Octonion Spectrum of 3D Octonion-Valued Signals – Properties and Possible Applications

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***Abstract*—The aim of this paper is to investigate the properties of octonion Fourier transform (OFT) of octonion-valued functions of three variables and its potential applications in signal processing. This work has been inspired by the original papers by Hahn and Snopek concerning octonion Fourier transform definition and its applications in the analysis of the hypercomplex analytic signals. First, the generalization of the OFT definition to the octonion-valued functions is provided, and then the octonion analogues of classical Fourier transform properties are derived, e.g. argument scaling, modulation and shift theorem. Finally, the results are illustrated with some examples that indicate possible applications.**

I. Introduction

The classical signal theory deals with real- or complex-valued time series (or images). However, in some practical applications, signals are represented by more abstract structures, e.g. hypercomplex algebras [10]. Quaternions and octonions deserve special attention in this considerations. Recently, they drew scientists' attention due to their numerous applications [7]. Quaternions are used in two different ways – to describe a vector-valued signal (with three or four coordinates) of one variable, i.e.

$$u(t) = u_0(t) + u_1(t)\mathbf{i} + u_2(t)\mathbf{j} + u_3(t)\mathbf{k},$$

where $u_0, u_1, u_2, u_3 \colon \mathbb{R} \to \mathbb{R}$, or to analyse a scalar signal of two variables, i.e. $u\colon \mathbb{R}^2 \to \mathbb{R}$. The basic tool in the second approach is the quaternion Fourier transform (QFT) [2]:

$$U_{\mathrm{QFT}}(f_1, f_2) = \int_{\mathbb{R}}\int_{\mathbb{R}} u(t_1, t_2) e^{-2\pi \mathbf{i} f_1 t_1} e^{-2\pi \mathbf{j} f_2 t_2}\,\mathrm{d}t_1 \mathrm{d}t_2. \qquad \text{(I.1)}$$

It allows us (in contrast to the classical 2D Fourier transform) to analyse two dimensions of the sampling grid independently. Each time-like dimension can be associated with a different dimension of the 4D quaternion space, while the complex transform mixes those two dimensions. This property enables us to use the Fourier transform in the analysis of some two-dimensional linear time-invariant systems described by systems of partial differential equations with constant coefficients [6]. It also allows us to study some symmetries present in certain signals (images), which was impossible before [7].

In the last few years some generalizations of the Fourier transform to the octonion and higher-order algebras appeared in the literature [8], [11]. The main goal of this paper is further development of such generalization based on octonions. Analysis of the current state of knowledge on applications of octonions in the signal processing shows some areas previously unexplored or requiring thorough theoretical and experimental studies.

In our previous investigations [1] we were able to show that the OFT is well defined (i.e. we proved the inverse transform theorem) for scalar (real-valued) functions of three variables. In our research we also derived some properties of the OFT, analogous to the properties of the classical (complex) and quaternion Fourier transform, e.g. symmetry properties (analogue to the Hermitian symmetry properties), shift theorem, Plancherel and Parseval theorems, and Wiener-Khintchine theorem. Proofs of those theorems were based on the previous research of Hahn and Snopek, who used the fact that real–valued functions can be expressed as a sum of components of different parity [8].

The paper is organized as follows. In Section II we recall the octonion algebra, its basic properties, the definition of the octonion Fourier transform and proof of its well-posedness. In Sections III and IV we focus on deriving some important properties of the OFT, e.g. argument scaling, modulation and shift theorems which lead to some remarks on Parseval and Plancherel Theorems. The paper is concluded with a short discussion of the obtained results.

II. Basic definitions

A. The octonion algebra

An octonion $o \in \mathbb{O}$ can be defined as [3]

$$o = r_0 + r_1\mathbf{e}_1 + \ldots + r_7\mathbf{e}_7, \quad r_0, r_1, \ldots, r_7 \in \mathbb{R}, \qquad \text{(II.1)}$$

where $\mathbf{e}_1, \ldots, \mathbf{e}_7$ are seven different imaginary units. Rules of octonion multiplication are presented in Tab. I. Number $r_0 \in \mathbb{R}$ is called the real part of o (and denoted as $\mathrm{Re}\, o$) and the pure imaginary octonion $r_1\mathbf{e}_1 + \ldots + r_7\mathbf{e}_7$ is called the imaginary part of o (and denoted as $\mathrm{Im}\, o$). Octonions form a *non-associative* and a *non-commutative* algebra, which means that in general for o_1, o_2, $o_3 \in \mathbb{O}$ we have

$$(o_1 \cdot o_2) \cdot o_3 \neq o_1 \cdot (o_2 \cdot o_3), \qquad o_1 \cdot o_2 \neq o_2 \cdot o_1.$$

However, for any o_1, $o_2 \in \mathbb{O}$ we have $(o_1 \cdot o_2)^* = o_2^* \cdot o_1^*$, where * is the octonion conjugate, i.e. $o^* = \mathrm{Re}\, o - \mathrm{Im}\, o$. We also define the octonion module by $|o| = \sqrt{o \cdot o^*}$. What is

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TABLE I
MULTIPLICATION RULES IN OCTONION ALGEBRA.

