Roundoff Noise Analysis for Generalized Direct-Form II Structure of 2-D Separable-Denominator Digital Filters

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***Abstract*—Based on the concept of polynomial operators, generalized direct-form II structure of two-dimensional (2-D) separable-denominator (SD) digital filters is explored. It is shown that 2-D SD digital filters can be modeled by a generalized SIMO direct-form II and a generalized MISO transposed direct-form II that are connected in cascade. Then an expression for the roundoff noise gain in the resulting structure is derived and investigated. Moreover, the roundoff noise gain is compared with that deduced in a recent study of generalized direct-form II realization of 2-D SD digital filters.**

I. Introduction

In the past decades, delta operator has widely been used in the realization of digital filters to improve finite-word-length (FWL) performance in systems with high sampling rate [1]-[5]. Li and Gevers have studied the roundoff noise gain for the delta-operator state-space realization of a transfer function and under what conditions the roundoff noise gain for the optimal delta-operator realization is smaller than that for the optimal shift-operator realization [1]. In [2], the filter was expressed by second-order sections connected in cascade, each section was implemented with a direct form in delta operator, and different direct forms in the delta operator were extensively compared. In [3], the concept of separately scaling the Δs and filter coefficients in the delta transposed direct-form II section was proposed to globally further minimize output roundoff noise gain, as compared to selecting a single optimal Δ only. In [4], a method for obtaining an optimal arbitrary-order delta-operator direct-form II transposed filter has been presented in terms of the roundoff noise gain and coefficient sensitivity. Based on the concept of polynomial operators, a new structure which is considered a generalization of the shift operator z-based direct-form II transposed structure has been explored for digital filter implementation [5]. Alternatively, due to their desirable properties and ease of stability test, 2-D SD digital filters have been widely studied as a very popular class for multidimensional signal processing in the past [6]-[12]. Recently, generalized direct-form II state-space realization of 2-D SD digital filters has been constructed, and an expression for the roundoff noise, an l_2-scaling method to avoid overflow, and a way to minimize the roundoff noise gain with respect to free parameters subject to l_2-scaling constraints have been examined [13].

In this paper, we present a detailed roundoff noise analysis for generalized direct-form II structure of 2-D SD digital filters using a different approach from [13]. The roundoff noise gain is compared with that deduced in a recent study of generalized direct-form II state-space realization of 2-D SD digital filters in [13].

II. Structure of 2-D Digital Filters

Consider a 2-D stable SD digital filter of order (m, n) described by

$$H(z_1, z_2) = \frac{\displaystyle\sum_{k=0}^{m}\sum_{l=0}^{n} c_{kl} z_1^{-k} z_2^{-l}}{\displaystyle\Big(1 + \sum_{k=1}^{m} a_k z_1^{-k}\Big)\Big(1 + \sum_{l=1}^{n} b_l z_2^{-l}\Big)} \qquad (1)$$

where the denominator and numerator are assumed to be co-prime. Let $\boldsymbol{P}_1$ and $\boldsymbol{P}_2$ be $(m+1)\times(m+1)$ and $(n+1)\times(n+1)$ nonsingular matrices, respectively, defined by

$$\begin{aligned} \big[\, q_0^h(z_1) \;\; q_1^h(z_1) \;\; \cdots \;\; q_m^h(z_1) \,\big]^T &= \boldsymbol{P}_1 \big[\, z_1^m \;\; \cdots \;\; z_1 \;\; 1 \,\big]^T \\ \big[\, q_0^v(z_2) \;\; q_1^v(z_2) \;\; \cdots \;\; q_n^v(z_2) \,\big]^T &= \boldsymbol{P}_2 \big[\, z_2^n \;\; \cdots \;\; z_2 \;\; 1 \,\big]^T \end{aligned} \qquad (2)$$

Next, scalars $\{\alpha_k |\, k = 1, 2, \cdots, m\}$, $\{\beta_l |\, l = 1, 2, \cdots, n\}$ and $\{r_{kl} |\, k = 0, 1, \cdots, m;\; l = 0, 1, \cdots, n\}$ are defined so that

$$\begin{aligned} \kappa_1 [1 \; \alpha_1 \; \alpha_2 \; \cdots \; \alpha_m] \boldsymbol{P}_1 &= [1 \; a_1 \; a_2 \; \cdots \; a_m] \\ \kappa_2 [1 \; \beta_1 \; \beta_2 \; \cdots \; \beta_n] \boldsymbol{P}_2 &= [1 \; b_1 \; b_2 \; \cdots \; b_n] \\ \kappa_1 \kappa_2 \boldsymbol{P}_1^T \begin{bmatrix} r_{00} & \cdots & r_{0n} \\ \vdots & \ddots & \vdots \\ r_{m0} & \cdots & r_{mn} \end{bmatrix} \boldsymbol{P}_2 &= \begin{bmatrix} c_{00} & \cdots & c_{0n} \\ \vdots & \ddots & \vdots \\ c_{m0} & \cdots & c_{mn} \end{bmatrix} \end{aligned} \qquad (3)$$

From (1)-(3), the transfer function in (1) can be expressed as

$$H(z_1, z_2) = \frac{\displaystyle\sum_{k=0}^{m}\sum_{l=0}^{n} r_{kl} q_k^h(z_1) q_l^v(z_2)}{\displaystyle\sum_{k=0}^{m} \alpha_k q_k^h(z_1) \sum_{l=0}^{n} \beta_l q_l^v(z_2)} \qquad (4)$$

