# A Closed-Form Transfer Function of 2-D Maximally Flat Half-Band FIR Digital Filters with Arbitrary Filter Orders

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***Abstract*—2-D maximally flat diamond-shaped half-band linear phase FIR digital filters are used in sampling structure conversion. In some cases, this filter is expected to have different filter order for each dimension. However, the conventional methods can realize such filters only if difference between each order is 2, 4 and 6. In this paper, we proposed a closed-form transfer function of 2-D low-pass maximally flat diamond-shaped half-band FIR digital filters with arbitrary filter orders. The constraints to treat arbitrary filter orders are firstly proposed. Then, a closed-form transfer function is achieved by using Bernstein polynomial.**

## I. Introduction

Two-dimensional (2-D) finite impulse response (FIR) digital filters are one of the important systems for 2-D signals such as image signals and plane waves. 2-D diamond-shaped half-band linear phase (DSHBLP) FIR digital filter is used to perform sampling rate conversion for quincuncial sampled 2-D signals [8]. It is very easy to implement the 2-D DSHBLP FIR digital filter because almost half filter coefficients are zero since impulse response of this filter has quincuncial sampling pattern.

Many design methods for 2-D low-pass DSHB FIR digital filters have been proposed [1]–[5], [7]. Especially, design methods for 2-D low-pass maximally flat (MF) DSHBLP FIR digital filters have been proposed to achieve high accurate extraction of signal [3]–[5], [7]. Yoshida et al. proposed the design method by solving the linear simultaneous equations obtained from the constraints about the magnitude flatness at $(\omega_1, \omega_2) = (0, 0)$ and $(\omega_1, \omega_2) = (\pi, \pi)$ [3]. However, the linear simultaneous equations need to be formulated and solved each time when design specification is changed. To solve this problem, Cooklev et al. gave the closed-form transfer function of same filter by using Bernstein polynomial [4], [5].

Above MFDSHBLP FIR digital filters are subjected to have the same filter orders for both dimensions. By contrast, 2-D MFDSHBLP FIR digital filters are required in some applications, e.g. interlace-to-noninterlace scanning converter in TV signal processing [6], [9]. Inui et al. proposed the design method of the 2-D MFDSHBLP FIR digital filters with different filter orders by using linear equations [7]. However, the method can not design the filter with arbitrary filter orders. Furthermore, the method also can not realize optimal solution since their magnitude response is not monotonically decreasing.

In this paper, we propose a design method for 2-D low-pass MFDSHBLP FIR digital filters with arbitrary filter orders. To design with arbitrary filter orders and realize monotonically decreasing magnitude response, we propose new constraints. The proposed transfer function is achieved as a closed-form solution by using Bernstein polynomial. The parameter of the proposed method is only the flatness degree for magnitude response at $(\omega_1, \omega_2) = (0, 0)$ and $(\omega_1, \omega_2) = (\pi, \pi)$. Finally, thought design examples, it is confirmed that the proposed method can design the filters regardless the filter order difference and realize the monotonically decreasing magnitude response.

## II. The proposed method

In general, the transfer function of an $(N_1 \times N_2)$ order 2-D FIR digital filter is given as

$$H(z_1, z_2) = \sum_{n_1=0}^{N_1} \sum_{n_2=0}^{N_2} h(n_1, n_2) z_1^{-n_1} z_2^{-n_2}, \tag{1}$$

where $N_1$ and $N_2$ are the filter orders, and $h(n_1, n_2)$ is the impulse response. Here, HB LP digital filter can be realized only with even orders and the impulse response sequence is symmetric, i.e., $h(n_1, n_2) = h(N_1 - n_1, n_2) = h(n_1, N_2 - n_2) = h(N_1 - n_1, N_2 - n_2)$. Consequently, the frequency response of 2-D zero-phase FIR digital filters is given as

$$H_0(\omega_1, \omega_2) = \sum_{n_1=0}^{N_1/2} \sum_{n_2=0}^{N_2/2} \tilde{h}(n_1, n_2) \cos n_1\omega_1 \cos n_2\omega_2, \tag{2}$$

where $\tilde{h}(n_1, n_2)$ is the filter coefficient [4].

