Arbitrary Length Perfect Integer Sequences Using Geometric Series

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***Abstract*—A novel method to construct perfect integer sequences based on geometric series is proposed. The method can be applied to arbitrary signal length. A closed form construction has been derived for a given ratio. Moreover, perfect Gaussian integer sequences can also be constructed by this method. The idea can be further generalized to obtain other perfect integer sequences from a given one by the Extended Euclidean algorithm. To the authors' knowledge, these sequences cannot be found by any previous work. Concrete examples are illustrated.**

***Index Terms*—Discrete Fourier transform, geometric series, zero autocorrelation, perfect integer sequences.**

I. Introduction

The zero autocorrelation (ZAC) or perfect Gaussian integer sequences have been studied a lot [1]–[13] recently since they have many applications such as code division multiple access (CDMA) [11], equalization, synchronization, channel estimation, cell search and CW radar [10], [14]. On the contrary, perfect integer sequences are less discussed because it is not easy to find a non-trivial one. As we can see, a perfect Gaussian integer sequence with length N has $2N$ variables since each value has the form $a + bi$, but for a perfect integer sequence there are only N variables. Thus, the methods in [2]–[4], [6]–[10], [12], [13] may not work well. In this paper we will reveal a novel method to solve this problem.

The benefits of our work can be summarized as follows.

- Integer ZAC can be constructed. In applications such as communication, transmitting integer instead of Gaussian integer can save the bandwidth.
- Suitable for arbitrary length.
- Closed form solution.

The remaining of this paper is organized as follows. Some useful notation and definition are given in Section II. Some related works are reviewed in Section III. The main result will be revealed in Section IV. Some extensions are introduced in Section V. The conclusion is in Section VI.

II. Preliminary Results

Let $x(n)$ be a sequence with length N and W_N be $e^{\frac{-2\pi i}{N}}$. The discrete Fourier transform (DFT) of $x(n)$ is defined as

$$\hat{x}(m) = \sum_{n=0}^{N-1} x(n) W_N^{nm} \tag{1}$$

where $n, m \in 0, 1, 2, ..., N-1$. And the inverse discrete Fourier transform (IDFT) of $\hat{x}$ is

$$x(n) = \frac{1}{N} \sum_{m=0}^{N-1} \hat{x}(m) W_N^{-mn} \tag{2}$$

which is denoted by $F\{x\} = \hat{x}$ and $F^{-1}\{\hat{x}\} = x$.

Let $\delta(n)$ be the delta function such that

$$\delta(n) = \begin{cases} 1 & n = 0 \\ 0 & n \neq 0 \end{cases} \tag{3}$$

And we can define constant amplitude (CA) and zero autocorrelation (ZAC) as follows.

Definition A sequence $x(n)$ is constant amplitude (CA) if

$$|x(n)| = A \tag{4}$$

for some constant A

Definition A sequence $x(n)$ is zero autocorrelation (ZAC) if

$$\sum_{n=0}^{N-1} x^*(n-m)x(n) = C\delta(m) \tag{5}$$

for some constant C, where x^* is the complex conjugate of x

An useful theorem about CA and ZAC is given as follows.

Theorem II.1. *A sequence x is CA if and only if its DFT $\hat{x}$ is ZAC. Similarly, a sequence x is ZAC if and only if its DFT $\hat{x}$ is CA.*

Proof. See [20] □

A Gaussian integer is a complex number $a+bi$ while both a and b are integers.

Definition A perfect Gaussian integer sequence is a ZAC sequence such that each value in the sequence is a Gaussian integer. Similarly, A perfect integer sequence is a ZAC sequence such that each value in the sequence is an integer.

Example

$$[-2, 3, 6]$$

is a perfect integer sequence.

Example

$$[3 - 3i, 1, 1 + i, 2i]$$

is a perfect Gaussian integer sequence.

III. Review on other methods

For any length N, there are two trivial perfect integer sequences [5]

$$x(n) = [1, 0, 0, 0, ..., 0] \tag{6}$$

$$x(n) = [N-2, -2, -2, -2, ..., -2] \tag{7}$$

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Moreover, if N is even, the following sequence is also a perfect integer sequence.

$$x(n) = \begin{cases} \frac{N-2}{2} & n = 0 \\ (-1)^{n-1} & n \neq 0 \end{cases} \tag{8}$$

We will generalize these trivial cases to obtain a surprisingly simple method to construct perfect integer sequence in Section IV.

