On the Angular Resolution Limit Uncertainty

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***Abstract*—The Angular Resolution Limit (ARL), denoted by δ, is a key statistical quantity to measure our ability to resolve two closely-spaced narrowband far-field complex sources. In the literature, the ARL, denoted by δ_0, is systematically assumed to be perfectly known for mathematical convenience. In this work, our knowledge on the ARL is supposed to be only partial, meaning that $\delta \sim \mathcal{N}(\delta_0, \sigma_\delta^2)$. The degree of uncertainty is quantified by the ratio $\xi = \delta_0^2/\sigma_\delta^2$. Based on the Chernoff Upper Bound (CUB) on the minimal error probability, we show that the CUB is highly dependent on the degree of uncertainty, ξ. As by-product, the optimal s-value for which the CUB is the tightest upper bound is analytically studied.**

***Index Terms*—Angular Resolution Limit, model of uncertainty, upper bound on the error probability.**

I. INTRODUCTION

The resolvability of closely spaced signals, in terms of parameter of interest, for a given scenario (*e.g.*, for a given Signal-to-Noise Ratio (SNR), a given number of snapshots and/or a given number of sensors) is a former and challenging problem which was recently updated by Smith [1], Liu and Nehorai [3], Amar and Weiss [2] or Sharman and Milanfar [11]. More precisely, the concept of Statistical Resolution Limit (SRL), *i.e.*, the minimum distance between two closely spaced signals embedded in an additive noise that allows a correct resolvability/parameter estimation, is rising in several applications especially in problems such array processing [7], [8], [10], MIMO radar [4], [5], [6], or multidimensional harmonic estimation [9]. In this literature, the Angular Resolution Limit (ARL), denoted by δ_0, is always modelled as a perfectly known deterministic parameter. In practice, this assumption is somewhat unrealistic since generally, the knowledge of the ARL is only partial. It is clear that assuming a perfect knowledge of the ARL leads to too optimistic conclusions. In this work, the uncertainty on the ARL is taken into account modelling the ARL as a random variable such that $\delta \sim \mathcal{N}(\delta_0, \sigma_\delta^2)$. Consequently, the degree of uncertainty is quantified by the ratio $\xi = \delta_0^2/\sigma_\delta^2$. Indeed, for $\xi \to \infty$, $\delta \to \delta_0$ can be considered as perfectly known. On the contrary, for $\xi \to 0$, our degree of uncertainty tends to be maximal.

The detection performance for a random quantity in terms of minimal error probability is analytically intractable [12]. To alleviate this technical difficulty, we exploit some powerful tools from the theory of Information Geometry [14] and in particular the Chernoff Upper Bound (CUB) on the minimal error probability [13].

II. INFORMATION GEOMETRY FRAMEWORK

A. The Bayes' detection theory

Let $\Pr(\mathcal{H}_i)$ be the a priori hypothesis probability with $\Pr(\mathcal{H}_0)+\Pr(\mathcal{H}_1) = 1$. Let $\Pr(\mathbf{y}|\mathcal{H}_i)$ and $\Pr(\mathcal{H}_i|\mathbf{y})$ be the i-th conditional hypothesis and the posterior probabilities, respectively. The Bayes' detection rule chooses the hypothesis $\mathcal{H}_i$ associated with the largest posterior probability $\Pr(\mathcal{H}_i|\mathbf{y})$. Introduce the indicator hypothesis function according to $\phi(\mathbf{y}) \sim \mathrm{Bernou}(\alpha)$ where $\mathrm{Bernou}(\alpha)$ stands for the Bernoulli distribution of success probability $\alpha = \Pr(\phi(\mathbf{y}) = 1) = \Pr(\mathcal{H}_1)$. Function $\phi(\mathbf{y})$ is defined on $\mathcal{X} \to \{0, 1\}$ where $\mathcal{X}$ is the data-set of cardinality $|\mathcal{X}|$ enjoying the following decomposition $\mathcal{X} = \mathcal{X}_0 \cup \mathcal{X}_1$ where $\mathcal{X}_0 = \{\mathbf{y} : \phi(\mathbf{y}) = 0\} = \mathcal{X} \setminus \mathcal{X}_1$ and