$\cdot$	1	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$	$\mathbf{e}_4$	$\mathbf{e}_5$	$\mathbf{e}_6$	$\mathbf{e}_7$
1	1	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$	$\mathbf{e}_4$	$\mathbf{e}_5$	$\mathbf{e}_6$	$\mathbf{e}_7$
$\mathbf{e}_1$	$\mathbf{e}_1$	-1	$\mathbf{e}_3$	$-\mathbf{e}_2$	$\mathbf{e}_5$	$-\mathbf{e}_4$	$-\mathbf{e}_7$	$\mathbf{e}_6$
$\mathbf{e}_2$	$\mathbf{e}_2$	$-\mathbf{e}_3$	-1	$\mathbf{e}_1$	$\mathbf{e}_6$	$\mathbf{e}_7$	$-\mathbf{e}_4$	$-\mathbf{e}_5$
$\mathbf{e}_3$	$\mathbf{e}_3$	$\mathbf{e}_2$	$-\mathbf{e}_1$	-1	$\mathbf{e}_7$	$-\mathbf{e}_6$	$\mathbf{e}_5$	$-\mathbf{e}_4$
$\mathbf{e}_4$	$\mathbf{e}_4$	$-\mathbf{e}_5$	$-\mathbf{e}_6$	$-\mathbf{e}_7$	-1	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$
$\mathbf{e}_5$	$\mathbf{e}_5$	$\mathbf{e}_4$	$-\mathbf{e}_7$	$\mathbf{e}_6$	$-\mathbf{e}_1$	-1	$-\mathbf{e}_3$	$\mathbf{e}_2$
$\mathbf{e}_6$	$\mathbf{e}_6$	$\mathbf{e}_7$	$\mathbf{e}_4$	$-\mathbf{e}_5$	$-\mathbf{e}_2$	$\mathbf{e}_3$	-1	$-\mathbf{e}_1$
$\mathbf{e}_7$	$\mathbf{e}_7$	$-\mathbf{e}_6$	$\mathbf{e}_5$	$\mathbf{e}_4$	$-\mathbf{e}_3$	$-\mathbf{e}_2$	$\mathbf{e}_1$	-1

more important, the algebra of octonions is *alternative*, which means that for any $o_1,\, o_2 \in \mathbb{O}$ we have

$$o_1 \cdot (o_1 \cdot o_2) = (o_1 \cdot o_1) \cdot o_2, \quad (o_1 \cdot o_2) \cdot o_2 = o_1 \cdot (o_2 \cdot o_2).$$

Similarly as for the complex numbers and quaternions, we define the octonion exponential function as the infinite sum $e^o := \sum_{k=0}^{\infty} \frac{o^k}{k!}$. Due to the fact, that octonions are non-commutative, for any $o_1, o_2 \in \mathbb{O}$ we have

$$e^{o_1+o_2} = e^{o_1} \cdot e^{o_2} \qquad \text{if and only if} \qquad o_1 \cdot o_2 = o_2 \cdot o_1.$$

We can also write for any $\alpha \in \mathbb{R}$ that

$$\cos\alpha = \frac{e^{\boldsymbol{\mu}\alpha} + e^{-\boldsymbol{\mu}\alpha}}{2}, \quad \sin\alpha = \frac{e^{\boldsymbol{\mu}\alpha} - e^{-\boldsymbol{\mu}\alpha}}{2\boldsymbol{\mu}}, \tag{II.2}$$

and $e^{\boldsymbol{\mu}\alpha} = \cos\alpha + \boldsymbol{\mu}\sin\alpha$, where $\boldsymbol{\mu}$ is any octonion such that $|\boldsymbol{\mu}| = 1$ and $\operatorname{Re}\boldsymbol{\mu} = 0$. For the clarity of the formulas that will follow, we introduce the notation:

$$\begin{aligned} &\mathrm{e}_j^{\alpha} = e^{\mathbf{e}_j 2\pi\alpha}, \quad j = 1, \ldots, 7, \\ &\mathrm{s}^{\alpha} = \sin(2\pi\alpha), \quad \mathrm{c}^{\alpha} = \cos(2\pi\alpha). \end{aligned}$$

B. The octonion Fourier Transform

This section is devoted to the definition of the octonion Fourier transform (OFT) of the $\mathbb{O}$-valued function of three variables. This definition was introduced in [9] and used in later publications concerning theory of hypercomplex analytic functions [8–11]. In [1] we proved that the OFT of $\mathbb{R}$-valued function is well-defined and has some interesting properties (such as the analogue of the Hermitian symetry), but (to the best of our knowledge) the general definition for the $\mathbb{O}$-valued functions has not yet been proved to be correct.

Consider the $\mathbb{O}$-valued function $u\colon \mathbb{R}^3 \to \mathbb{O}$, i.e.

$$u(\mathbf{x}) = u_0(\mathbf{x}) + u_1(\mathbf{x})\mathbf{e}_1 + \ldots + u_7(\mathbf{x})\mathbf{e}_7,$$

where $u_i\colon \mathbb{R}^3 \to \mathbb{R}$, $i = 0, 1, \ldots, 7$, $\mathbf{x} = (x_1, x_2, x_3)$. The octonion Fourier transform of $u(\mathbf{x})$ is defined by

$$U_{\mathrm{OFT}}(\mathbf{f}) = \int_{\mathbb{R}^3} u(\mathbf{x}) \cdot \mathrm{e}_1^{-f_1 x_1} \cdot \mathrm{e}_2^{-f_2 x_2} \cdot \mathrm{e}_4^{-f_3 x_3}\, \mathrm{d}\mathbf{x}, \tag{II.3}$$

where $\mathbf{f} = (f_1, f_2, f_3)$. Since the octonion algebra is non-associative it should be noted that the multiplication in the above integrals is done from left to right. Choice and order of imaginary units in the exponents is not accidental, as we explained in [1]. Conditions of existence of the integral (II.3) are the same as for the classical (complex) Fourier transform and we will omit the details here. We will focus on the question of invertibility of the OFT. For the special case of $u\colon \mathbb{R}^3 \to \mathbb{R}$ we proved the following theorem in [1].

Theorem II.1. *Let $u\colon \mathbb{R}^3 \to \mathbb{O}$ be a continuous and square-integrable function. Then*

$$u(\mathbf{x}) = \int_{\mathbb{R}^3} U_{\mathrm{OFT}}(\mathbf{f}) \cdot \mathrm{e}_4^{f_3 x_3} \cdot \mathrm{e}_2^{f_2 x_2} \cdot \mathrm{e}_1^{f_1 x_1}\, \mathrm{d}\mathbf{f}$$

(where multiplication is performed from left to right).

The result follows from the Fourier Integral Theorem [4].