For most other methods, the main concept to generate perfect integer or Gaussian integer sequences $x(n), n = 0, 1, 2, ..., N-1$ of length N is based on categorization. The index n is categorized into many classes, and the value of sequence $x(n)$ is dependent on the class n belongs. For example, in [2], one of the base sequences $\mathbf{y}_2$ is defined as

$$\mathbf{y}_2 = \begin{cases} 1 & n \equiv 0 \mod 4 \\ i & n \equiv 1 \mod 4 \\ -1 & n \equiv 2 \mod 4 \\ -i & n \equiv 3 \mod 4 \end{cases} \tag{9}$$

The base sequence can be linearly combined with other base sequences to generate perfect Gaussian integer sequence.

The categorization method is not unique. In [3], the sequence length $N = p$ where p is a prime number. All the nonzero index $n \neq 0$ is thus mapping into cyclic subgroup of the Galios field $\mathbf{GF}(p)$. A concrete example is $p = 13$ with primitive root 2. The cyclic subgroup of $\mathbf{GF}(13)$ is

$$\begin{aligned} &2^0 = 1 \\ &2^2 = 4 \\ &2^4 = 16 \equiv 3 \mod 13 \\ &2^6 = 64 \equiv 12 \mod 13 \\ &2^8 = 256 \equiv 9 \mod 13 \\ &2^{10} = 1024 \equiv 10 \mod 13 \end{aligned}$$

The sequence is then constructed by

$$x(n) = \begin{cases} a_2 & n = 0 \\ a_0 & n = 1, 3, 4, 9, 10, 12 \\ a_1 & otherwise \end{cases} \tag{10}$$

with carefully designed Gaussian integer a_0, a_1 and a_2. A numerical example can be found as the Example 2 in [3]. The main drawback of the above methods is that they cannot applied to construct perfect integer sequences.

In [5], another categorization method is used. Let $gcd(N, n)$ be the greatest common divisor of N and n. And let $d_1, d_2, ..., d_t$ be all divisor for N. The index in the frequency domain (by taking discrete Fourier transform, DFT) is categorized by $gcd(N, n) = d_s,\ s = 1, 2, ..., t$. The result is equal to the summation of the circular shifted Ramanujan's Sum. Some non-trivial perfect integer sequence can be formed. A concrete example is $N = 6$ and $d_1 = 6, d_2 = 3, d_3 = 2, d_4 = 1$. The perfect integer sequence may be

$$x(n) = [0, 1, 3, 4, -3, 1] \tag{11}$$

Although this method can generate non-trivial perfect integer sequences, it can only applied to composite length. In other words, when N is a prime number, the method only gives trivial solutions.

The difference set and its application on low correlation code have been studied for decades [15]–[19]. However, these results have not applied to generate perfect Gaussian integer sequences until [8]. The method is similar to others by determining a difference set $\mathbb{D}$ which is a subset of $\{0, 1, 2, ..., N-1\}$. Then the sequence can be formed as

$$x(n) = \begin{cases} a_0 & n \in \mathbb{D} \\ a_1 & otherwise \end{cases} \tag{12}$$

with carefully designed Gaussian integer a_0 and a_1. The advantage of the difference set method is that the sequence is binary, which means only two kinds of value a_0 and a_1 are used. The main drawback of this method is that not every length N has a difference set.

IV. Perfect integer sequences by geometric series

Observe Equations (6) to (8), we can find two interest things. The first is that only the element $x(0)$ is different from the others. The second is that each $x(n), n \neq 0$ forms a geometric series. In Equation (6), $[0, 0, ..., 0]$ can be viewed as geometric series with any ratio r. In Equation (7), the ratio $r = 1$ while in Equation (8), the ratio $r = -1$.

Another example can be found in $N = 4$. The closed form solution for perfect integer sequence is

$$\left[-b, a, b, \frac{b^2}{a}\right] \tag{13}$$

where $a, b, \frac{b^2}{a}$ are integers. This is solved by brute force. The last three terms $\left[a, b, \frac{b^2}{a}\right]$ can also be viewed as geometric series, with ratio $r = \frac{b}{a}$.