$$\mathcal{X}_1 = \{\mathbf{y} : \phi(\mathbf{y}) = 1\} = \left\{\mathbf{y} : \Omega(\mathbf{y}) = \log \frac{\Pr(\mathcal{H}_1|\mathbf{y})}{\Pr(\mathcal{H}_0|\mathbf{y})} > 0\right\}$$

where $\Omega(\mathbf{y})$ is the log posterior-odds ratio. The average probability of error is

$$P_e = E_{\mathbf{y}}\left\{\Pr(\mathrm{Error}|\mathbf{y})\right\} \quad (1)$$

with

$$\Pr(\mathrm{Error}|\mathbf{y}) = \begin{cases} \Pr(\mathcal{H}_0|\mathbf{y}) & \text{if} \quad \mathbf{y} \in \mathcal{X}_1, \\ \Pr(\mathcal{H}_1|\mathbf{y}) & \text{if} \quad \mathbf{y} \in \mathcal{X}_0. \end{cases}$$

The standard strategy to minimize $\Pr(\mathrm{Error}|\mathbf{y})$ for a given $\mathbf{y}$ is $\min\{\Pr(\mathcal{H}_0|\mathbf{y}), \Pr(\mathcal{H}_1|\mathbf{y})\}$ [12]. So using (1), the minimal average error probability can be expressed as

$$P_e = E_{\mathbf{y}}\Big\{\min\{\Pr(\mathcal{H}_0|\mathbf{y}), \Pr(\mathcal{H}_1|\mathbf{y})\}\Big\}$$

Using Bayes' relation, we obtain

$$P_e = \int_{\mathcal{X}} \min\Big\{(1-\alpha)p_0(\mathbf{y}), \alpha p_1(\mathbf{y})\Big\} d\mathbf{y} \quad (2)$$

where $p_i(\mathbf{y}) = \Pr(\mathbf{y}|\mathcal{H}_i)$.

B. Chernoff Upper Bound (CUB) and asymptotic error exponent

Using the property that $\min\{x, z\} \leq x^s z^{1-s}$ with $x, z > 0$ and $s \in (0, 1)$ in (2), the minimal error probability is upper bounded according to

$$P_e \leq \frac{1-\alpha}{\beta^s} E_{\mathbf{y}}\left\{\exp[-C_{\mathbf{y}}(s)]\right\} \quad (3)$$

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where $\beta = \frac{1-\alpha}{\alpha}$ and

$$C_{\mathbf{y}}(s) = -\log \int_{\mathcal{X}} p_0(\mathbf{y})^{1-s} p_1(\mathbf{y})^s \mathrm{d}\mathbf{y} \quad (4)$$

is the (Chernoff) s-divergence. The term $C_{\mathbf{y}}(s)$ characterizes the exponential rate of the error exponent of P_e. The Chernoff information, denoted by $C_{\mathbf{y}}(s)$, is an asymptotic characterization on the best achievable Bayes' error probability. It is worth observing that the integral in (4) can be reformulated as

$$\begin{aligned}\int_{\mathcal{X}} p_0(\mathbf{y})^{1-s} p_1(\mathbf{y})^s \mathrm{d}\mathbf{y} &= \int_{\mathcal{X}} \frac{p_1(\mathbf{y})^s}{p_0(\mathbf{y})^s} p_0(\mathbf{y}) \mathrm{d}\mathbf{y} \\ &= \int_{\mathcal{X}} \exp\left[s\Gamma(\mathbf{y})\right] p_0(\mathbf{y}) \mathrm{d}\mathbf{y} \\ &= E_{\mathbf{y}|\mathcal{H}_0}\left\{\exp\left(s\Gamma(\mathbf{y})\right)\right\} \\ &= M_{\Gamma(\mathbf{y}|\mathcal{H}_0)}(s) \quad (5)\end{aligned}$$

where $\Gamma(\mathbf{y}) = \log\left(\frac{p_1(\mathbf{y})}{p_0(\mathbf{y})}\right)$ and $M_X(s)$ is the Moment Generating Function (MGF) of the random variable X.