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where scaling factors κ_1 and κ_2 are determined by requiring $\alpha_0 = 1$ and $\beta_0 = 1$. We now define

$$\rho_k^h(z_1) = \frac{z_1 - \overline{\gamma}_k}{\overline{\Delta}_k} \quad \text{for} \quad k = 1, 2, \cdots, m$$
$$\rho_l^v(z_2) = \frac{z_2 - \hat{\gamma}_l}{\hat{\Delta}_l} \quad \text{for} \quad l = 1, 2, \cdots, n \tag{5}$$

where $\{\overline{\gamma}_k\}$, $\{\hat{\gamma}_l\}$, $\{\overline{\Delta}_k > 0\}$ and $\{\hat{\Delta}_l > 0\}$ are four sets of constants [13] and polynomial operators are chosen as

$$q_k^h(z_1) = \rho_{k+1}^h(z_1)\rho_{k+2}^h(z_1) \cdots \rho_m^h(z_1), \quad k = 0, 1, \cdots, m-1$$
$$q_l^v(z_2) = \rho_{l+1}^v(z_2)\rho_{l+2}^v(z_2) \cdots \rho_n^v(z_2), \quad l = 0, 1, \cdots, n-1 \tag{6}$$

and $q_m^h(z_1) = q_n^v(z_2) = 1$. Using (5), we can specify the corresponding transformation matrices $\boldsymbol{P}_1$, $\boldsymbol{P}_2$ and scalars $\kappa_1 = \overline{\Delta}_1 \overline{\Delta}_2 \cdots \overline{\Delta}_m$, $\kappa_2 = \hat{\Delta}_1 \hat{\Delta}_2 \cdots \hat{\Delta}_n$. Making use of (6), the transfer function in (4) can be written as [13]

$$H(z_1, z_2) = \frac{\displaystyle\sum_{k=0}^{m} \sum_{l=0}^{n} r_{kl} \prod_{p=0}^{k} \rho_p^h(z_1)^{-1} \prod_{q=0}^{l} \rho_q^v(z_2)^{-1}}{\Big(\displaystyle\sum_{k=0}^{m} \alpha_k \prod_{p=0}^{k} \rho_p^h(z_1)^{-1}\Big)\Big(\displaystyle\sum_{l=0}^{n} \beta_l \prod_{q=0}^{l} \rho_q^v(z_2)^{-1}\Big)} \tag{7}$$

where $\alpha_0 = \beta_0 = 1$ and $\rho_0^h(z_1)^{-1} = \rho_0^v(z_2)^{-1} = 1$.

The implementations of $\rho_k^h(z_1)^{-1}$ and $\rho_l^v(z_2)^{-1}$ are depicted in Fig. 1. As an illustrative example, the structure of (7) for a 2-D filter with $(m, n) = (3, 3)$ is depicted in Fig. 2 where $u(i, j)$ is a scalar input and $y(i, j)$ is a scalar output.

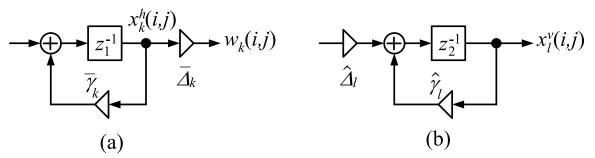

Fig. 1. (a) Implementation of $\rho_k^h(z_1)^{-1}$ and (b) that of $\rho_l^v(z_2)^{-1}$.

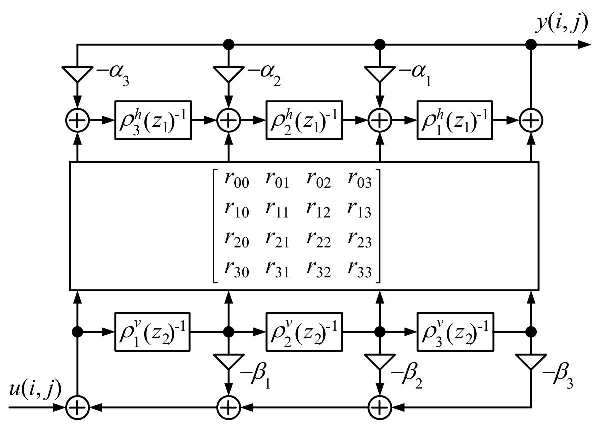

Fig. 2. The 2-D filter structure of (7) with $(m, n) = (3, 3)$.

From Figures 1 and 2, we deduce

$$y(i,j) = w_1(i,j) + r_{00}\Big[u(i,j) - \sum_{l=1}^{n} \beta_l x_l^v(i,j)\Big] + \sum_{l=1}^{n} r_{0l} x_l^v(i,j)$$
$$w_k(i,j) = \rho_k^h(z_1)^{-1}\Big\{w_{k+1}(i,j) + \sum_{l=1}^{n} r_{kl} x_l^v(i,j) + r_{k0}\Big[u(i,j) - \sum_{l=1}^{n} \beta_l x_l^v(i,j)\Big] - \alpha_k y(i,j)\Big\} \tag{8}$$

for $k = 1, 2, \cdots, m$ where $w_{m+1}(i, j) = 0$. In addition,

$$x_1^v(i,j+1) = \hat{\gamma}_1 x_1^v(i,j) + \hat{\Delta}_1\Big[u(i,j) - \sum_{l=1}^{n} \beta_l x_l^v(i,j)\Big]$$
$$x_l^v(i,j+1) = \hat{\Delta}_l x_{l-1}^v(i,j) + \hat{\gamma}_l x_l^v(i,j) \tag{9}$$

for $l = 2, 3, \cdots, n$.