From the observation above, a natural assumption is that any geometric series can be used to generate perfect integer sequence. With some calculation the assumption can be proved. The result is given as follows. Let r be an integer. A closed form perfect integer sequences is given as

$$x(n) = \begin{cases} -\sum_{k=2}^{N-1} r^k & n = 0 \\ (1+r)r^n & n \neq 0 \end{cases} \tag{14}$$

Note that $x(n)$ is a geometric series from $n = 1$ to $N-1$. Since $r = 0$ or $r = \pm 1$ gives only trivial result, from now on only the cases $r \neq 0, \pm 1$ are discussed.

For a concrete example, let $N = 5$ and $r = 2$

$$\begin{aligned} x(n) &= [-(2^2 + 2^3 + 2^4), 3 \cdot 2^1, 3 \cdot 2^2, 3 \cdot 2^3, 3 \cdot 2^4] \quad (15) \\ &= [-28, 6, 12, 24, 48] \quad (16) \end{aligned}$$

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For another example, let $N = 6$ and $r = -3$

$$x(n) = \begin{bmatrix} -\sum_{k=2}^{5}(-3)^k \\ (-2)\cdot(-3)^1 \\ (-2)\cdot(-3)^2 \\ (-2)\cdot(-3)^3 \\ (-2)\cdot(-3)^4 \\ (-2)\cdot(-3)^5 \end{bmatrix} \tag{17}$$

$$= [180, 6, -18, 54, -162, 486] \tag{18}$$

The two examples above illustrate that both prime length $N = 5$ and composite length $N = 6$ can be used. In fact, there is no limitation on signal length.

To prove Equation (14), by Theorem II.1 we should check that $\hat{x}$ is CA. By definition,

$$\hat{x}(m) = \sum_{n=0}^{N-1} x(n) W_N^{nm} \tag{19}$$

$$= -\sum_{k=2}^{N-1} r^k + \sum_{n=1}^{N-1} (1+r) r^n W_N^{nm} \tag{20}$$

$$= \left(-\sum_{k=2}^{N-1} r^k\right) - (1+r) + (1+r)$$

$$+ (1+r)\sum_{n=1}^{N-1} r^n W_N^{nm} \tag{21}$$

$$= -\sum_{k=0}^{N-1} r^k + (1+r)\sum_{n=0}^{N-1} r^n W_N^{nm} \tag{22}$$

$$= -\frac{1-r^N}{1-r} + (1+r)\frac{1-(rW_N^m)^N}{1-rW_N^m} \tag{23}$$

$$= -\frac{1-r^N}{1-r} + (1+r)\frac{1-r^N}{1-rW_N^m} \tag{24}$$

$$= (1+r)(1-r^N)\left(\frac{1}{1-rW_N^m} - \frac{1}{1-r^2}\right) \tag{25}$$

To prove $\hat{x}$ is CA, the following lemma is needed.

Lemma IV.1. *For any complex number $z = re^{i\theta}$, where $r = |z| \neq 1$,*

$$\left|\frac{1}{1-z} - \frac{1}{1-r^2}\right| = \left|\frac{r}{1-r^2}\right| \tag{26}$$

In other words, the magnitude of $\frac{1}{1-z} - \frac{1}{1-r^2}$ only depends on r and is independent from θ.

Proof.

$$\frac{1}{1-re^{i\theta}} - \frac{1}{1-r^2} = \frac{1}{1-r\cos(\theta)+ir\sin(\theta)} - \frac{1}{1-r^2} \tag{27}$$

$$= \frac{1-r\cos(\theta)-ir\sin(\theta)}{(1-r\cos(\theta))^2 + r^2\sin^2(\theta)} - \frac{1}{1-r^2} \tag{28}$$

$$= \left(\frac{1-r\cos(\theta)}{1-2r\cos(\theta)+r^2} - \frac{1}{1-r^2}\right) - i\frac{r\sin(\theta)}{1-2r\cos(\theta)+r^2} \tag{29}$$