A 2-D zero-phase FIR digital filter is said to be 2-D HB FIR digital filters if

$$H_0(\omega_1, \omega_2) + H_0(\pi - \omega_1, \pi - \omega_2) = 1 \tag{3}$$

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is satisfied for arbitrary $\omega_1$ and $\omega_2$. The impulse response of the filter is subjected to be

$$h(n_1,n_2) = \begin{cases} 0 & \left(n_1 \neq \frac{N_1}{2}, n_2 \neq \frac{N_2}{2}, n_1+n_2 = \text{even}\right) \\ 0.5 & \left(n_1 = \frac{N_1}{2}, n_2 = \frac{N_2}{2}\right) \end{cases}. \quad (4)$$

The frequency response of $(N_1 \times N_1)$ order 2-D low-pass MFDSHB FIR digital filter must satisfy the following constraints [3]:

$$H_0(\omega_1,\omega_2)|_{(\omega_1,\omega_2)=(0,0)} = 1 \quad (5a)$$

$$H_0(\omega_1,\omega_2)|_{(\omega_1,\omega_2)=(0,\pi)} = 0.5 \quad (5b)$$

$$H_0(\omega_1,\omega_2)|_{(\omega_1,\omega_2)=(\pi,0)} = 0.5 \quad (5c)$$

$$\left.\frac{\partial^i H_0}{\partial\omega_1^j \partial\omega_2^{i-j}}\right|_{(\omega_1,\omega_2)=(0,0)} = 0 \begin{pmatrix} i = 2,4,\cdots,N_1-2 \\ j = 0,2,\cdots,i \end{pmatrix} \quad (5d)$$

$$\left.\frac{\partial^i H_0}{\partial\omega_1^j \partial\omega_2^{i-j}}\right|_{(\omega_1,\omega_2)=(\pi,\pi)} = 0 \begin{pmatrix} i = 0,2,\cdots,N_1-2 \\ j = 0,2,\cdots,i \end{pmatrix}. \quad (5e)$$

The number of constraints are equal to the degree of freedom of the transfer function. That is, the number of constraints are equal to the number of non zero coefficients.

Now, we consider to design $(N_1 \times (N_1+d))$ order 2-D low-pass MFDSHBLP FIR digital filter. Note that $d$ is the difference of the filter orders, and its value is even. It is clear that $N_1 \times d$ additional constraints are needed to design such filters since the degree of freedom of the transfer function increases. The conventional method gave the following additional constraints [7]:

$$\sum_{j=0}^{i/2} \binom{i}{2j} \left.\frac{\partial^i H_0}{\partial\omega_1^{2j} \partial\omega_2^{i-2j}}\right|_{(\omega_1,\omega_2)=(0,0)} = 0 \ (i = \text{even}) \quad (6a)$$

$$\left.\frac{\partial^i \omega_2}{\partial\omega_1^i}\right|_{(\omega_1,\omega_2)=(\pi/2,\pi/2)} = \begin{cases} -1 & (i=1) \\ 0 & (i = 2,3,4,\cdots) \end{cases}. \quad (6b)$$

However, it is very difficult to derive these constraints for an arbitrary $d$. Moreover, design which all MF FIR digital filters obtained are not necessary monotonically decreasing magnitude response.

To solve this problem, the proposed method gives the following new additional constraints:

$$\left.\frac{\partial^i H_0}{\partial\omega_1^j \partial\omega_2^{i-j}}\right|_{(\omega_1,\omega_2)=(0,0)} = 0 \begin{pmatrix} i = 2,4,\cdots,N_1+d-2 \\ j = 0,2,\cdots,2s_i-2 \end{pmatrix} \quad (7a)$$

$$\left.\frac{\partial^i H_0}{\partial\omega_1^j \partial\omega_2^{i-j}}\right|_{(\omega_1,\omega_2)=(\pi,\pi)} = 0 \begin{pmatrix} i = 0,2,\cdots,N_1+d-2 \\ j = 0,2,\cdots,2s_i-2 \end{pmatrix}, \quad (7b)$$

where $s_i$ is the flatness degree for the magnitude response at $(\omega_1,\omega_2) = (0,0)$ and $(\omega_1,\omega_2) = (\pi,\pi)$ and given by

$$s_i = \begin{cases} \frac{i}{2}+1 & , i \leq N_1-2 \\ m_i & , N_1 \leq i \leq N_1+\frac{d}{2} \\ \frac{N_1}{2} - s_{N_1+N_2-i}+1 & , i > N_1+\frac{d}{2} \end{cases}, \quad (8)$$

where $\lfloor x \rfloor$ denotes the maximum integer not exceeding $x$, and $m_i$ is an integer parameter to control the shape of equi-amplitude line. The number of $m_i$ is depending on

$$\frac{N_1}{2} = m_{N_1} \geq m_{N_1+2} \geq m_{N_1+4} \geq \cdots \geq \left\lfloor \frac{N_1+2}{4} \right\rfloor. \quad (9)$$