We note that the model in (7) contains $2(m+n) + (m+1) \cdot (n+1)$ nontrivial parameters $\{\alpha_k\}$, $\{\overline{\Delta}_k\}$, $\{\beta_l\}$, $\{\hat{\Delta}_l\}$ and $\{r_{kl}\}$ plus $m + n$ free parameters $\{\overline{\gamma}_k\}$ and $\{\hat{\gamma}_l\}$.

III. Roundoff Noise Analysis

We begin by examining the roundoff noise caused by the term $\alpha_k y(i, j)$ for $1 \leq k \leq m$ at the output. Due to the product quantization, for the actual filter implemented by a FWL system, (8) can be written as

$$\tilde{y}(i,j) = \tilde{w}_1(i,j) + r_{00}\Big[u(i,j) - \sum_{l=1}^{n} \beta_l x_l^v(i,j)\Big] + \sum_{l=1}^{n} r_{0l} x_l^v(i,j)$$
$$\tilde{w}_s(i,j) = \rho_s^h(z_1)^{-1}\Big\{\tilde{w}_{s+1}(i,j) + \sum_{l=1}^{n} r_{sl} x_l^v(i,j) + r_{s0}\Big[u(i,j) - \sum_{l=1}^{n} \beta_l x_l^v(i,j)\Big] - \big[\alpha_s \tilde{y}(i,j) + \overline{\varepsilon}_s(i,j)\big]\Big\} \tag{10}$$

for $1 \leq s \leq m$ where $\tilde{w}_{m+1}(i, j) = 0$, $\overline{\varepsilon}_s(i, j) = 0$ unless $s = k$, $\tilde{y}(i, j)$ is the actual output, $\tilde{w}_s(i, j)$ is the actual signal of $w_s(i, j)$ and $\overline{\varepsilon}_k(i, j) = Q[\alpha_k \tilde{y}(i, j)] - \alpha_k \tilde{y}(i, j)$ is the roundoff noise due to quantizer $Q[\,\cdot\,]$. Subtracting (8) from (10) yields

$$\delta y(i,j) = \delta w_1(i,j)$$
$$\delta w_s(i,j) = \rho_s^h(z_1)^{-1}\big[\delta w_{s+1}(i,j) - \alpha_s \delta y(i,j) - \overline{\varepsilon}_s(i,j)\big] \tag{11a}$$

where

$$\delta y(i,j) = \tilde{y}(i,j) - y(i,j)$$
$$\delta w_s(i,j) = \tilde{w}_s(i,j) - w_s(i,j) = \overline{\Delta}_s[\tilde{x}_s^h(i,j) - x_s^h(i,j)] = \overline{\Delta}_s\, \delta x_s^h(i,j) \tag{11b}$$

If a 1-D state-space model $(\boldsymbol{A}_1, \overline{\boldsymbol{e}}_k, \boldsymbol{c}_1)_m$ is realized using (5) from (11a), the transfer function from $-\overline{\varepsilon}_k(i, j)$ to $\delta y(i, j)$ is given by

$$H_{1k}(z_1) = \boldsymbol{c}_1(z_1 \boldsymbol{I}_m - \boldsymbol{A}_1)^{-1}\overline{\boldsymbol{e}}_k \quad \text{for} \quad k = 1, 2, \cdots, m \tag{12}$$

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where $\overline{\boldsymbol{e}}_k$ is the kth column of an identity matrix $\boldsymbol{I}_m$ and

$$\boldsymbol{c}_1 = [\, \overline{\Delta}_1 \;\; 0 \;\; \cdots \;\; 0 \,]$$

$$\boldsymbol{A}_1 = \begin{bmatrix} -\alpha_1\overline{\Delta}_1 & \overline{\Delta}_2 & \cdots & 0 \\ -\alpha_2\overline{\Delta}_1 & 0 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \overline{\Delta}_m \\ -\alpha_m\overline{\Delta}_1 & 0 & \cdots & 0 \end{bmatrix} + \begin{bmatrix} \overline{\gamma}_1 & 0 & \cdots & 0 \\ 0 & \overline{\gamma}_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \overline{\gamma}_m \end{bmatrix}$$

Based on above analysis, it is natural to define the roundoff noise gain in terms of $H_{1k}(z_1)$ as

$$J_1(\alpha_k) = \frac{E[\delta y(i,j)^2]}{E[\overline{\varepsilon}_k(i,j)^2]} = \frac{1}{2\pi j}\oint_{|z_1|=1} H_{1k}^H(z_1)\, H_{1k}(z_1)\frac{dz_1}{z_1} \tag{13}$$

where $H_{1k}^H(z_1)$ denotes the conjugate transpose of $H_{1k}(z_1)$. Substituting (12) into (13), it follows that for $k = 1, 2, \cdots, m$

$$J_1(\alpha_k) = \overline{\boldsymbol{e}}_k^T \left[\sum_{i=0}^{\infty}(\boldsymbol{c}_1\overline{\boldsymbol{A}}_1^i)^T\boldsymbol{c}_1\overline{\boldsymbol{A}}_1^i\right]\overline{\boldsymbol{e}}_k = \overline{\boldsymbol{e}}_k^T\boldsymbol{W}^h\overline{\boldsymbol{e}}_k \tag{14}$$

where $\boldsymbol{W}^h$ is the horizontal observability Grammian which can be obtained by solving the Lyapunov equation [9],[13]

$$\boldsymbol{W}^h = \boldsymbol{A}_1^T\boldsymbol{W}^h\boldsymbol{A}_1 + \boldsymbol{c}_1^T\boldsymbol{c}_1$$