III. Binary hypothesis test and large deviation analysis

A. Signal and noise models

Consider two far-field and narrowband complex sources denoted by $s_1(t)$ and $s_2(t)$ measured for the t-th snapshot. The observation on the ℓ-th sensor of an uniform linear array and for the t-th snapshot is given by

$$y_\ell(t) = s_1(t) \cdot [\mathbf{a}(\omega_1)]_\ell + s_2(t) \cdot [\mathbf{a}(\omega_2)]_\ell + w_\ell(t)$$

where $[\mathbf{a}(\omega_m)]_\ell = \exp[j \cdot \omega_m \cdot (\ell - 1)]$ for $1 \leq m \leq 2$, $1 \leq \ell \leq L$ with L the number of sensors. We are interested in quantifying our ability to resolve the two closely spaced sources $s_1(t)$ and $s_2(t)$. Each source collected over T snapshots is denoted by the vector $\mathbf{s}_1$ and $\mathbf{s}_2$, respectively. The noise $w_\ell(t)$ is assumed to be Gaussian distributed, temporally white (each noise snapshot is independent of the others), but spatially correlated such that, the collected noise over T snapshots $\mathbf{w}$ is $\mathbf{w} \sim \mathcal{CN}(\mathbf{0}, \sigma^2\mathbf{M})$.

Let $\delta = \omega_2 - \omega_1$ be the ARL between the two sources. The closely-spaced assumption means that δ is small. The detection problem of interest can be formulated as a binary hypothesis test as follows:

$$\begin{cases} \mathcal{H}_0 : & \delta = 0, \\ \mathcal{H}_1 : & \delta \neq 0. \end{cases} \quad (6)$$

As δ is small, by using the first order Taylor expansion around the so-called centre parameters $\omega_c = \frac{\omega_1+\omega_2}{2}$, we obtain $\mathbf{a}(\omega_1) \overset{1}{\approx} \mathbf{a}(\omega_c) - \frac{j}{2}\delta\dot{\mathbf{a}}(\omega_c)$ and $\mathbf{a}(\omega_2) \overset{1}{\approx} \mathbf{a}(\omega_c) + \frac{j}{2}\delta\dot{\mathbf{a}}(\omega_c)$, where symbol $\overset{1}{\approx}$ stands for first-order approximation and $\dot{\mathbf{a}}(\omega_c) = \frac{\partial \mathbf{a}(\omega_c)}{\partial \omega_c}$. We can write the linear $(TL) \times 1$ approximated vector as follows[1]

$$\mathbf{y} \overset{1}{\approx} \boldsymbol{\mu}_\delta + \mathbf{w}$$

[1]See [11], [8] for the full derivations and calculus.

where $\boldsymbol{\mu}_\delta = \mathbf{a}(\omega_c) \otimes (\mathbf{s}_1 + \mathbf{s}_2) + \frac{j}{2}\delta\dot{\mathbf{a}}(\omega_c) \otimes (\mathbf{s}_2 - \mathbf{s}_1)$.

Let us consider the case in which the two sources $\mathbf{s}_1$ and $\mathbf{s}_2$ and ω_c are known. We define the new observation vector $\mathbf{z} = \mathbf{y} - \mathbf{a}(\omega_\mathbf{c}) \otimes (\mathbf{s_1} + \mathbf{s_2})$. This assumption is realistic in a supervised system where the sources are pilot-assisted. In this case the hypothesis test (6) becomes

$$\begin{cases} \mathcal{H}_0 : & \mathbf{z} = \mathbf{w} \\ \mathcal{H}_1 : & \mathbf{z} = \delta\mathbf{p} + \mathbf{w} \end{cases}$$

where $\mathbf{p} = -\frac{j}{2}\dot{\mathbf{a}}(\omega_c) \otimes (\mathbf{s}_2 - \mathbf{s}_1)$.

In our scenario we suppose that we do not have full knowledge of the true angular distance δ between the two sources. We only known its mean value δ_0. To deal with this uncertainty, we model the amplitude of the vector $\mathbf{p}$ as a Gaussian random variable with mean value δ_0 and variance σ_δ^2, *i.e.* $\delta \sim \mathcal{N}(\delta_0, \sigma_\delta^2)$.