In this paper, a closed-form transfer function satisfying (5) and (7) is proposed using Bernstein polynomial as

$$H_0(\omega_1,\omega_2) = \sum_{n_1=0}^{N_1/2} \sum_{n_2=0}^{N_2/2} \binom{N_1/2}{n_1} \binom{N_2/2}{n_2} f\left(\frac{2n_1}{N_1}, \frac{2n_2}{N_2}\right) \times x^{n_1}(1-x)^{\frac{N_1}{2}-n_1} y^{n_2}(1-y)^{\frac{N_2}{2}-n_2}, \quad (10)$$

where,

$$\binom{a}{n} = \frac{a(a-1)(a-2)\cdots(a-n+1)}{n!}$$

$$x = \frac{1-\cos\omega_1}{2}$$

$$y = \frac{1-\cos\omega_2}{2},$$

and $f(\cdot)$ is the Bernstein coefficient. From (3) and (10), $f(\cdot)$ satisfies the following relationship:

$$f\left(\frac{2n_1}{N_1}, \frac{2n_2}{N_2}\right) + f\left(1-\frac{2n_1}{N_1}, 1-\frac{2n_2}{N_2}\right) = 1. \quad (11)$$

From (11), it is clear that there are only three combinations between $f(2n_1/N_1, 2n_2/N_2)$ and $f(1-2n_1/N_1, 1-2n_2/N_2)$:

$$f\left(\frac{2n_1}{N_1}, \frac{2n_2}{N_2}\right) = 1 \text{ and } f\left(1-\frac{2n_1}{N_1}, 1-\frac{2n_2}{N_2}\right) = 0$$

$$f\left(\frac{2n_1}{N_1}, \frac{2n_2}{N_2}\right) = 0 \text{ and } f\left(1-\frac{2n_1}{N_1}, 1-\frac{2n_2}{N_2}\right) = 1$$

$$f\left(\frac{2n_1}{N_1}, \frac{2n_2}{N_2}\right) = 0.5 \text{ and } f\left(1-\frac{2n_1}{N_1}, 1-\frac{2n_2}{N_2}\right) = 0.5.$$

From the above discussion, $f(\cdot)$ can be derived the following:

$$f\left(\frac{2n_1}{N_1}, \frac{2n_2}{N_2}\right) = \begin{cases} 1 & , (n_1,n_2) \in R \\ 0 & , \left(\frac{N_1}{2}-n_1, \frac{N_2}{2}-n_2\right) \in R \\ 0.5 & , \text{otherwise} \end{cases} \quad (12a)$$

$$R = \left\{ (n_1,n_2) \in \mathbb{N} \,\middle|\, 0 \leq n_1+n_2 \leq \frac{N_2}{2}-1, \; 0 \leq n_1 \leq s_{2(n_1+n_2)}-1 \right\}. \quad (12b)$$

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## III. Design Examples

*Example 1:* In this example, we compare the proposed method and the conventional method [7]. Figures 1(a) and 1(b) show the magnitude responses of the conventional filter with $N_1 = 14$ and $N_2 = 18$. Figures 2(a) and 2(b) show the magnitude response of the proposed filter with the same filter orders. It is clear from fig. 1(b) that the magnitude response of conventional filter is not monotonically decreasing at $(\omega_1, \omega_2) = (0, 0)$ and $(\omega_1, \omega_2) = (\pi, \pi)$. On the other hand, it is clear from fig. 2(b) that the magnitude response of the proposed filter is monotonically decreasing over the whole area. From fig. 3, it is confirmed that the proposed method can achieve the quincuncial sampling pattern as same as the conventional method.

*Example 2:* In this example, we illustrate to realize the 2-D low-pass MFDSHBLP FIR digital filter designed by the proposed method when changing the combination of constraints. For example, if we will design $(10 \times 18)$ order 2-D low-pass MFDSHBLP FIR digital filter which can only realize by using the proposed method because of $d = 8$. In this case, the parameter $m_i$ is appeared only if $i = 12$. From (8), the value of $m_{12}$ is 3, 4, or 5. Figures 4(a), 4(b), 5(a), 5(b), 6(a) and 6(b) show magnitude responses of proposed filter with $N_1 = 10$, $N_2 = 18$ and $s_{12} = \{3, 4, 5\}$, respectively. From figs. 4(a), 4(b), 5(a), 5(b), 6(a) and 6(b), it is confirmed that $m_{12}$ controls the shape of the equi-amplitude lines. Note that all of these filters are HB characteristics and the filter coefficients of each filter has a quincuncial sampling pattern.