Theorem II.2. *Let $u\colon \mathbb{R}^n \to \mathbb{R}$ be a continuous and square-integrable function (in the Lebesgue sense). Then*

$$u(\mathbf{x}) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} u(\mathbf{y}) \cdot e^{2\pi \mathbf{i}\, \mathbf{f}\cdot(\mathbf{x}-\mathbf{y})}\, \mathrm{d}\mathbf{y}\, \mathrm{d}\mathbf{f},$$

where $\mathbf{i} = \mathbf{e}_1$ is complex imaginary unit, $\mathbf{y} = (y_1, \ldots, y_n)$.

Proof of Theorem II.1. We need to prove the following:

$$\begin{aligned} u(\mathbf{x}) = \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} & u(\mathbf{y}) \cdot \mathrm{e}_1^{-f_1 y_1} \cdot \mathrm{e}_2^{-f_2 y_2} \cdot \mathrm{e}_4^{-f_3 y_3} \\ & \cdot \mathrm{e}_4^{f_3 x_3} \cdot \mathrm{e}_2^{f_2 x_2} \cdot \mathrm{e}_1^{f_1 x_1}\, \mathrm{d}\mathbf{y}\mathrm{d}\mathbf{f}, \end{aligned}$$

where $\mathbf{y} = (y_1, y_2, y_3)$ and multiplication is done from left to right. Writing u as a sum $u = u_0 + u_1\mathbf{e}_1 + \ldots + u_7\mathbf{e}_7$ and using the distributive law on the set of $\mathbb{O}$ we see that the claim of the theorem is equivalent to the system of equations

$$\begin{aligned} u_0(\mathbf{x}) = \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} & u_0(\mathbf{y}) \cdot \mathrm{e}_1^{-f_1 y_1} \cdot \mathrm{e}_2^{-f_2 y_2} \cdot \mathrm{e}_4^{-f_3 y_3} \\ & \cdot \mathrm{e}_4^{f_3 x_3} \cdot \mathrm{e}_2^{f_2 x_2} \cdot \mathrm{e}_1^{f_1 x_1}\, \mathrm{d}\mathbf{y}\, \mathrm{d}\mathbf{f}, \end{aligned} \tag{II.4}$$

$$\begin{aligned} u_i(\mathbf{x})\mathbf{e}_i = \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} & u_i(\mathbf{y})\mathbf{e}_i \cdot \mathrm{e}_1^{-f_1 y_1} \cdot \mathrm{e}_2^{-f_2 y_2} \cdot \mathrm{e}_4^{-f_3 y_3} \\ & \cdot \mathrm{e}_4^{f_3 x_3} \cdot \mathrm{e}_2^{f_2 x_2} \cdot \mathrm{e}_1^{f_1 x_1}\, \mathrm{d}\mathbf{y}\, \mathrm{d}\mathbf{f}, \end{aligned} \tag{II.5}$$

where $i = 1, \ldots, 7$.

Proof of (II.4) can be found in [1]. We only need to prove (II.5). We use the fact that for any imaginary unit $\mathbf{e}_i$, $i = 1, \ldots, 7$, we have

$$\begin{aligned} &\Big(\big((\mathbf{e}_i \cdot \mathrm{e}_1^{-f_1 y_1}) \cdot \mathrm{e}_2^{-f_2 y_2} \big) \cdot \mathrm{e}_4^{-f_3 y_3} \Big) \cdot \mathrm{e}_4^{f_3 x_3} \\ &\qquad = \big((\mathbf{e}_i \cdot \mathrm{e}_1^{-f_1 y_1}) \cdot \mathrm{e}_2^{-f_2 y_2} \big) \cdot \big(\mathrm{e}_4^{-f_3 y_3} \cdot \mathrm{e}_4^{f_3 x_3} \big), \end{aligned} \tag{II.6}$$

$$\begin{aligned} &\big((\mathbf{e}_i \cdot \mathrm{e}_1^{-f_1 y_1}) \cdot \mathrm{e}_2^{-f_2 y_2} \big) \cdot \mathrm{e}_2^{f_2 x_2} \\ &\qquad = (\mathbf{e}_i \cdot \mathrm{e}_1^{-f_1 y_1}) \cdot (\mathrm{e}_2^{-f_2 y_2} \cdot \mathrm{e}_2^{f_2 x_2}) \end{aligned} \tag{II.7}$$

$$(\mathbf{e}_i \cdot \mathrm{e}_1^{-f_1 y_1}) \cdot \mathrm{e}_1^{f_1 x_1} = \mathbf{e}_i \cdot (\mathrm{e}_1^{-f_1 y_1} \cdot \mathrm{e}_1^{f_1 x_1}). \tag{II.8}$$

By following the same steps as in [1] and using for each $i = 1, \ldots, 7$ equation (II.6), Fubini's Theorem and Theorem II.2, then (II.7), Fubini's Theorem and Theorem II.2, and finally equation (II.8), Fubini's Theorem and Theorem II.2 we prove (II.5). □

In the remaining part of this paper we will assume that functions u and v have well-defined and invertible octonion Fourier transforms which will be denoted by U_{OFT} and V_{OFT}, respectively. We will also denote $\mathcal{F}_{\mathrm{OFT}}\{u\}$ as the OFT of function u. Considerations about properties of OFT should begin with the elementary result – octonion Fourier transofrm is $\mathbb{R}$-linear operation, i.e. for $a, b \in \mathbb{R}$ we have

$$\mathcal{F}_{\text{OFT}}\{a \cdot u + b \cdot v\} = a \cdot \mathcal{F}_{\text{OFT}}\{u\} + b \cdot \mathcal{F}_{\text{OFT}}\{v\}.$$

It should be noted here that OFT is not $\mathbb{O}$-linear, i.e. the abovementioned formula is not true for every $a, b \in \mathbb{O}$.