Similarly, the roundoff noise gain produced by the coefficient r_{kl} for $l = 0, 1, \cdots, n$ in the second equation of (8) can be expressed as

$$J_2(r_{kl}) = \overline{\boldsymbol{e}}_k^T\boldsymbol{W}^h\overline{\boldsymbol{e}}_k \quad \text{for} \quad k = 1, 2, \cdots, m \tag{15}$$

With $w_{k+1}(i,j)$ replaced by $\overline{\Delta}_{k+1}x_{k+1}^h(i,j)$ in the second equation of (8), the roundoff noise gain due to $\overline{\Delta}_{k+1}$ can be viewed as a function of r_{kl}, i.e.,

$$J_2(\overline{\Delta}_{k+1}) = J_2(r_{kl}) = \overline{\boldsymbol{e}}_k^T\boldsymbol{W}^h\overline{\boldsymbol{e}}_k \quad \text{for} \quad k = 1, 2, \cdots, m-1 \tag{16}$$

As shown in Fig. 1(a), parameter $\overline{\gamma}_k$ induces a multiplication $\overline{\gamma}_k x_k^h(i,j)$ which produces no roundoff noise if $\overline{\gamma}_k = 0, \pm 1$. Let $\psi(\overline{\gamma}_k)\epsilon_k^h(i,j)$ denote the roundoff noise due to $\overline{\gamma}_k$ where $\psi(\overline{\gamma}_k) = 1$ for all $\overline{\gamma}_k$ except $\overline{\gamma}_k = 0, \pm 1$ for which $\psi(\overline{\gamma}_k) = 0$, and $\delta y(i,j)$ be the corresponding output deviation. Then the transfer function from $\psi(\overline{\gamma}_k)\epsilon_k^h(i,j)$ to $\delta y(i,j)$ becomes $H_{1k}(z_1)$ in (12). Actually, this roundoff noise can be viewed as that generated by the term $r_{kl}x_l^v(i,j)$. Hence

$$J_3(\overline{\gamma}_k) = \psi(\overline{\gamma}_k)J_2(r_{kl}) = \psi(\overline{\gamma}_k)\overline{\boldsymbol{e}}_k^T\boldsymbol{W}^h\overline{\boldsymbol{e}}_k \tag{17}$$

for $k = 1, 2, \cdots, m$ where

$$\psi(\gamma) = \begin{cases} 1 & \text{for} \quad \gamma \neq 0\ , \pm 1 \\ 0 & \text{for} \quad \gamma = 0\ , \pm 1 \end{cases}$$

Concerning the roundoff noise due to coefficient r_{0l} for $l = 0, 1, \cdots, n$ in the first equation of (8), the first equation in (11a) needs to be changed to

$$\delta y(i,j) = \delta w_1(i,j) + \varepsilon_{0l}(i,j) \tag{18}$$

When a 1-D state-space model $(\boldsymbol{A}_1, \boldsymbol{\alpha}, -\boldsymbol{c}_1, 1)_m$ is realized from (11a) whose first equation was replaced by (18) and $\overline{\varepsilon}_s(i,j) = 0$ in the second equation, the transfer function $H_{10}(z_1)$ from $\varepsilon_{0l}(i,j)$ to $\delta y(i,j)$ is given by

$$H_{10}(z_1) = -\boldsymbol{c}_1(z_1\boldsymbol{I}_m - \boldsymbol{A}_1)^{-1}\boldsymbol{\alpha} + 1 \tag{19}$$

where $\boldsymbol{\alpha} = [\alpha_1, \alpha_2, \cdots, \alpha_m]^T$. Hence the roundoff noise gain caused by the coefficient r_{0l} is found to be

$$J_4(r_{0l}) = \boldsymbol{\alpha}^T\boldsymbol{W}^h\boldsymbol{\alpha} + 1 \quad \text{for} \quad l = 0, 1, \cdots, n \tag{20}$$

Supposing that $w_1(i,j)$ in the first equation of (8) is replaced by $\overline{\Delta}_1 x_1^h(i,j)$, the roundoff noise gain produced by $\overline{\Delta}_1$ is identical to that by r_{0l}, which leads to

$$J_4(\overline{\Delta}_1) = J_4(r_{0l}) = \boldsymbol{\alpha}^T\boldsymbol{W}^h\boldsymbol{\alpha} + 1 \tag{21}$$