With this model we can now derive $\Gamma(\mathbf{z})$ and solve the integral in (5). It is possible to prove that

$$\begin{aligned}\Gamma(\mathbf{z}) &= \log\left(\frac{p_1(\mathbf{z})}{p_0(\mathbf{z})}\right) \\ &= \log\frac{|\mathbf{M_0}|}{|\mathbf{M_1}|} - (\mathbf{z} - \delta_0\mathbf{p})^H \mathbf{M}_1^{-1} (\mathbf{z} - \delta_0\mathbf{p}) + \mathbf{z}^H\mathbf{M}_0^{-1}\mathbf{z} \\ &= \log\frac{|\mathbf{M_0}|}{|\mathbf{M_1}|} - \mathbf{z}^H\left(\mathbf{M}_1^{-1} - \mathbf{M}_0^{-1}\right)\mathbf{z} \\ &+ 2\mathrm{Re}\left\{\delta_0\mathbf{z}^H\mathbf{M}_1^{-1}\mathbf{z}\right\} - \delta_0^2\mathbf{p}^H\mathbf{M}_1^{-1}\mathbf{p} \quad (7)\end{aligned}$$

where $\mathbf{M}_0 = \sigma^2\mathbf{M}$, $\mathbf{M}_1 = \sigma^2\mathbf{M} + \sigma_\delta^\mathbf{2}\mathbf{p}\mathbf{p}^H$ and $|\mathbf{R}|$ stands for the determinant of the matrix $\mathbf{R}$. Using Woodbury's identity [12] we can derive that

$$\mathbf{M}_1^{-1} = \mathbf{M}_0^{-1} - \frac{\sigma_\delta^2\mathbf{M}_0^{-1}\mathbf{p}\mathbf{p}^H\mathbf{M}_0^{-1}}{1 + \sigma_\delta^2\mathbf{p}^H\mathbf{M}_0^{-1}\mathbf{p}}. \quad (8)$$

Replacing eq. (8) in (7), we obtain

$$\begin{aligned}\Gamma(\mathbf{z}) &= \log\frac{|\mathbf{M_0}|}{|\mathbf{M_1}|} + \frac{\sigma_\delta^2\left|\mathbf{z}^H\mathbf{M}_0^{-1}\mathbf{p}\right|^2}{1 + \sigma_\delta^2\mathbf{p}^H\mathbf{M}_0^{-1}\mathbf{p}} \\ &+ 2\mathrm{Re}\left\{\frac{\delta_0\mathbf{z}^H\mathbf{M}_0^{-1}\mathbf{p}}{1 + \sigma_\delta^2\mathbf{p}^H\mathbf{M}_0^{-1}\mathbf{p}}\right\} - \frac{\delta_0^2\mathbf{p}^H\mathbf{M}_0^{-1}\mathbf{p}}{1 + \sigma_\delta^2\mathbf{p}^H\mathbf{M}_0^{-1}\mathbf{p}}. \quad (9)\end{aligned}$$

The key statistic that appears in the previous equations is $t = \mathbf{z}^H\mathbf{M}^{-1}\mathbf{p} = t_I + jt_q$, the output of a whitening matched filter [12], where $t_I = \mathrm{Re}\left\{\mathbf{z}^H\mathbf{M}^{-1}\mathbf{p}\right\}$ and $t_Q = \mathrm{Im}\left\{\mathbf{z}^H\mathbf{M}^{-1}\mathbf{p}\right\}$. Under the hypothesis $\mathcal{H}_0$, $E\left\{t_I|\mathcal{H}_0\right\} = E\left\{t_Q|\mathcal{H}_0\right\} = 0$, $var\left(t_I|\mathcal{H}_0\right) = var\left(t_I|\mathcal{H}_0\right) = \frac{\sigma^2}{2}\mathbf{p}^H\mathbf{M}^{-1}\mathbf{p}$ and the random variables t_I and t_Q are Gaussian distributed and independent [12].