Octonion Fourier transform can be computed using formula (II.3), however in the case of $\mathbb{R}$- and $\mathbb{C}$-valued functions we can use classical (complex) Fourier transform. In particular, the following theorem holds, which is the generalization of the result of [11], where it was proved for $\mathbb{R}$-valued functions. For the clarity of the formulas we will use the notation:

$$\mathbf{f}_{ijk} = (if_1, jf_2, kf_3),$$

where $i, j, k \in \{+, -\}$, e.g. $\mathbf{f}_{+-+} = (f_1, -f_2, f_3)$.

Theorem II.3. *Let $u \colon \mathbb{R}^3 \to \mathbb{C}$, and denote $U = \mathcal{F}_{\text{CFT}}\{u\}$, $U_{\text{OFT}} = \mathcal{F}_{\text{OFT}}\{u\}$. Then*

$$\begin{aligned} &U_{\text{OFT}}(\mathbf{f}) \\ &= \frac{1}{4}\big(U(\mathbf{f}_{+++}) \cdot (1 - \mathbf{e}_3) + U(\mathbf{f}_{+-+}) \cdot (1 + \mathbf{e}_3)\big) \cdot (1 - \mathbf{e}_5) \\ &\quad + \frac{1}{4}\big(U(\mathbf{f}_{++-}) \cdot (1 - \mathbf{e}_3) + U(\mathbf{f}_{+--}) \cdot (1 + \mathbf{e}_3)\big) \cdot (1 + \mathbf{e}_5) \end{aligned}$$

where octonion multiplication is done from left to right.

Proof. In this proof we will carefully follow and modify steps presented in [11]. From the definition of the classical Fourier transform and equation (II.2) with $\boldsymbol{\mu} = \mathbf{e}_1$ we get

$$U(f_1, f_2, f_3) = \int_{\mathbb{R}^3} u(\mathbf{x}) \mathrm{e}_1^{-f_1 x_1} \mathrm{e}_1^{-f_2 x_2} \mathrm{e}_1^{-f_3 x_3} \, \mathrm{d}\mathbf{x}.$$

Then

$$\frac{1}{2}\big(U(\mathbf{f}_{+++}) + U(\mathbf{f}_{+-+})\big) = \int_{\mathbb{R}^3} u(\mathbf{x}) \mathrm{e}_1^{-f_1 x_1} \mathrm{c}^{f_2 x_2} \mathrm{e}_1^{-f_3 x_3} \, \mathrm{d}\mathbf{x}, \tag{II.9}$$

$$\begin{aligned} &\frac{1}{2}\big(U(\mathbf{f}_{+++}) - U(\mathbf{f}_{+-+})\big) \\ &\quad = \int_{\mathbb{R}^3} u(\mathbf{x}) \mathrm{e}_1^{-f_1 x_1} (-\mathbf{e}_1 \mathrm{s}^{f_2 x_2}) \mathrm{e}_1^{-f_3 x_3} \, \mathrm{d}\mathbf{x}. \end{aligned} \tag{II.10}$$

From the fact that for $o = r_0 + r_1 \mathbf{e}_1$ we have

$$\big((o \cdot \mathbf{e}_1) \cdot \mathrm{e}_1^{f_3 x_3}\big) \cdot \mathbf{e}_3 = \big(o \cdot (\mathbf{e}_1 \cdot \mathbf{e}_3)\big) \cdot \mathrm{e}_1^{-f_3 x_3}$$

(because the multiplication is alternative) it follows that

$$\begin{aligned} &\frac{1}{2}\big(U(\mathbf{f}_{++-}) - U(\mathbf{f}_{+--})\big)\mathbf{e}_3 \\ &\quad = \int_{\mathbb{R}^3} u(\mathbf{x}) \mathrm{e}_1^{-f_1 x_1} (\mathbf{e}_2 \mathrm{s}^{f_2 x_2}) \mathrm{e}_1^{-f_3 x_3} \, \mathrm{d}\mathbf{x}. \end{aligned} \tag{II.11}$$

Subtracting (II.11) from (II.9) we then obtain

$$\begin{aligned} &\frac{1}{2}\big(U(\mathbf{f}_{+++}) + U(\mathbf{f}_{+-+})\big) + \frac{1}{2}\big(U(\mathbf{f}_{+--}) - U(\mathbf{f}_{++-})\big)\mathbf{e}_3 \\ &\quad = \int_{\mathbb{R}^3} u(\mathbf{x}) \mathrm{e}_1^{-f_1 x_1} \mathrm{e}_2^{-f_2 x_2} \mathrm{e}_1^{-f_3 x_3} \, \mathrm{d}\mathbf{x} =: V(\mathbf{f}). \end{aligned} \tag{II.12}$$

By following similar steps applied to function V we get

$$\begin{aligned} &\frac{1}{2}\big(V(\mathbf{f}_{+++}) + V(\mathbf{f}_{++-})\big) + \frac{1}{2}\big(V(\mathbf{f}_{+--}) - V(\mathbf{f}_{+-+})\big)\mathbf{e}_5 \\ &\quad = \int_{\mathbb{R}^3} u(\mathbf{x}) \mathrm{e}_1^{-f_1 x_1} \mathrm{e}_2^{-f_2 x_2} \mathrm{e}_4^{-f_3 x_3} \, \mathrm{d}\mathbf{x}. \end{aligned} \tag{II.13}$$

We conclude the proof by substituting formula (II.12) in (II.13) and regrouping all terms. □

III. Properties of the Octonion Spectrum

Properties of the complex Fourier transform and its quaternion counterpart are well known in literature [2], [4]. In this section we will prove analogues of classical properties such as argument scaling and modulation theorem. In [1] we already proved the shift theorem so we will omit the details here. Proof of the first theorem we present in this section is identical to the classical case and utilizes integration by substitution, which we leave to the reader.