We now examine the roundoff noise caused by the term $\beta_p x_p^v(i,j)$ for $1 \leq p \leq n$ at the output. Due to the product quantization, for the actual filter implemented by a FWL system, (8) and (9) can be written as

$$\begin{aligned} \tilde{y}(i,j) &= \tilde{w}_1(i,j) + \sum_{l=1}^{n} r_{0l}\tilde{x}_l^v(i,j) \\ &\quad + r_{00}\Big[u(i,j) - \sum_{l=1}^{n}\beta_l\tilde{x}_l^v(i,j) - \hat{\varepsilon}_p(i,j)\Big] \\ \tilde{w}_k(i,j) &= \rho_k^h(z_1)^{-1}\Big\{\tilde{w}_{k+1}(i,j) + \sum_{l=1}^{n} r_{kl}\tilde{x}_l^v(i,j) \\ &\quad + r_{k0}\Big[u(i,j) - \sum_{l=1}^{n}\beta_l\tilde{x}_l^v(i,j) - \hat{\varepsilon}_p(i,j)\Big] - \alpha_k\tilde{y}(i,j)\Big\} \end{aligned} \tag{22}$$

for $k = 1, 2, \cdots, m$ where $\tilde{w}_{m+1}(i,j) = 0$, and

$$\begin{aligned} \tilde{x}_1^v(i,j+1) &= \hat{\gamma}_1\tilde{x}_1^v(i,j) \\ &\quad + \hat{\Delta}_1\Big[u(i,j) - \sum_{l=1}^{n}\beta_l\tilde{x}_l^v(i,j) - \hat{\varepsilon}_p(i,j)\Big] \\ \tilde{x}_l^v(i,j+1) &= \hat{\Delta}_l\tilde{x}_{l-1}^v(i,j) + \hat{\gamma}_l\tilde{x}_l^v(i,j) \end{aligned} \tag{23}$$

for $l = 2, 3, \cdots, n$, respectively, where $\tilde{x}_l^v(i,j)$ denotes the actual signal of $x_l^v(i,j)$ and $\hat{\varepsilon}_p(i,j) = Q[\beta_p\tilde{x}_p^v(i,j)] - \beta_p\tilde{x}_p^v(i,j)$ is the roundoff noise due to quantizer $Q[\cdot]$. Subtracting (8) from (22) yields

$$\begin{aligned} \delta y(i,j) &= \delta w_1(i,j) + \sum_{l=1}^{n} r_{0l}\delta x_l^v(i,j) \\ &\quad + r_{00}\Big[-\hat{\varepsilon}_p(i,j) - \sum_{l=1}^{n}\beta_l\delta x_l^v(i,j)\Big] \\ \delta w_k(i,j) &= \rho_k^h(z_1)^{-1}\Big\{\delta w_{k+1}(i,j) + \sum_{l=1}^{n} r_{kl}\delta x_l^v(i,j) \\ &\quad + r_{k0}\Big[-\hat{\varepsilon}_p(i,j) - \sum_{l=1}^{n}\beta_l\delta x_l^v(i,j)\Big] - \alpha_k\delta y(i,j)\Big\} \end{aligned} \tag{24}$$

for $k = 1, 2, \cdots, m$ where $\delta w_{m+1}(i,j) = 0$ and $\delta x_l^v(i,j) = \tilde{x}_l^v(i,j) - x_l^v(i,j)$ for $l = 1, 2, \cdots, n$. Using (5) and (11b),

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we can write (24) as

$$
\begin{aligned}
\delta y(i,j) &= \overline{\Delta}_1 \delta x_1^h(i,j) + \sum_{l=1}^{n} (r_{0l} - r_{00}\beta_l)\delta x_l^v(i,j) \\
&\quad + r_{00} \cdot -\hat{\varepsilon}_p(i,j) \\
\delta x_k^h(i+1,j) &= -\alpha_k \overline{\Delta}_1 \delta x_1^h(i,j) + \overline{\Delta}_{k+1}\delta x_{k+1}^h(i,j) \\
&\quad + \sum_{l=1}^{n}\big(r_{kl} - \alpha_k r_{0l} - r_{k0}\beta_l + r_{00}\alpha_k\beta_l\big)\delta x_l^v(i,j) \\
&\quad + \overline{\gamma}_k \delta x_k^h(i,j) + (r_{k0} - r_{00}\alpha_k)\cdot -\hat{\varepsilon}_p(i,j)
\end{aligned} \tag{25}
$$

for $k = 1, 2, \cdots, m$ where $x_{m+1}^h(i,j) = 0$. By subtracting (9) from (23), we obtain

$$
\begin{aligned}
\delta x_1^v(i,j+1) &= \hat{\gamma}_1 \delta x_1^v(i,j) + \hat{\Delta}_1\Big[-\hat{\varepsilon}_p(i,j) - \sum_{l=1}^{n}\beta_l \delta x_l^v(i,j)\Big] \\
\delta x_l^v(i,j+1) &= \hat{\Delta}_l \delta x_{l-1}^v(i,j) + \hat{\gamma}_l \delta x_l^v(i,j)
\end{aligned} \tag{26}
$$

