Another compressed sensing-based approach is carried out in [19]. The authors utilize a different kind of sparsity, which comes from the low number of generating sources (in particular, they use the analysis formulation of compressed sensing and co-sparsity concept). The proposed framework is then applied to acoustic and EEG source localization corresponding to sound wave and Poisson’s equation, respectively [20].

Browsing the literature, theoretical guarantees are found relatively scarcely. Furthermore, most scenarios assume a discretized setting. This can cause severe problems when dealing with sources which are not located on the discretization grid [21].

In this work, we will address these two issues. We propose to use ideas from infinite-dimensional (”off-the-grid“) compressed sensing to attack the thermal source localization problem. Although these methods are not per se new, this is the first time it is applied to thermal source localization. For this method, we are able to derive rigorous recovery statements using the so-called soft recovery framework of one of the authors [22]. We present and discuss the results of these endeavors in Section II. In Section III, we sketch a heuristic way of attentively discretizing the infinite-dimensional method we propose and test it in several settings (in particular both for one- and two-dimensional fields.)

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II. Problem Formulation and Theoretical Guarantees

The governing equation for thermal propagation is the heat equation:

$$\begin{cases} \partial_t u(t,x) - \Delta u(t,x) &= 0,\ t \in (0,T),\ x \in \mathbb{R}^2 \\ u(0,x) &= u_0,\ x \in \mathbb{R}^2. \end{cases} \tag{1}$$

Here $\Delta = \nabla^2 = \partial_{x_1}^2 + \partial_{x_2}^2$ is the Laplace operator and u_0 is the initial heat distribution. Assuming point-like initial (instantaneous) sources, it is reasonable to model u_0 as a short linear combinations of δ-peaks,

$$u_0 = \mu_0 = \sum_{i=1}^{s} c_i^0 \delta_{p_i}, \tag{2}$$

where $s \in \mathbb{N}$, $c_i^0 > 0$ and $p_i \in \mathbb{R}^2$ denote the number of sources, their amplitudes and locations, respectively. For simplicity, we will throughout the paper adopt the normalization assumption $\sum_{i=1}^{s} c_i^0 = 1$.

Using common inverse problem notions, μ_0 is the ground truth signal we want to recover. The indirect measurements from which we try to resolve μ_0 are samples of the solution of (1), say on a set $\mathcal{S}$. Considering that, as is well known, the solution of (1) is given as the convolution of the initial distribution with the Green function $G(x,t) = (4\pi t)^{-1} e^{-\frac{1}{2t}|x|^2}$, the measurements b are hence given by

$$b = \left(\int_{\mathbb{R}^2} G(p-x,t) d\mu_0(p) \right)_{(x,t)\in\mathcal{S}},$$

where $\mathcal{S}$ is a set of sampling points, whose structure we will specify later. Let M denote the operator which maps μ_0 to the measurement vector b.

Considering the structure (2) of the ground truth signal, recent findings in the field of infinite dimensional compressed sensing (super-resolution) [23], [24] suggest that TV-minimization should be used to recover it, where the TV-norm is defined as:

$$\|\mu\|_{TV} = \sup_{\substack{\bigcup_{i=1}^N U_i = \mathbb{R}^2 \\ U_i \text{ disjoint.}}} \sum_{i=1}^{N} |\mu(U_i)|;$$

As an example, the total variation norm of the measure μ_0 as in (2) is equal to $\|c\|_1$. Hence, one can intuitively view the TV norm as the infinite dimensional version of its counterpart ℓ_1-norm in the discrete setting. Therefore, in the noiseless case, the program

$$\min \|\mu\|_{TV} \text{ subject to } M\mu = b \tag{$\mathcal{P}_{TV}$}$$

suffices. In the case of b being contaminated with noise, one should instead use a regularized version of the above problem, e.g.

$$\min \|M\mu - b\|_2 \text{ subject to } \|\mu\|_{TV} \leq \rho. \tag{$\mathcal{P}_{TV}^{\rho,e}$}$$

Note that the optimal choice of $\rho = \|\mu_0\|_{TV}$. Since the TV-norm of μ_0 is most often unknown in practice, this is however not feasible, so that one has to calibrate ρ.