Thus, let $Q = 1 - 2r\cos(\theta) + r^2$

$$\left|\frac{1}{1-re^{i\theta}} - \frac{1}{1-r^2}\right|^2 \tag{30}$$

$$= \left(\frac{1-r\cos(\theta)}{Q} - \frac{1}{1-r^2}\right)^2 + \left(\frac{r\sin(\theta)}{Q}\right)^2 \tag{31}$$

$$= \frac{(1-r\cos(\theta))^2}{Q^2} - 2\frac{1-r\cos(\theta)}{Q(1-r^2)} + \frac{1}{(1-r^2)^2} + \frac{r^2\sin^2(\theta)}{Q^2} \tag{32}$$

Note that $(1-r\cos(\theta))^2 + r^2\sin^2(\theta) = Q$

$$\left|\frac{1}{1-re^{i\theta}} - \frac{1}{1-r^2}\right|^2 \tag{33}$$

$$= \frac{Q}{Q^2} - 2\frac{1-r\cos(\theta)}{Q(1-r^2)} + \frac{1}{(1-r^2)^2} \tag{34}$$

$$= \frac{1-r^2}{Q(1-r^2)} - 2\frac{1-r\cos(\theta)}{Q(1-r^2)} + \frac{1}{(1-r^2)^2} \tag{35}$$

$$= \frac{-1+2r\cos(\theta)-r^2}{Q(1-r^2)} + \frac{1}{(1-r^2)^2} \tag{36}$$

$$= \frac{-Q}{Q(1-r^2)} + \frac{1}{(1-r^2)^2} = \frac{1-(1-r^2)}{(1-r^2)^2} = \left(\frac{r}{(1-r^2)}\right)^2 \tag{37}$$

which completes the proof. □

We can now prove $x(n)$ in Equation (14) is ZAC. By (25) and Lemma IV.1,

$$|\hat{x}(m)| = \left|(1+r)(1-r^N)\left(\frac{1}{1-rW_N^m} - \frac{1}{1-r^2}\right)\right| \tag{38}$$

$$= |(1+r)(1-r^N)| \cdot \left|\frac{1}{1-rW_N^m} - \frac{1}{1-r^2}\right| \tag{39}$$

$$= \left|\frac{r(1+r)(1-r^N)}{1-r^2}\right| = \left|\frac{r(1-r^N)}{1-r}\right| \tag{40}$$

in other words, $|\hat{x}|$ is constant when r is fixed. Thus, $\hat{x}$ is CA and by Theorem II.1, x is ZAC.

V. Some extensions on the proposed method

A. Other perfect integer sequences by rational $r = \frac{p}{q}$

Although in the Equation (14) r is assumed to be an integer, the assumption can be extended to any rational number $r = \frac{p}{q}$. The result sequence will be rational and can become an integer sequence by multiplying the least common multiple (lcm) of the denominators. For example, let $r = \frac{2}{3}$ and $N = 5$, Equation (14) becomes

$$x(n) = \begin{bmatrix} -\sum_{k=2}^{4}(\frac{2}{3})^k \\ \frac{5}{3}\cdot(\frac{2}{3})^1 \\ \frac{5}{3}\cdot(\frac{2}{3})^2 \\ \frac{5}{3}\cdot(\frac{2}{3})^3 \\ \frac{5}{3}\cdot(\frac{2}{3})^4 \end{bmatrix} \tag{41}$$

$$= [\frac{-76}{81}, \frac{10}{9}, \frac{20}{27}, \frac{40}{81}, \frac{80}{243}] \tag{42}$$

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The lcm of the denominator is 243. Thus, the final perfect integer sequence becomes

$$x(n) = [-228, 270, 180, 120, 80] \tag{43}$$

For another example let $r = \frac{-2}{3}$ and $N = 5$,

$$x(n) = \begin{bmatrix} -\sum_{k=2}^{4}(\frac{-2}{3})^k \\ \frac{1}{3}\cdot(\frac{-2}{3})^1 \\ \frac{1}{3}\cdot(\frac{-2}{3})^2 \\ \frac{1}{3}\cdot(\frac{-2}{3})^3 \\ \frac{1}{3}\cdot(\frac{-2}{3})^4 \end{bmatrix} \tag{44}$$

$$= [\frac{-28}{81}, \frac{-2}{9}, \frac{4}{27}, \frac{-8}{81}, \frac{16}{243}] \tag{45}$$