Observing that $|\mathbf{M}_1|$=$|\mathbf{M}_0|\left(1 + \sigma_\delta^2\mathbf{p}^H\mathbf{M}_0^{-1}\mathbf{p}\right)$ and recalling that $\mathbf{M}_0 = \sigma^2\mathbf{M}$ we can rewrite eq. (9) as follows

$$\begin{aligned}\Gamma(\mathbf{z}) &= \log\frac{\sigma^2}{\sigma^2 + a} + \frac{\sigma_\delta^2}{\sigma^2(\sigma^2 + a)}\left(t_I^2 + t_Q^2\right) \\ &+ \frac{2\delta_0}{\sigma^2 + a}t_I - \frac{b}{\sigma^2 + a}\end{aligned}$$

where $a = \sigma_\delta^2 \mathbf{p}^H \mathbf{M}^{-1} \mathbf{p}$ and $b = \delta_0^2 \mathbf{p}^H \mathbf{M}^{-1} \mathbf{p}$. Grouping all the terms with t_I , after some calculations, we obtain

$$\Gamma(\mathbf{z}) = \log \frac{\sigma^2}{\sigma^2 + a} + \frac{\sigma_\delta^2}{\sigma^2 (\sigma^2 + a)} \left(t_I + \frac{\delta_0 \sigma^2}{\sigma_\delta^2} \right)^2$$

$$+ \frac{\sigma_\delta^2}{\sigma^2 (\sigma^2 + a)} t_Q^2 - \frac{\delta_0^2 \sigma^2}{\sigma_\delta^2 (\sigma^2 + a)} - \frac{b}{\sigma^2 + a}$$

then

$$E_{\mathbf{z}|\mathcal{H}_0} \{\exp(s\Gamma(\mathbf{z}))\} = \frac{(\sigma^2)^s}{(\sigma^2 + a)^s} \exp\left(-s \frac{\delta_0^2 \sigma^2 + b\sigma_\delta^2}{\sigma_\delta^2 (\sigma^2 + a)} \right) \cdot M_{y_1|\mathcal{H}_0}(s) M_{y_2|\mathcal{H}_0}(s) \quad (10)$$

where $Y_1|\mathcal{H}_0 = \frac{\sigma_\delta^2}{\sigma^2(\sigma^2+a)} \left(t_I + \frac{\delta_0 \sigma^2}{\sigma_\delta^2} \right)^2$ and $Y_2|\mathcal{H}_0 = \frac{\sigma_\delta^2}{\sigma^2(\sigma^2+a)} t_Q^2$.

We can prove that $Y_1|\mathcal{H}_0$ is a non-central random variable $\chi_1^2(d, \lambda)$ where $d = \frac{a}{2(\sigma^2+a)}$ is the scale parameter and $\lambda = 2\frac{\delta_0^2 \sigma^2}{a\sigma_\delta^2}$ in the non-centrality parameter.

Conversely $Y_2|\mathcal{H}_0$ is a central $\chi_1^2(d)$ random variable.

Now we are able to write $M_{y_1|\mathcal{H}_0}(s)$ and $M_{y_2|\mathcal{H}_0}(s)$ according to

$$M_{y_1|\mathcal{H}_0}(s) = \frac{1}{\sqrt{1 - \frac{a}{\sigma^2+a} s}} \exp\left(\frac{\delta_0^2}{\sigma_\delta^2} \frac{s\sigma^2}{\sigma^2 + a} \frac{1}{\left(1 - \frac{a}{\sigma^2+a} s\right)} \right) \quad (11)$$

and

$$M_{y_2|\mathcal{H}_0}(s) = \frac{1}{\sqrt{1 - \frac{a}{\sigma^2+a} s}} \quad (12)$$

For ease, let define the SNR at the output of the whitening matched filter $t = \mathbf{z}^H \mathbf{M}^{-1} \mathbf{p}$ according to

$$SNR = \gamma \frac{E\left\{ \left| \delta \mathbf{p}^H \mathbf{M}^{-1} \mathbf{p} \right|^2 \right\}}{E\left\{ \left| \mathbf{d}^H \mathbf{M}^{-1} \mathbf{p} \right|^2 \right\}} = \frac{(\delta_0^2 + \sigma_\delta^2)}{\sigma^2} \mathbf{p}^H \mathbf{M}^{-1} \mathbf{p} = \frac{a + b}{\sigma^2}.$$