It is relatively easy to write down conditions which guarantee that the solution μ_0 of the problem $(\mathcal{P}_{TV})$ (or $(\mathcal{P}_{TV}^{\rho,e})$) is equal (or close to, respectively) to μ_0 sense (see e.g. [24]). It is however too hard to rigorously check this condition, and only a few examples of measurement operators where this is explicitly possible exist in the literature (most are variations of the ones considered in [23]).

Here, we instead aim to apply the soft recovery framework of one of the authors [22]. This framework describes a condition which guarantees that, for a specific peak p_{i_0} in the ground truth (2), there is a peak in the solution of $(\mathcal{P}_{TV})$ which is close to said peak. Although this statement is not as strong as the ones described in the previous paragraph, it is still of major importance, since it shows that an approximate initial source positions recovery is possible.

The publication [22] does not cover the case of noisy measurements. We are however able to extend the theory also to this case, as is presented in the extended version of this paper (see [25]).

Let us first describe the assumptions we need to make in order to prove our main result.

- The peak p_{i_0} we wish to recover is located in the rectangle $[-1/2, 1/2]^2$.
- The samples are all taken at a fixed time t and spatially on a uniform grid over $[-1,1]^2$ with spacing $\frac{1}{m}$ for some $m \in \mathbb{N}$, i.e.

$$x_n = \left(\tfrac{n_1}{m}, \tfrac{n_2}{m}\right), \quad n \in \{-m, \dots m\}^2. \tag{3}$$

The main result now reads as follows:

Theorem 1 (Main Result)**.** *Let $t > 0$ and suppose that the sample set $\mathcal{S}$ has the structure (3) with $m \gtrsim (1 + t^{-1/2})(c_{i_0}^0)^{-1}$. Then if $b = M\mu_0$, then for every minimizer μ_* of $\mathcal{P}_{TV}$, there exists a $p_* \in \operatorname{supp}\mu_*$ with $|p_0 - p^*| \leq \sqrt{4t^{-1}\log\left(2/c_{i_0}^0\right)}$.*

In fact, also in the case that $b = M\mu_0 + e$ with $\|e\|_2 \leq \epsilon$, the regularized problem $(\mathcal{P}_{TV}^{e,\rho})$ for every $\rho \geq 1$ has the following property: For every minimizer μ_* of $\mathcal{P}_{TV}^{e,\rho}$, there exists a $p_* \in \operatorname{supp}\mu_*$ with

$$|p^* - p_{i_0}| \leq \sqrt{4\Lambda \log\left(\left(\frac{c_{i_0}^0}{2} - \frac{6(2\epsilon + (\rho - 1))}{8\rho}\right)^{-1}\right)}.$$

Put shortly, the bound for the noiseless case deteriorates gracefully with a non-optimal choice of ρ and increasing noise level.

Theorem 1 provides a lower bound on the number of measurements $d = m^2$ needed to secure approximate recovery of the source positions. The bound grows with increasing t and decreasing amplitude c_{i_0}. This makes sense: the smaller $c_{i_0}^0$ is, the less significant is the peak and an increasing time t leads to the thermal field "smearing out, making the reconstruction of μ_0 harder. Also note that the bound on the source reconstruction precision deteriorates with increasing time and decreasing amplitude, due to the same reasons.

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Note that if all peaks are equally large, $c_{i_0}^{-1}$ exactly equals the sparsity s of the signal (remember that we assumed $\sum c_i = 1$) $m \gtrsim c_{i_0}^{-1}$ hence corresponds to $d \gtrsim s^2$, which unfortunately (in contrast of a linear dependence), is sub-optimal. This may well be an artifact of the proof, as may be the quality of the source reconstruction precision. We leave the question of whether an improvement is possible as an open problem.