Theorem III.1. *Let $u \colon \mathbb{R}^3 \to \mathbb{O}$ and $U = \mathcal{F}_{\text{OFT}}\{u\}$. Moreover, let $a, b, c \in \mathbb{R} \setminus \{0\}$ and function $v \colon \mathbb{R}^3 \to \mathbb{O}$ be defined by $v(x_1, x_2, x_3) = u(\frac{x_1}{a}, \frac{x_2}{b}, \frac{x_3}{c})$, $V = \mathcal{F}_{\text{OFT}}\{v\}$. Then*

$$V(f_1, f_2, f_3) = |abc| \, U(af_1, bf_2, cf_3).$$

Theorem III.1 can be generalized to all linear maps of $\mathbf{x}$. In the case of quaternion Fourier transform one can find similar result in [2] for functions $u \colon \mathbb{R}^2 \to \mathbb{R}$ and $v = u(\mathbf{A}\mathbf{x})$, where $\mathbf{A}$ is a $\mathbb{R}$-valued 2×2 matrix such that $\det(\mathbf{A}) \neq 0$. In the octonion setup, considering $v(\mathbf{x}) = u(\mathbf{A}\mathbf{x})$, where $\mathbf{A}$ is some arbitrary nonsingular 3×3 matrix, we would get a result containing 64 different terms. Due to the complication of calculations and slight significance for further research we skip this formula.

We will now prove a series of theorems, known in system theory as the modulation theorem. It is worth noticing that the claim of cosine modulation theorem (i.e. where the carrier is a cosine function) is exactly the same as in the case of complex Fourier transform, although for sine carrier function there are some significant differences. We will use the notation of central difference with respect to the i-th variable:

$$\delta_{1,f_0} U(\mathbf{f}) = U(f_1 + f_0, f_2, f_3) - U(f_1 - f_0, f_2, f_3)$$

and analogously for $\delta_{2,f_0} U$ and $\delta_{3,f_0} U$.

Theorem III.2. *Let $u \colon \mathbb{R}^3 \to \mathbb{O}$ and $U = \mathcal{F}_{\text{OFT}}\{u\}$. Moreover let $f_0 \in \mathbb{R}$ and denote $u^{\sin,i}(\mathbf{x}) = u(\mathbf{x}) \cdot \mathrm{s}^{f_0 x_i}$, $U^{\sin,i} = \mathcal{F}_{\text{OFT}}\{u^{\sin,i}\}$, $i = 1, 2, 3$. Then*

$$\begin{aligned} U^{\sin,1}(\mathbf{f}) &= \delta_{1,f_0} U(\mathbf{f}_{+--}) \cdot \frac{\mathbf{e}_1}{2}, \\ U^{\sin,2}(\mathbf{f}) &= \delta_{2,f_0} U(\mathbf{f}_{++-}) \cdot \frac{\mathbf{e}_2}{2}, \\ U^{\sin,3}(\mathbf{f}) &= \delta_{3,f_0} U(\mathbf{f}_{+++}) \cdot \frac{\mathbf{e}_4}{2}. \end{aligned}$$

Proof. We will use the equivalent definition of the sine function, i.e. equation (II.2) with $\boldsymbol{\mu} = \mathbf{e}_j$, $j = 1, \ldots, 7$. We will

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also use following properties of octonions. For any $o \in \mathbb{O}$ and $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$ we have

$$\Big(\big(o \cdot (\mathrm{e}_1^{-\alpha_1} \cdot \mathbf{e}_1)\big) \cdot \mathrm{e}_2^{-\alpha_2}\Big) \cdot \mathrm{e}_4^{-\alpha_3} = \Big(\big((o \cdot \mathrm{e}_1^{-\alpha_1}) \cdot \mathrm{e}_2^{\alpha_2}\big) \cdot \mathrm{e}_4^{\alpha_3}\Big) \cdot \mathbf{e}_1, \tag{III.1}$$

$$\big((o \cdot \mathrm{e}_1^{-\alpha_1}) \cdot (\mathrm{e}_2^{-\alpha_2} \cdot \mathbf{e}_2)\big) \cdot \mathrm{e}_4^{-\alpha_3} = \Big(\big((o \cdot \mathrm{e}_1^{-\alpha_1}) \cdot \mathrm{e}_2^{-\alpha_2}\big) \cdot \mathrm{e}_4^{\alpha_3}\Big) \cdot \mathbf{e}_2, \tag{III.2}$$

$$\big((o \cdot \mathrm{e}_1^{-\alpha_1}) \cdot \mathrm{e}_2^{-\alpha_2}\big) \cdot (\mathrm{e}_4^{-\alpha_3} \cdot \mathbf{e}_4) = \Big(\big((o \cdot \mathrm{e}_1^{-\alpha_1}) \cdot \mathrm{e}_2^{-\alpha_2}\big) \cdot \mathrm{e}_4^{-\alpha_3}\Big) \cdot \mathbf{e}_4. \tag{III.3}$$

Then, for $i = 1$ we have

$$\begin{aligned}
&U^{\sin,1}(f_1, f_2, f_3) \\
&= \int_{\mathbb{R}^3} u(\mathbf{x})\big(\mathrm{e}_1^{-f_1 x_1} \mathrm{s}^{f_0 x_1}\big) \mathrm{e}_2^{-f_2 x_2} \mathrm{e}_4^{-f_3 x_3} \, \mathrm{d}\mathbf{x} \\
&= -\int_{\mathbb{R}^3} u(\mathbf{x})\big(\mathrm{e}_1^{-f_1 x_1} \cdot \mathbf{e}_1 \mathrm{s}^{f_0 x_1} \cdot \mathbf{e}_1\big) \mathrm{e}_2^{-f_2 x_2} \mathrm{e}_4^{-f_3 x_3} \, \mathrm{d}\mathbf{x} \\
&\overset{\text{(III.1)}}{=} -\frac{1}{2} \int_{\mathbb{R}^3} u(\mathbf{x})\big(\mathrm{e}_1^{-f_1 x_1}(\mathrm{e}_1^{f_0 x_1} - \mathrm{e}_1^{-f_0 x_1})\big) \\
&\qquad \cdot \mathrm{e}_2^{f_2 x_2} \mathrm{e}_4^{f_3 x_3} \, \mathrm{d}\mathbf{x} \cdot \mathbf{e}_1 \\
&= -\frac{1}{2} \int_{\mathbb{R}^3} u(\mathbf{x})\big(\mathrm{e}_1^{-(f_1 - f_0) x_1} - \mathrm{e}_1^{-(f_1 + f_0) x_1}\big) \\
&\qquad \cdot \mathrm{e}_2^{f_2 x_2} \mathrm{e}_4^{f_3 x_3} \, \mathrm{d}\mathbf{x} \cdot \mathbf{e}_1 \\
&= \frac{1}{2} \cdot \delta_{1,f_0} U(f_1, -f_2, -f_3) \cdot \mathbf{e}_1,
\end{aligned}$$