To further simplify eq. (10) we observe that $\xi \delta_0^2 / \sigma_\delta^2 = b/a$, then $a = b/\xi$ and $\sigma^2 = b(\xi + 1)/(\xi\gamma)$. With this notation and replacing (11) and (12) in (10) we obtain

$$E_{\mathbf{z}|\mathcal{H}_0} \{\exp(s\Gamma(\mathbf{z}))\} = \frac{(1 + \xi)^s}{(1 + \xi + \gamma)^s} \frac{\exp(-s\xi)}{1 - s\frac{\gamma}{1+\xi+\gamma}} \cdot \exp\left[\frac{\xi(1 + \xi)s}{1 + \xi + \gamma(1 - s)} \right] \quad (13)$$

Replacing now (13) in (3), we finally get the CUB

$$P_e \le \frac{1 - \alpha}{\beta^s} \frac{(1 + \xi)^s}{(1 + \xi + \gamma)^s} \frac{1}{1 - s\frac{\gamma}{1+\xi+\gamma}} \cdot \exp\left[-\frac{\xi\gamma(1 - s)s}{1 + \xi + \gamma(1 - s)} \right]. \quad (14)$$

TABLE I

s_{min} VS ξ

ξ	0.1	0.2	0.5	1	2	5	10
$\beta = 1$	0.8565	0.857	0.858	0.861	0.869	0.883	0.884
$\beta = 0.333$	0.8291	0.8293	0.831	0.837	0.848	0.870	0.875

From eq. (14) it is evident that the CUB does not depend on δ_0^2 and σ_δ^2 separately, but only on their ratio ξ and on γ.

For low values of SNR, $\gamma << 1$. Under this hypothesis eq. (14) can be approximated as

$$P_e \le \frac{1 - \alpha}{\beta^s} \exp\left[\frac{\xi\gamma}{1 + \xi} s(-1 + s) \right]$$

and the value of s for which the bound is minimum is $s = \frac{1}{2} + \frac{1+\xi}{2\xi\gamma} \log(\beta)$ (provided that $s > 0$). When $\Pr(\mathcal{H}_0) = \Pr(\mathcal{H}_1)$, $\beta = 1$ and $s = \frac{1}{2}$.

In the more general case s_{min} can be calculated as the unique solution in the range $(0, 1)$ of the 2nd order equation

$$s^2 - \frac{\gamma^2 - 2K_0\gamma(1 + \xi + \gamma)}{K_0\gamma^2} s + \frac{K_0(1 + \xi + \gamma)^2 - (1 + \xi + \gamma)((1 + \xi)\xi + \gamma)}{K_0\gamma^2} = 0 \quad (15)$$

where $K_0 = \left(\xi - log \frac{1+\xi}{(1+\xi+\gamma)\beta} \right)$. This equation has been derived by calculating the log of the CUB and then by derivating it with respect to s.

B. Analysis of the results

In order to vary the uncertainty on the value of the ARL and to analyse its impact on the CUB, we have defined the parameter $\xi = \delta_0^2 / \sigma_\delta^2$, that is, the ratio between the square mean value of the ARL and its variance. For low values of ξ the variance of the ARL is large, so the uncertainty is high. Conversely, for high values of ξ the uncertainty is small.

In the following figures 1 and 2 we show the s_{min} as a function of the SNR, with $\beta = 1$ ($\alpha = 0.5$, the two hypotheses are equiprobable) and $\beta = 0.333$ ($\alpha = 0.75$) respectively, for different values of the parameter ξ. As expected for low values of SNR and $\beta = 1$ $s_{min} = 0.5$. It is worth observing that the behaviour of s_{min} , derived from eq. (15), highly depends on ξ . The values of s_{min} increases monotonically with SNR for every value of ξ and of β.