At this point, we consider a 2-D local state-space realization of (25) and (26). The transfer function from $-\hat{\varepsilon}_p(i,j)$ to $\delta y(i,j)$ is then found to be [13]

$$
\begin{aligned}
H(z_1,z_2) &= d + \boldsymbol{c}_1(z_1\boldsymbol{I}_m - \boldsymbol{A}_1)^{-1}\boldsymbol{b}_1 \\
&\quad + \big[\boldsymbol{c}_1(z_1\boldsymbol{I}_m - \boldsymbol{A}_1)^{-1}\boldsymbol{A}_2 + \boldsymbol{c}_2\big](z_2\boldsymbol{I}_n - \boldsymbol{A}_4)^{-1}\boldsymbol{b}_2
\end{aligned} \tag{27}
$$

where

$$
\begin{aligned}
\boldsymbol{A}_2 &= \begin{bmatrix} r_{11} & r_{12} & \cdots & r_{1n} \\ r_{21} & r_{22} & \cdots & r_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ r_{m1} & r_{m2} & \cdots & r_{mn} \end{bmatrix} - \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_m \end{bmatrix} \begin{bmatrix} r_{01} & r_{02} & \cdots & r_{0n} \end{bmatrix} \\
&\quad - \begin{bmatrix} r_{10} \\ r_{20} \\ \vdots \\ r_{m0} \end{bmatrix} \begin{bmatrix} \beta_1 & \beta_2 & \cdots & \beta_n \end{bmatrix} + r_{00} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_m \end{bmatrix} \begin{bmatrix} \beta_1 & \beta_2 & \cdots & \beta_n \end{bmatrix} \\
\boldsymbol{A}_4 &= \begin{bmatrix} -\beta_1\hat{\Delta}_1 & -\beta_2\hat{\Delta}_1 & \cdots & -\beta_n\hat{\Delta}_1 \\ \hat{\Delta}_2 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & \hat{\Delta}_n & 0 \end{bmatrix} + \begin{bmatrix} \hat{\gamma}_1 & 0 & \cdots & 0 \\ 0 & \hat{\gamma}_2 & \ddots & \vdots \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \cdots & \hat{\gamma}_n \end{bmatrix} \\
\boldsymbol{b}_1 &= \begin{bmatrix} r_{10} & r_{20} & \cdots & r_{m0} \end{bmatrix}^T - r_{00}\begin{bmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_m \end{bmatrix}^T \\
\boldsymbol{b}_2 &= \begin{bmatrix} \hat{\Delta}_1 & 0 & \cdots & 0 \end{bmatrix}^T \qquad d = r_{00} \\
\boldsymbol{c}_2 &= \begin{bmatrix} r_{01} & r_{02} & \cdots & r_{0n} \end{bmatrix} - r_{00}\begin{bmatrix} \beta_1 & \beta_2 & \cdots & \beta_n \end{bmatrix}
\end{aligned}
$$

Based on this, the roundoff noise gain defined by $J_5(\beta_p) = E[\delta y(i,j)^2]/E[\hat{\varepsilon}_p(i,j)^2]$ can be expressed as

$$
J_5(\beta_p) = \frac{1}{(2\pi j)^2}\oint_{|z_1|=1}\oint_{|z_2|=1} |H(z_1,z_2)|^2 \frac{dz_1 dz_2}{z_1 z_2} \tag{28}
$$

Substituting (27) into (28) yields

$$
\begin{aligned}
J_5(\beta_p) &= \boldsymbol{b}_1^T \boldsymbol{W}^h \boldsymbol{b}_1 + \boldsymbol{b}_2^T \boldsymbol{W}^v \boldsymbol{b}_2 + d^2 \\
&= \boldsymbol{b}_1^T \boldsymbol{W}^h \boldsymbol{b}_1 + \hat{\Delta}_1^2 \hat{\boldsymbol{e}}_1^T \boldsymbol{W}^v \hat{\boldsymbol{e}}_1 + r_{00}^2
\end{aligned} \tag{29}
$$

where $\boldsymbol{W}^v$ is the vertical observability Grammian which can be obtained by solving the Lyapunov equation [9], [13]

$$
\boldsymbol{W}^v = \boldsymbol{A}_4^T \boldsymbol{W}^v \boldsymbol{A}_4 + \boldsymbol{A}_2^T \boldsymbol{W}^h \boldsymbol{A}_2 + \boldsymbol{c}_2^T \boldsymbol{c}_2
$$

Due to the product quantization caused by $\hat{\Delta}_1$ in the first equation of (9), the actual filter implemented by a FWL system can be written as

$$
\begin{aligned}
\tilde{y}(i,j) &= \tilde{w}_1(i,j) + \sum_{l=1}^{n} r_{0l}\tilde{x}_l^v(i,j) \\
&\quad + r_{00}\Big[u(i,j) - \sum_{l=1}^{n}\beta_l \tilde{x}_l^v(i,j)\Big] \\
\tilde{w}_k(i,j) &= \rho_k^h(z_1)^{-1}\Big\{\tilde{w}_{k+1}(i,j) + \sum_{l=1}^{n} r_{kl}\tilde{x}_l^v(i,j) \\
&\quad + r_{k0}\Big[u(i,j) - \sum_{l=1}^{n}\beta_l \tilde{x}_l^v(i,j)\Big] - \alpha_k \tilde{y}(i,j)\Big\}
\end{aligned} \tag{30}
$$

for $k = 1, 2, \cdots, m$ where $\tilde{w}_{m+1}(i,j) = 0$, and

$$
\begin{aligned}
\tilde{x}_1^v(i,j+1) &= \hat{\gamma}_1 \tilde{x}_1^v(i,j) + \hat{\Delta}_1\Big[u(i,j) - \sum_{l=1}^{n}\beta_l \tilde{x}_l^v(i,j)\Big] \\
&\quad + \varepsilon_1^v(i,j) \\
\tilde{x}_l^v(i,j+1) &= \hat{\Delta}_l \tilde{x}_{l-1}^v(i,j) + \hat{\gamma}_l \tilde{x}_l^v(i,j)
\end{aligned} \tag{31}
$$