which concludes the proof. For $i = 2, 3$ the property is proved analogously, using equations (III.2) and (III.3). ☐

Proof of the following theorem is similar to that of Theorem III.2 and uses the equivalent definition of the cosine function, i.e. equation (II.2) with $\boldsymbol{\mu} = \mathbf{e}_j$, $j = 1, \ldots, 7$. We will omit the details.

Theorem III.3. *Let $u \colon \mathbb{R}^3 \to \mathbb{O}$ and $U = \mathcal{F}_{\mathrm{OFT}}\{u\}$. Moreover, let $f_0 \in \mathbb{R}$ and denote $u^{\cos,i}(\mathbf{x}) = u(\mathbf{x}) \cdot \mathrm{c}^{f_0 x_i}$, $U^{\cos,i} = \mathcal{F}_{\mathrm{OFT}}\{u^{\cos,i}\}$, $i = 1, 2, 3$. Then*

$$U^{\cos,i}(\mathbf{f}) = \delta_{i,f_0} U(\mathbf{f}_{+++}) \cdot \frac{1}{2}, \quad i = 1, 2, 3.$$

For the completeness of our considerations, we should also state the shift theorem. As we said earlier, this theorem was already proved in our earlier work, i.e. article [1], in the case of the $\mathbb{R}$-valued functions. The proof in the general case is very straightforward and uses integration by substitution and equations (III.1)–(III.3). We omit the details here.

Theorem III.4. *Let $u \colon \mathbb{R}^3 \to \mathbb{O}$ and $U = \mathcal{F}_{\mathrm{OFT}}\{u\}$. Moreover let $\alpha, \beta, \gamma \in \mathbb{R}$ and denote $u^{\alpha}(\mathbf{x}) = u(x_1 - \alpha, x_2, x_3)$, $u^{\beta}(\mathbf{x}) = u(x_1, x_2 - \beta, x_3)$ and $u^{\gamma}(\mathbf{x}) = u(x_1, x_2, x_3 - \gamma)$. Let $U^{\ell} = \mathcal{F}_{\mathrm{OFT}}\{u^{\ell}\}$, $\ell = \alpha, \beta, \gamma$. Then*

$$\begin{aligned}
U^{\alpha}(\mathbf{f}) &= \mathrm{c}^{f_1 \alpha} U(\mathbf{f}_{+++}) - \mathrm{s}^{f_1 \alpha} U(\mathbf{f}_{+--}) \cdot \mathbf{e}_1, \\
U^{\beta}(\mathbf{f}) &= \mathrm{c}^{f_2 \beta} U(\mathbf{f}_{+++}) - \mathrm{s}^{f_2 \beta} U(\mathbf{f}_{++-}) \cdot \mathbf{e}_2, \\
U^{\gamma}(\mathbf{f}) &= \mathrm{c}^{f_3 \gamma} U(\mathbf{f}_{+++}) - \mathrm{s}^{f_3 \gamma} U(\mathbf{f}_{+++}) \cdot \mathbf{e}_4.
\end{aligned}$$

IV. Remarks on Parseval-Plancherel theorems

In [1] we proved octonion analogues of Parseval-Plancherel Theorems for $\mathbb{R}$-valued functions. We will now generalize some of this results and show that in some cases such generalization is not possible. First, recall the Plancherel Theorem, as stated in [1].

Theorem IV.1. *Let $u, v \colon \mathbb{R}^3 \to \mathbb{R}$ be square-integrable functions and $U_{\mathrm{OFT}} = \mathcal{F}_{\mathrm{OFT}}\{u\}$, $V_{\mathrm{OFT}} = \mathcal{F}_{\mathrm{OFT}}\{v\}$. Then*

$$\langle u, v \rangle = \langle U_{\mathrm{OFT}}, V_{\mathrm{OFT}} \rangle,$$

where $\langle u, v \rangle = \int_{\mathbb{R}^3} u(\mathbf{x}) \cdot v^*(\mathbf{x}) \, \mathrm{d}\mathbf{x}$.

It should be noted that in case of $\mathbb{O}$-valued functions, $\langle \cdot, \cdot \rangle$ does not satisfy the axioms of the scalar product (due to the non-associativity of octonion multiplication).

The assumption that u and v are $\mathbb{R}$-valued is relevant. For the $\mathbb{O}$-valued functions the claim of Theorem IV.1 doesn't hold, which will be discussed in detail in Remark IV.3. In case of $\mathbb{R}$-valued functions Theorem IV.2 (known as Rayleigh Theorem) is direct corollary of Theorem IV.1. However, we will prove the more general case, for $\mathbb{O}$-valued functions, thus showing that the OFT preserves the energy of $\mathbb{O}$-valued signal.

Theorem IV.2. *L^2-norm of any function $u \colon \mathbb{R}^3 \to \mathbb{O}$ (square-integrable) is equal to the L^2-norm of its octonion Fourier transform U_{OFT}, i.e.*

$$\|u\|_{L^2(\mathbb{R}^3)} = \|U_{\mathrm{OFT}}\|_{L^2(\mathbb{R}^3)},$$

where $\|v\|_{L^2(\mathbb{R}^3)} = \left(\int_{\mathbb{R}^3} |v(\mathbf{x})|^2 \, \mathrm{d}\mathbf{x}\right)^{1/2}$ for any square-integrable function $v \colon \mathbb{R}^3 \to \mathbb{O}$.