In figure 3 the corresponding CUB is plotted as a function of ξ for $SNR = 30\,\text{dB}$. It is clear again from these curves that the impact of the uncertainty on the CUB is very large. Passing from $\xi = 1$, for which, for instance, $\delta_0 = 0.1$ and $\sigma_\delta^2 = 10^{-2}$, to $\xi = 10$, for which $\delta_0 = 0.1$ and $\sigma_\delta^2 = 10^{-3}$, the CUB changes of more than 2 orders of magnitude. We can then conclude that, considering the ARL always deterministic and known can be very optimistic.

The values of s_{min} corresponding to some of the tested values of ξ in figure 3 are reported in Table I

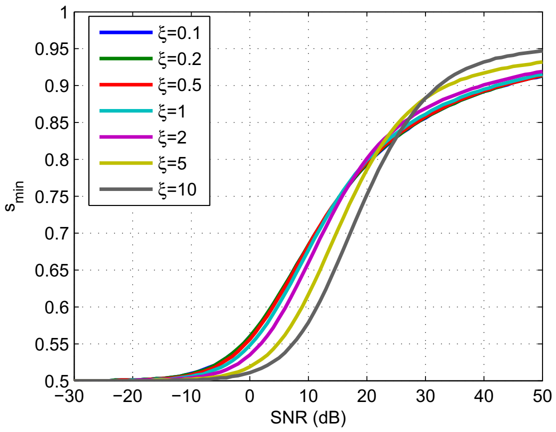

Fig. 1. s_{min} vs SNR, $\beta = 1$

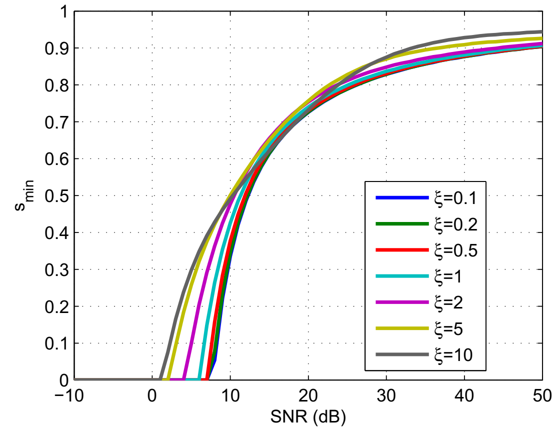

Fig. 2. s_{min} vs SNR, $\beta = 0.333$

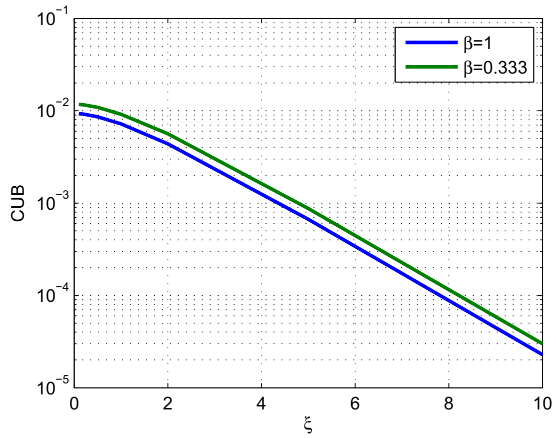

Fig. 3. CUB vs ξ, $SNR = 30\,\text{dB}$

IV. Conclusion

Quantifying the resolution of two closed-spaced sources is a fundamental problem at the heart of challenging applications. The Angular Resolution Limit (ARL) measures our ability to resolve two closely-spaced sources in the context of array processing. Usually in the literature, the ARL is supposed perfectly known, *i.e.*, modelled as a deterministic variable. In this work, we choose to relax this too severe assumption. Indeed, our new ARL is modelled as a random variable such as $\delta \sim \mathcal{N}(\delta_0, \sigma_\delta^2)$. In this paper the degree of uncertainty has been quantified by the ratio $\xi = \delta_0^2/\sigma_\delta^2$. Large (small) ratio means low (high) uncertainty. Based on the Chernoff Upper Bound (CUB) on the minimal error probability, we show that the ARL is highly dependent on the ratio ξ. As a by product, the optimal s-value for which the CUB is the tightest upper bound is analytically studied and its independence of δ_0 is proved. In future work we will consider the more general case of non-Gaussian noise.

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