Let us conclude this section by briefly commenting on the proof of Theorem 1 (a detailed proof is provided in the extended version of this paper [25]). To prove that the condition described in [22] is satisfied, one needs to construct a function lying in the range of the operator M^*, $\operatorname{ran} M^* = \operatorname{span}\left(G(\cdot - x, t), x, t \in \mathcal{S}\right)$, having certain properties - qualitatively, it needs to approximate the Gaussian $G(\cdot - p_i, t)$. Hence, the problem reduces to approximating the latter using the functions $G(\cdot - x, t)$, $(x, t) \in \mathcal{S}$. This problem can the further be transformed further into a Fourier series-approximation problem. This final task can then be tackled using classical techniques.

III. Experiments

The method we propose is of infinite-dimensional nature. In a few special cases (most notably for Fourier measurements [23], but see also e.g. [26], [27]), it is possible to rewrite the problem into a finite-dimensional one. In our case, it is not clear if this is possible, whence we instead propose the following heuristic:

For a grid X, we define the operator M_X as the restriction of M onto atomic measures with supports in X, or equivalently the following map:

$$M_X : \mathbb{R}^{|X|} \to \mathbb{R}^m, c \mapsto \sum_{x \in X} c_x M \delta_x$$

Note that the TV norm of a measure of the form $\sum_{x \in X} c_x \delta_x$ is equal to the ℓ_1 norm of the vector $c \in \mathbb{R}^{|X|}$. Hence, the problems $(\mathcal{P}_{TV})$ and $(\mathcal{P}_{TV}^{\rho,e})$ have the following discretized forms:

$$\min \|c\|_1 \text{ subject to } M_X c = b$$
$$\min \|M_X c - b\|_2 \text{ subject to } \|c\|_1 \leq \rho.$$

Choosing the grid is a delicate matter. Hoping to keep the size of X down while still putting many grid points in proximity of the actual sources, we use the following iterative approach: Starting with a coarse grid, we adaptively add grid points as follows: At every iteration, we consider the dual problem to our discretization procedure. The solution to that problem induces a so-called dual certificate. For the infinite dimensional problem, the locations of the δ-peaks in a solution $\mu_* = \sum_{i=1}^{s} c_i^* \delta_{x_i^*}$ are given as the set of points x_i for which the dual certificate ν has absolute value 1. Hence, by considering points in which the discrete dual certificate is large, we get an idea of where the actual peaks of the solutions are, and consequently add grid points close to them (See also figure 1).

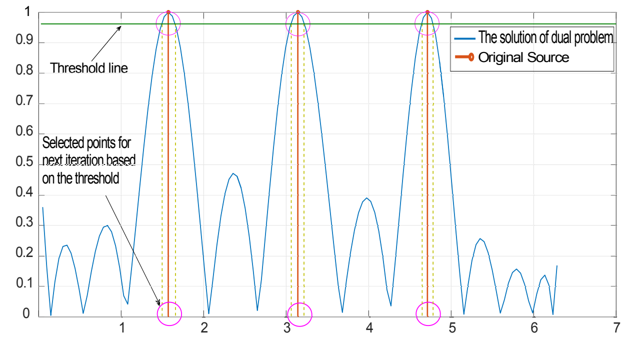

Fig. 1. As it can be seen, the points are selected based on the threshold which put on the peak solutions of the dual problem. The system model dictionary matrix is refined for the next iteration by adding extra atoms corresponding to these points.

The dual problem of the non-regularized problem $(\mathcal{P}_{TV})$ has a closed form:

$$\max_p \; Re\{< b, p >\} \text{ subject to } \|M_X^* p\|_\infty \leq 1. \quad (\mathcal{P}_{Dual,TV})$$

This is no longer the case for the problem $\mathcal{P}_{TV}^{\rho,e}$ [28]. Therefore, we deviate a bit from the theory presented above and instead, as is usual, consider the following, unconstrained version of the LASSO:

$$c^* = \min_c \tfrac{1}{2}\|M_X c - b\|_2^2 + \lambda \|c\|_1. \quad (4)$$

It is well known that for each parameter ρ, there is a λ such that the solution of $(\mathcal{P}_{TV}^{\rho,e})$ is equal to the one of $(\mathcal{P}_{LASSO})$ [29, Theorem B.28, p.562]. Considering the fact that the parameter ρ (or λ, respectively) anyhow needs to be fine-tuned, making this transition is justified. The dual problem of $(\mathcal{P}_{LASSO})$ again has a simple form:

$$\min_p \; \|\tfrac{b}{\lambda} - p\|_2 \text{ subject to } \|M_X^* p\|_\infty \leq 1. \quad (5)$$