for $l = 2, 3, \cdots, n$, respectively, where $\varepsilon_1^v(i,j)$ is the roundoff noise caused by $\hat{\Delta}_1$. Subtracting (8) from (30) yields

$$
\begin{aligned}
\delta y(i,j) &= \delta w_1(i,j) + \sum_{l=1}^{n} r_{0l}\delta x_l^v(i,j) - r_{00}\sum_{l=1}^{n}\beta_l \delta x_l^v(i,j) \\
\delta w_k(i,j) &= \rho_k^h(z_1)^{-1}\Big\{\delta w_{k+1}(i,j) + \sum_{l=1}^{n} r_{kl}\delta x_l^v(i,j) \\
&\quad - r_{k0}\sum_{l=1}^{n}\beta_l \delta x_l^v(i,j) - \alpha_k \delta y(i,j)\Big\}
\end{aligned} \tag{32}
$$

for $k = 1, 2, \cdots, m$ where $\delta w_{m+1}(i,j) = 0$. Subtracting (9) from (31), we obtain

$$
\begin{aligned}
\delta x_1^v(i,j+1) &= \hat{\gamma}_1 \delta x_1^v(i,j) - \hat{\Delta}_1 \sum_{l=1}^{n}\beta_l \delta x_l^v(i,j) + \varepsilon_1^v(i,j) \\
\delta x_l^v(i,j+1) &= \hat{\Delta}_l \delta x_{l-1}^v(i,j) + \hat{\gamma}_l \delta x_l^v(i,j)
\end{aligned} \tag{33}
$$

for $l = 2, 3, \cdots, n$, respectively. Using (5) and (11b), we consider a 2-D local state-space realization of (32) and (33). The transfer function from $\varepsilon_1^v(i,j)$ to $\delta y(i,j)$ is then found to be

$$
H_{21}(z_1,z_2) = \big[\boldsymbol{c}_1(z_1\boldsymbol{I}_m - \boldsymbol{A}_1)^{-1}\boldsymbol{A}_2 + \boldsymbol{c}_2\big](z_2\boldsymbol{I}_n - \boldsymbol{A}_4)^{-1}\hat{\boldsymbol{e}}_1 \tag{34}
$$

where $\hat{\boldsymbol{e}}_l$ denotes the lth column of an identity matrix $\boldsymbol{I}_n$. Hence the roundoff noise gain due to $\hat{\Delta}_1$ becomes

$$
J_6(\hat{\Delta}_1) = \hat{\boldsymbol{e}}_1^T \boldsymbol{W}^v \hat{\boldsymbol{e}}_1 \tag{35}
$$

Similarly, the roundoff noise gain due to $\hat{\Delta}_l$ for $l = 2, 3, \cdots, n$ is given by

$$
J_6(\hat{\Delta}_l) = \hat{\boldsymbol{e}}_l^T \boldsymbol{W}^v \hat{\boldsymbol{e}}_l \quad \text{for} \quad l = 2, 3, \cdots, n \tag{36}
$$

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and the roundoff noise gain due to $\hat{\gamma}_l$ for $l = 1, 2, \cdots, n$ can be written as

$$J_6(\hat{\gamma}_l) = \psi(\hat{\gamma}_l)\hat{\boldsymbol{e}}_l^T \boldsymbol{W}^v \hat{\boldsymbol{e}}_l \quad \text{for} \quad l = 1, 2, \cdots, n \tag{37}$$

Based on the above analysis, the total roundoff noise gain of the filter structure in Fig. 2 can be defined as

$$\begin{aligned} J_\rho = & \sum_{k=1}^{m}\Big[J_1(\alpha_k)+\sum_{l=0}^{n} J_2(r_{kl})+J_3(\overline{\gamma}_k)\Big] \\ & + \sum_{k=1}^{m-1} J_2(\overline{\Delta}_{k+1}) + J_4(\overline{\Delta}_1) + J_4(r_{00}) \\ & + \sum_{l=1}^{n}\Big[J_4(r_{0l}) + J_5(\beta_l) + J_6(\hat{\Delta}_l) + J_6(\hat{\gamma}_l)\Big] \end{aligned} \tag{38}$$

which can be written as

$$\begin{aligned} J_\rho = & (n+3)\,\mathrm{tr}\big[\boldsymbol{W}^h\big] - \overline{\boldsymbol{e}}_m^T \boldsymbol{W}^h \overline{\boldsymbol{e}}_m + \mathrm{tr}\big[\overline{\boldsymbol{\Psi}}\boldsymbol{W}^h\big] \\ & + (n+2)\big(\boldsymbol{\alpha}^T \boldsymbol{W}^h \boldsymbol{\alpha}+1\big)+\mathrm{tr}\big[\boldsymbol{W}^v\big]+\mathrm{tr}\big[\hat{\boldsymbol{\Psi}}\boldsymbol{W}^v\big] \\ & + n\big(\boldsymbol{b}_1^T \boldsymbol{W}^h \boldsymbol{b}_1 + \hat{\Delta}_1^2 \hat{\boldsymbol{e}}_1^T \boldsymbol{W}^v \hat{\boldsymbol{e}}_1 + r_{00}^2\big) \end{aligned} \tag{39}$$

where

$$\begin{aligned} \overline{\boldsymbol{\Psi}} &= \mathrm{diag}\{\psi(\overline{\gamma}_1), \psi(\overline{\gamma}_2), \cdots, \psi(\overline{\gamma}_m)\} \\ \hat{\boldsymbol{\Psi}} &= \mathrm{diag}\{\psi(\hat{\gamma}_1), \psi(\hat{\gamma}_2), \cdots, \psi(\hat{\gamma}_n)\} \end{aligned}$$