Proof. Any function $u \colon \mathbb{R}^3 \to \mathbb{O}$ can be represented as

$$u(\mathbf{x}) = u_0(\mathbf{x}) + u_1(\mathbf{x})\mathbf{e}_1 + \ldots + u_7(\mathbf{x})\mathbf{e}_7,$$

where $u_i \colon \mathbb{R}^3 \to \mathbb{R}$, $i = 0, 1, \ldots, 7$.

Then, using the following properties of octonions, i.e.

$$\begin{aligned}
\big((\mathbf{e}_1 \cdot \mathrm{e}_1^{-\alpha_1}) \cdot \mathrm{e}_2^{-\alpha_2}\big) \cdot \mathrm{e}_4^{-\alpha_3} &= \big((\mathrm{e}_1^{-\alpha_1} \cdot \mathrm{e}_2^{\alpha_2}) \cdot \mathrm{e}_4^{\alpha_3}\big) \cdot \mathbf{e}_1, \\
\big((\mathbf{e}_2 \cdot \mathrm{e}_1^{-\alpha_1}) \cdot \mathrm{e}_2^{-\alpha_2}\big) \cdot \mathrm{e}_4^{-\alpha_3} &= \big((\mathrm{e}_1^{\alpha_1} \cdot \mathrm{e}_2^{-\alpha_2}) \cdot \mathrm{e}_4^{\alpha_3}\big) \cdot \mathbf{e}_2, \\
\big((\mathbf{e}_4 \cdot \mathrm{e}_1^{-\alpha_1}) \cdot \mathrm{e}_2^{-\alpha_2}\big) \cdot \mathrm{e}_4^{-\alpha_3} &= \big((\mathrm{e}_1^{\alpha_1} \cdot \mathrm{e}_2^{\alpha_2}) \cdot \mathrm{e}_4^{-\alpha_3}\big) \cdot \mathbf{e}_4, \\
\big((\mathbf{e}_j \cdot \mathrm{e}_1^{-\alpha_1}) \cdot \mathrm{e}_2^{-\alpha_2}\big) \cdot \mathrm{e}_4^{-\alpha_3} &= \big((\mathrm{e}_1^{\alpha_1} \cdot \mathrm{e}_2^{\alpha_2}) \cdot \mathrm{e}_4^{\alpha_3}\big) \cdot \mathbf{e}_j
\end{aligned}$$

for $j \in \{3, 5, 6, 7\}$, we can write

$$\begin{aligned}
U_{\mathrm{OFT}}(f_1, f_2, f_3) = {} & U^0(\mathbf{f}_{+++}) + U^1(\mathbf{f}_{+--})\mathbf{e}_1 \\
&+ U^2(\mathbf{f}_{-+-})\mathbf{e}_2 + U^3(\mathbf{f}_{---})\mathbf{e}_3 + U^4(\mathbf{f}_{--+})\mathbf{e}_4 \\
&+ U^5(\mathbf{f}_{---})\mathbf{e}_5 + U^6(\mathbf{f}_{---})\mathbf{e}_6 + U^7(\mathbf{f}_{---})\mathbf{e}_7,
\end{aligned}$$

where $U^i = \mathcal{F}_{\mathrm{OFT}}\{u_i\}$, $i = 0, 1, \ldots, 7$, are OFTs of $\mathbb{R}$-valued functions. In [8] it was proved that they can be expressed as sums of components of different parity, i.e.

$$\begin{aligned}
U^i = {} & U_{eee} - U^i_{oee}\mathbf{e}_1 - U^i_{eoe}\mathbf{e}_2 + U^i_{ooe}\mathbf{e}_3 \\
& - U^i_{eeo}\mathbf{e}_4 + U^i_{oeo}\mathbf{e}_5 + U^i_{eoo}\mathbf{e}_6 - U^i_{ooo}\mathbf{e}_7, \quad i = 0, \ldots, 7,
\end{aligned}$$

where U^i_{xyz}, $x, y, z \in \{e, o\}$ are components of different parity as in [1]: e – even, o – odd (with respect to proper variable).

Since the claim has already been proved in case of $\mathbb{R}$-valued functions in [1], let us notice that

$$\|u\|^2_{L^2(\mathbb{R}^3)} = \sum_{i=0}^{7} \|u_i\|^2_{L^2(\mathbb{R}^3)} = \sum_{i=0}^{7} \|U^i\|^2_{L^2(\mathbb{R}^3)}.$$

It suffices to prove that $\|U_{\mathrm{OFT}}\|^2_{L^2(\mathbb{R}^3)} = \sum_{i=0}^{7} \|U^i\|^2_{L^2(\mathbb{R}^3)}$. It can easily be shown that the real part and each of the imaginary parts of U_{OFT} contains exactly one of the components of different parity of each function U^i, e.g.

$$\begin{aligned}\mathrm{Re}(U_{\mathrm{OFT}}) = U^0_{eee} + U^1_{oee} + U^2_{eoe} - U^3_{ooe} \\ + U^4_{eeo} - U^5_{oeo} - U^6_{eoo} + U^7_{ooo},\end{aligned}$$

where each component is taken in approriate point. Moreover, each component has different parity, which leads to

$$\begin{aligned}\int_{\mathbb{R}^3} (\mathrm{Re}(U_{\mathrm{OFT}}))^2 \, \mathrm{d}\mathbf{x} = \int_{\mathbb{R}^3} \big((U^0_{eee})^2 + (U^1_{oee})^2 + (U^2_{eoe})^2 \\ + (U^3_{ooe})^2 + (U^4_{eeo})^2 + (U^5_{oeo})^2 + (U^6_{eoo})^2 + (U^7_{ooo})^2\big) \, \mathrm{d}\mathbf{f}.\end{aligned}$$