Repeating the above procedure until a stopping criterion (of one's choice) is satisfied, we arrive at a final dual certificate ν_{final}. The points $X = (x_i)_{i=1}^p$ where this function has absolute value one are the estimates of the source localization. The amplitudes of the sources can then be directly estimated as $(M_X)^\dagger b$, where $(M_X)^\dagger$ denotes the Moore Penrose inverse of the operator M restricted to measures supported on the set X.

For the one-dimensional case, we conveniently use CVX [30], [31] to solve the optimization problem. Due to problems with e.g. storage in the two-dimensional setting, we instead use the primal-dual method [32] (since it in particular only needs matrix multiplications to be implemented as operators). The final grid is found by computing the local maximal points of the final certificate $|\nu_{final}|$, using gradient descent.

Figure 2 compares the performance of proposed method with the method proposed in [33]. The authors in [33] proposed a bound for sampling density which makes the forward operator well-conditioned, allowing the use of compressed sensing machinery to solve the inverse problem. In order to

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Algorithm 1: Summary of the method we propose

Data: A measurement operator $M : \mathcal{M} \to \mathbb{R}^d$, and (noisy) measurements $b \in \mathbb{R}^d$.

Result: An estimate μ_* of a sparse approximate solution to $M\mu = b$.

Initialize a course grid X.

repeat

Find a solution p to the dual problem of $(\mathcal{P}_{TV})$ or $(\mathcal{P}_{LASSO})$, respectively (depending on whether b is contaminated with noise or not), discretized to the grid X.

Find points $(q_i)_{i=1}^r$ in which $\nu = M_X^* p$ has large absolute value.

Refine X by adding points close to $(q_i)_{i=1}^r$

until *Stopping condition satisfied*;

Use final dual certificate to define final grid X_*

Output $\mu_* = (M_{X_*})^\dagger b$

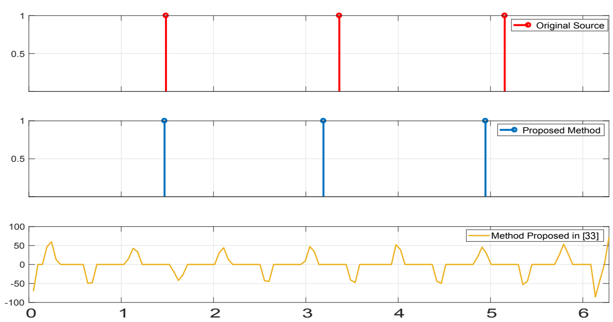

Fig. 2. The source localization performance results in a noisy scenario with SNR= $40dB$. In the experiment, the sampling density, ρ is chosen in the middle of the bound described in [33].

have a fair comparison, we select the sampling density in the middle of proposed bound in [33] and we carry our the experiment in one dimensional noisy setting with the signal to noise ratio (SNR) SNR $= 40\ dB$. Here, we define the SNR as the power of signal divided by the power of noise. As it can be seen the peak obtained using proposed method is almost close to the ground truth, while the reconstruction performance in [33] shows a period signal with a range between -100 to 100 due to the ill-conditioning property of the Green function. Figure 3 demonstrates the performance of the method in the case of two dimensions for different noise levels. The first row shows the performance of the method in (a) SNR= $0\ dB$, (b) SNR= $10\ dB$ and (c) SNR= $20\ dB$. As it can be seen, the method shows a good performance even in low SNR regions like $0\ dB$, and the localization performance is getting close to being exact as we increase the SNR. For having a better understanding of the performance of the proposed method in SNR= $0\ dB$, the second row illustrates (d) the original thermal field, (e) the course and noisy version of the thermal field which is captured by sensors and (f) the reconstructed filed using proposed method. The 2D view of this reconstruction for SNR= $0\ dB$ is also demonstrated in figure 4.