Remark 1: At this point, it is of interest to note that the roundoff noise gain for state-space realization of the filter structure in (7) can be evaluated as [13]

$$\begin{aligned} J_{S\rho} = & (n+3)\mathrm{tr}[\boldsymbol{W}^h] - \overline{\boldsymbol{e}}_m^T \boldsymbol{W}^h \overline{\boldsymbol{e}}_m \\ & + \sum_{k=2}^{m} \psi(\overline{\gamma}_k)\overline{\boldsymbol{e}}_k^T \boldsymbol{W}^h \overline{\boldsymbol{e}}_k + n\hat{\boldsymbol{e}}_1^T \boldsymbol{W}^v \hat{\boldsymbol{e}}_1 \\ & + \mathrm{tr}[\boldsymbol{W}^v] + \sum_{l=2}^{n} \psi(\hat{\gamma}_l)\hat{\boldsymbol{e}}_l^T \boldsymbol{W}^v \hat{\boldsymbol{e}}_l + n + 2 \end{aligned} \tag{40}$$

From (39) and (40), it follows that

$$\begin{aligned} J_\rho - J_{S\rho} = & \psi(\overline{\gamma}_1)\overline{\boldsymbol{e}}_1^T \boldsymbol{W}^h \overline{\boldsymbol{e}}_1 + \psi(\hat{\gamma}_1)\hat{\boldsymbol{e}}_1^T \boldsymbol{W}^v \hat{\boldsymbol{e}}_1 + (n+2)\boldsymbol{\alpha}^T \boldsymbol{W}^h \boldsymbol{\alpha} \\ & + n\Big[\boldsymbol{b}_1^T \boldsymbol{W}^h \boldsymbol{b}_1 + (\hat{\Delta}_1^2 - 1)\hat{\boldsymbol{e}}_1^T \boldsymbol{W}^v \hat{\boldsymbol{e}}_1 + r_{00}^2\Big] \end{aligned} \tag{41}$$

It is noted that the difference $J_\rho - J_{S\rho}$ evaluated in (41) is due to the different number of parameters (coefficients) between the filter structure in (7) and its state-space realization.

IV. A Numerical Example

Consider a 2-D stable SD digital filter of order $(m, n) = (3, 3)$ in (1) with

$$\begin{aligned} \begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix} &= \begin{bmatrix} -2.173645 & 1.836929 & -0.599655 \end{bmatrix} \\ \begin{bmatrix} b_1 & b_2 & b_3 \end{bmatrix} &= \begin{bmatrix} -2.280029 & 1.887939 & -0.564961 \end{bmatrix} \end{aligned}$$

$$\begin{bmatrix} c_{kl} \end{bmatrix} = \begin{bmatrix} 0.019421 & -0.027724 & 0.011468 & -0.000087 \\ 0.004839 & 0.017545 & -0.050267 & 0.033061 \\ -0.004328 & -0.008847 & 0.096260 & -0.083801 \\ -0.000138 & 0.007979 & -0.052927 & 0.062607 \end{bmatrix}$$

The numerical results obtained by applying the technique in [13] were summarized in comparison with two cases of $\boldsymbol{\gamma}_z = [0,\ 0,\ \cdots,\ 0]^T$ and $\boldsymbol{\gamma}_\delta = [1,\ 1,\ \cdots,\ 1]^T$ in Table I where

$$\begin{aligned} \boldsymbol{\gamma}\big(J_\rho^{opt}\big) &= \begin{bmatrix} 1.000 & 0.625 & 0.750 & 0.000 & 1.000 & 0.625 \end{bmatrix} \\ \boldsymbol{\gamma}\big(J_{S\rho}^{opt}\big) &= \begin{bmatrix} 0.250 & 0.625 & 0.750 & -0.750 & 0.750 & 0.750 \end{bmatrix} \end{aligned}$$

TABLE I
Performance Comparison Among Various $\boldsymbol{\gamma}$

	$\boldsymbol{\gamma}_z$	$\boldsymbol{\gamma}_\delta$	$\boldsymbol{\gamma}(J_\rho^{opt})$	$\boldsymbol{\gamma}(J_{S\rho}^{opt})$
J_ρ	922.4951	85.6754	51.2935	55.9660
$J_{S\rho}$	958.0212	81.6083	54.2407	51.5502

V. Conclusion

An expression of the roundoff noise gain for the resulting structure has been derived and investigated. Moreover, the roundoff noise gain has been compared with that deduced in a recent study of generalized direct-form II state-space realization of 2-D SD digital filters. In a numerical example, the roundoff noise gains have been minimized with respect to the free parameters subject to l_2-scaling constraints through exhaustive search in a finite element space [13].

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