The lcm of the denominator is 243 again. Thus, the final perfect integer sequence becomes

$$x(n) = [-84, -54, 36, -24, 16] \tag{46}$$

B. Perfect Gaussian integer sequences by geometric series

One natural extension to the Equation (14) is setting r as a Gaussian integer $a+bi$. However, the formula needs to be modified since when r is complex, $|r^2| \neq r^2$, so Lemma IV.1 is no longer suitable. Luckily the idea of geometric series still works. Let $x(0) = x_1 + x_2 i$, it is not hard to prove that

$$x = [x(0), r, r^2, r^3, ..., r^{N-1}] \tag{47}$$

has a solution for any $r = a + bi$, $|r| \neq 1$. The proof is similar to the proof of Lemma IV.1 so we omit it here. As a concrete example, let $r = 1 + i$ and $N = 5$,

$$x = [x_1 + x_2 i, 1+i, 2i, -2+2i, -4] \tag{48}$$

In order to construct ZAC sequence, by definition from Equation (5), the following equation must be satisfied.

$$\begin{aligned} &(x_1 - x_2 i)(1+i) + (1-i)2i + (-2i)(-2+2i) \\ &+(-2-2i)(-4) + (-4)(x_1 + x_2 i) = 0 \end{aligned} \tag{49}$$

The equation above can be rearranged by real part and imaginary part.

$$(-3x_1 + x_2 + 14) + i(x_1 - 5x_2 + 14) = 0 \tag{50}$$

The unique solution of the above equation is $x_1 = 6$ and $x_2 = 4$. The result perfect Gaussian integer sequence is

$$x = [6+4i, 1+i, 2i, -2+2i, -4] \tag{51}$$

C. Generating another perfect integer sequence by a given one

If we compare Equation (25) and Lemma IV.1, we can discover that Equation (25) is a special case when the θ in Lemma IV.1 is equally spaced, that is,

$$\theta = \frac{-2\pi m}{N} \tag{52}$$

This constraint is unnecessary and we will see how to extend Lemma IV.1 and construct another perfect integer sequence by a given one.

Let $x(n)$ be a perfect integer sequence with length N. Define the associate polynomial of x, $f_x(z)$ as

$$f_x(z) = \sum_{n=0}^{N-1} x(n) z^n \tag{53}$$

Clearly by definition $f_x(W_N^m) = \hat{x}(m)$, and since x is ZAC, $\hat{x}$ is CA. In other words,

$$|f_x(W_N^m)| = C \tag{54}$$

for some constant C and any integer m. Note that $C = |f_x(W_N^0)| = |f_x(1)| = \left|\sum_{n=0}^{N-1} x(n)\right|$ and x is an integer sequence, C must be an integer.

The idea to construct new perfect integer sequence is that $f_x(W_N^m)$ can be viewed as $Ce^{i\theta}$ and although W_N^m must be equally spaced in phase, the phase of $f_x(W_N^m)$ may be not. Thus, from another point of view, $f_x(z)$ can be a generalization to z. However, there is an inverse $(1-z)^{-1}$ in Lemma IV.1 and the following lemma shows the inverse $(1 - f_x(z))^{-1}$ exists.

Lemma V.1. *Let $x(n)$ be a perfect integer sequence with length N, $f_x(z)$ be the associate polynomial of x and $C = |f_x(1)|$. If $C \neq 1$ then $z^N - 1$ and $1 - f_x(z)$ are coprime. In other words, $z^N - 1$ and $1 - f_x(z)$ have no common divisor $d(z)$ such that the degree of $d(z)$ is greater than 0.*

Proof. (By contradiction)

Let $d(z)$ be the common divisor of $z^N - 1$ and $1 - f_x(z)$. Thus, there are $g(z)$ and $h(z)$ such that

$$z^N - 1 = h(z)d(z) \tag{55}$$

$$1 - f_x(z) = g(z)d(z) \tag{56}$$

Since the N roots of $z^N - 1$ are $1, W_N, W_N^2, ..., W_N^{N-1}$ and $d(z)$ is non zero degree polynomial, there must be some W_N^k, $k \in \{0, 1, ..., N-1\}$ such that $d(W_N^k) = 0$. Equation (56) becomes

$$1 - f_x(W_N^k) = g(W_N^k)d(W_N^k) = 0 \tag{57}$$

$$\Rightarrow |f_x(W_N^k)| = 1 \tag{58}$$

However by Equation (54) $C = |f_x(W_N^m)|$ for any integer m. Thus $C = 1$ which contradicts the assumption $C \neq 1$. ☐