The above equation follows from the fact that all other components created as a result of squaring $\mathrm{Re}(U_{\mathrm{OFT}})$ are odd with respect to at least one variable. Similar result will be obtained for each imaginary part of U_{OFT}. After adding all integrals and regrouping all terms we get

$$\begin{aligned}\int_{\mathbb{R}^3} |U_{\mathrm{OFT}}|^2 \, \mathrm{d}\mathbf{x} = \sum_{i=0}^{7} \int_{\mathbb{R}^3} \big((U^i_{eee})^2 + (U^i_{oee})^2 + (U^i_{eoe})^2 \\ + (U^i_{ooe})^2 + (U^i_{eeo})^2 + (U^i_{oeo})^2 + (U^i_{eoo})^2 + (U^i_{ooo})^2\big) \, \mathrm{d}\mathbf{f} \\ = \sum_{i=0}^{7} \|U^i\|^2_{L^2(\mathbb{R}^3)},\end{aligned}$$

which concludes the proof. □

Remark IV.3. As we already stated, the claim of Theorem IV.2 is valid for $\mathbb{O}$-valued functions, but Theorem IV.1 is true only for $\mathbb{R}$-valued functions. Consider the following example. Define two functions:

$$u(\mathbf{x}) = \frac{1}{(2\pi)^{3/2}} \exp\left\{-\frac{1}{2}\left(x_1^2 + (x_2-1)^2 + (x_3-1)^2\right)\right\},$$

$$v(\mathbf{x}) = \frac{\mathbf{e}_1}{(2\pi)^{3/2}} \exp\left\{-\frac{1}{2}\left(x_1^2 + (x_2+1)^2 + (x_3+1)^2\right)\right\},$$

which have OFT equal to

$$\begin{aligned}U_{\mathrm{OFT}}(\mathbf{f}) &= \exp\left\{-2\pi^2(f_1^2+f_2^2+f_3^2)\right\} \\ &\quad \cdot (\mathrm{c}^{f_2}\mathrm{c}^{f_3} - \mathrm{s}^{f_2}\mathrm{c}^{f_3}\mathbf{e}_2 - \mathrm{c}^{f_2}\mathrm{s}^{f_3}\mathbf{e}_4 + \mathrm{s}^{f_2}\mathrm{s}^{f_3}\mathbf{e}_5), \\ V_{\mathrm{OFT}}(\mathbf{f}) &= \exp\left\{-2\pi^2(f_1^2+f_2^2+f_3^2)\right\} \\ &\quad \cdot (\mathrm{c}^{f_2}\mathrm{c}^{f_3} - \mathrm{s}^{f_2}\mathrm{c}^{f_3}\mathbf{e}_2 - \mathrm{c}^{f_2}\mathrm{s}^{f_3}\mathbf{e}_4 + \mathrm{s}^{f_2}\mathrm{s}^{f_3}\mathbf{e}_5) \cdot \mathbf{e}_1 \\ &= \exp\left\{-2\pi^2(f_1^2+f_2^2+f_3^2)\right\} \\ &\quad \cdot (\mathrm{c}^{f_2}\mathrm{c}^{f_3}\mathbf{e}_1 + \mathrm{s}^{f_2}\mathrm{c}^{f_3}\mathbf{e}_3 + \mathrm{c}^{f_2}\mathrm{s}^{f_3}\mathbf{e}_5 + \mathrm{s}^{f_2}\mathrm{s}^{f_3}\mathbf{e}_7).\end{aligned}$$

Then

$$\begin{aligned}\langle u, v\rangle = -\mathbf{e}_1 \cdot \int_{\mathbb{R}^3} \frac{1}{(2\pi)^3} \exp\Big\{ -\frac{1}{2}\big(2x_1^2 + (x_2-1)^2 \\ + (x_2+1)^2 + (x_3-1)^2 + (x_3+1)^2\big)\Big\} \, \mathrm{d}\mathbf{x} = -\frac{\mathbf{e}_1}{8e^2\pi^{3/2}}.\end{aligned}$$

But on the other hand

$$\begin{aligned}\langle U_{\mathrm{OFT}}, V_{\mathrm{OFT}}\rangle &= \mathbf{e}_1 \cdot \int_{\mathbb{R}^3} \exp\left\{-4\pi^2(f_1^2+f_2^2+f_3^2)\right\} \\ &\quad \cdot \big(-(\mathrm{c}^{f_2}\mathrm{c}^{f_3})^2 + (\mathrm{s}^{f_2}\mathrm{c}^{f_3})^2 + (\mathrm{c}^{f_2}\mathrm{s}^{f_3})^2 + (\mathrm{s}^{f_2}\mathrm{s}^{f_3})^2\big) \, \mathrm{d}\mathbf{f} \\ &= -\frac{\mathbf{e}_1}{8e^2\pi^{3/2}} \cdot \left(\frac{1}{2} + e + \frac{e^2}{2}\right).\end{aligned}$$

This counterexample proves that in general the claim of Theorem IV.1 is not valid for all $u, v \colon \mathbb{R}^3 \to \mathbb{O}$.

V. Discussion and conclusion

We showed that the theory of OFT can be generalized to the case of functions with hypercomplex values. Their spectra have properties that are similar to their complex counterparts and can be applied in the analysis of vector-valued signals of three (time- or space-like) variables using a true 3D Fourier Transform that separates all symmetries in its spectrum [1].

Presented results form the foundation of octonion-based signal and system theory. It remains to study the properties of the OFT in context of other signal-domain operations, i.e. derivation and convolution of $\mathbb{R}$-valued functions. There are known results for quaternion Fourier transform (see [6]), but they use the notion of other hypercomplex algebra, i.e. double-complex numbers. Finding similar results for octonion Fourier transform requires defining other higher-order hypercomplex structures.

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