IV. Conclusion

In this paper, our aim is to exploit the infinite dimensional compressed sensing theory to solve thermal source localization problem. By utilizing soft recovery framework and its extension to noisy scenarios, we provided mathematical guarantees for showing that the placements of the thermal sources will be approximately recovered by our method. Based on the theory and its implementation, we can address different challenges, such as an insufficient number of spatiotemporal samples, high-coherent structure the measurements, discretization errors and finally off-grid sources positioning. Theoretically analyzing the convergence properties of the proposed algorithm, as well as experimentally test its performance on real data, are questions that should be addressed in future work.

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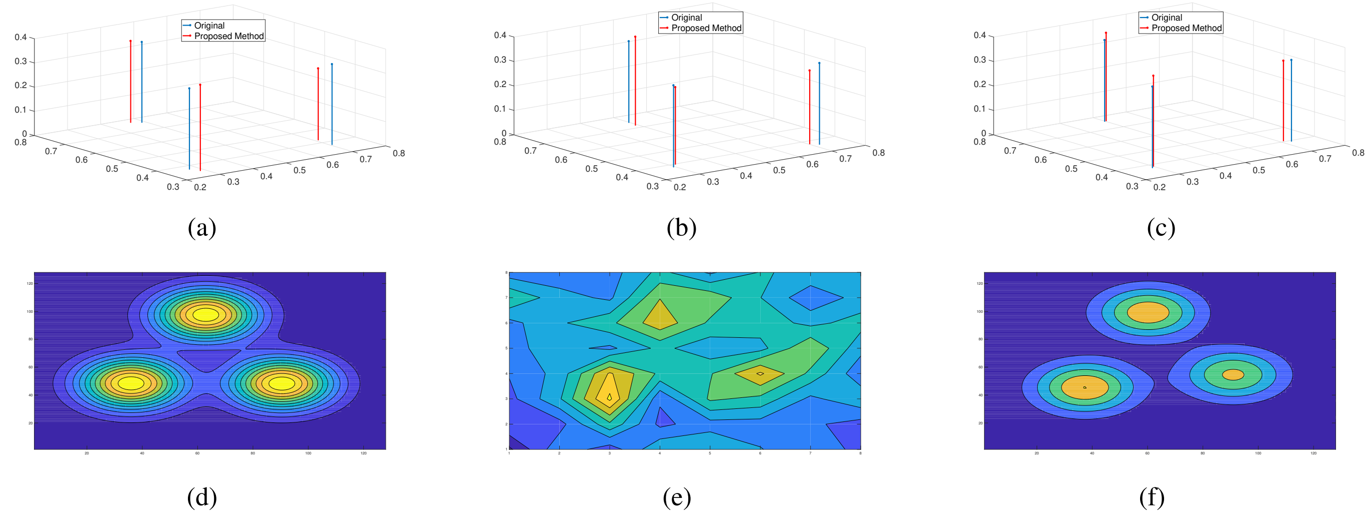

Fig. 3. The first row demonstrates the source localization performance of the proposed method in (a) SNR= $0\ dB$, (b) SNR= $10\ dB$ and (c) SNR= $20\ dB$. The second row illustrates (d) the original thermal field, (e) the courser and noisy version of thermal field which is captured by sensors and (f) the reconstructed filed using proposed method in SNR= $0\ dB$.

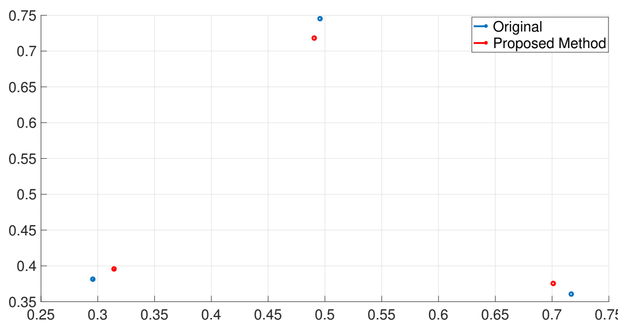

Fig. 4. Source localization performance of proposed method in two-dimensional setting for SNR= $0\ dB$.

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