Corollary V.2. *Under the same assumptions in the above lemma. There are rational coefficient polynomials $g_x(z)$ and $h_x(z)$ such that*

$$(z^N - 1)h_x(z) + (1 - f_x(z))\, g_x(z) = 1 \tag{59}$$

Proof. This is the well-known Bezout's lemma [21]. Moreover, these polynomials can be calculated by Extended Euclidean algorithm. ☐

We can construct another perfect sequence by x.

Theorem V.3. *$g_x(z) - \frac{1}{1-C^2}$ is an associate polynomial to a rational ZAC sequence.*

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Proof. By Corollary V.2 and let $z = W_N^m$,

$$(W_N^{mN} - 1)h_x(W_N^m) + (1 - f_x(W_N^m))\, g_x(W_N^m) = 1 \tag{60}$$

$$\Rightarrow 0 \cdot h_x(W_N^m) + (1 - f_x(W_N^m))\, g_x(W_N^m) = 1 \tag{61}$$

$$\Rightarrow \left| g_x(W_N^m) - \frac{1}{1 - C^2} \right| = \left| \frac{1}{1 - f_x(W_N^m)} - \frac{1}{1 - C^2} \right| \tag{62}$$

$$= \left| \frac{C}{1 - C^2} \right| \tag{63}$$

The final equation is due to $|f_x(W_N^m)| = C$ and Lemma IV.1. Thus, the sequence associate to the polynomial $g_x(z) - \frac{1}{1-C^2}$ is ZAC because its DFT is CA, which completes the proof. □

The perfect integer sequence x is not necessary constructed by the method proposed in Section IV. For example, let $N = 7$ and $x = [0, -1, 1, 1, 0, 1, 0]$

$$1 - f_x(z) = -z^5 - z^3 - z^2 + z + 1 \tag{64}$$

By Bezout's lemma [21] we can find

$$h_x(z) = \frac{1}{71}\left(9z^4 - 21z^3 - 13z^2 - 8z - 85\right)$$
$$g_x(z) = \frac{1}{71}\left(9z^6 - 21z^5 - 22z^4 + 4z^3 - 33z^2 + 6z - 14\right)$$
$$h_x(z)(z^7 - 1) + g_x(z)(1 - f_x(z)) = 1$$

Also note that $C = |f_x(1)| = 2$ thus

$$g_x(z) - \frac{1}{1 - C^2}$$
$$= \frac{1}{71}\left(9z^6 - 21z^5 - 22z^4 + 4z^3 - 33z^2 + 6z - 14\right) + \frac{1}{3} \tag{65}$$

$$= \frac{1}{213}\left(27z^6 - 63z^5 - 66z^4 + 12z^3 - 99z^2 + 18z + 29\right) \tag{66}$$

And we can easily check that

$$[29, 18, -99, 12, -66, -63, 27]$$

is a perfect integer sequence. To the authors' knowledge, this sequence cannot be found by any previous work.

In the next example let $N = 6$ and $x = [0, \frac{4}{3}, 0, \frac{4}{3}, 0, \frac{-2}{3}]$. Note that we can scale x by any rational number in order to make $C = |f_x(z)| \neq 1$. The calculation is skipped and the result perfect sequence is $[-26, -4, -20, -16, 4, 20]$. These two examples illustrate that the method works for both prime length $N = 7$ and composite length $N = 6$. In general, the proposed method will work for any length.

VI. Conclusion

In this work a novel method to construct perfect integer sequences is revealed. Unlike other methods which is based on index categorization, this method is based on geometric series and can work on arbitrary signal length. The result can extend to complex number in order to generate perfect Gaussian integer sequence. When proving the sequence is perfect, a generalization has been discovered. The new method can generate another perfect sequence by a given one.

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