## IV. Conclusion

In this paper, we introduced a design method for 2-D low-pass MFDSHBLP FIR digital filters with an arbitrary filter orders. Conventionally, the difference of filter orders for the filters permitted to be 2, 4 and 6. To solve this problem, we proposed the new flatness constraints were firstly formulated. Then, a closed-form transfer function with arbitrary filter orders were derived the parameters of the proposed method are $m_i$. $m_i$ determines the shape of the equi-amplitude line. Through design examples, it is confirmed that the proposed method can realize the monotonically decreasing magnitude response. Furthermore, it is also confirmed that the proposed method can realize various equi-amplitude line by setting $m_i$, and the line approaches to the straight with appropriate $m_i$.

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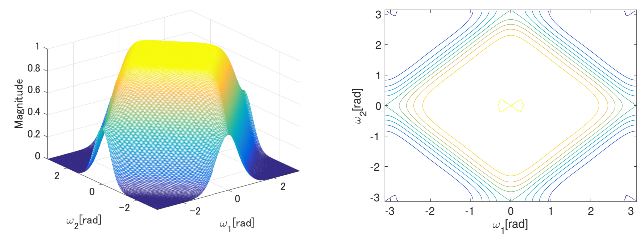
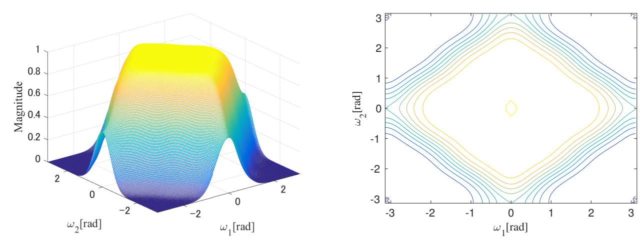


(a) The perspective plot (b) The contour plot

Fig. 1. The frequency responses of $(14 \times 18)$ order MFDSHBLP FIR digital filters designed by the conventional method



(a) The perspective plot (b) The contour plot

Fig. 2. The frequency responses of $(14 \times 18)$ order MFDSHBLP FIR digital filters designed by the proposed method

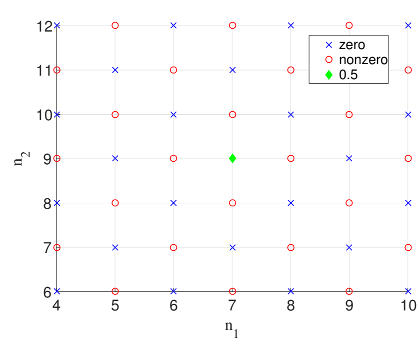
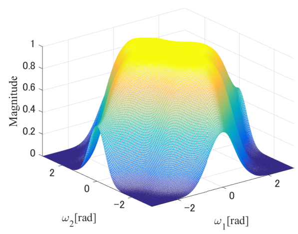
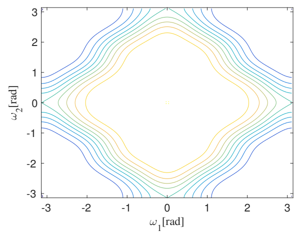



Fig. 3. The impulse response of $(14 \times 18)$ order MFDSHBLP FIR digital filters designed by the proposed method

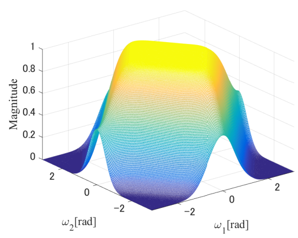



(a) The perspective plot

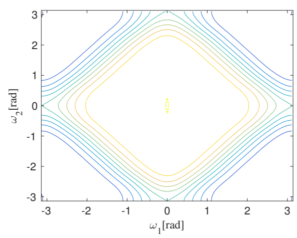



(b) The contour plot

Fig. 4. The frequency responses of $(10 \times 18)$ order MFDSH-BLP FIR digital filters with $s_{12} = 3$

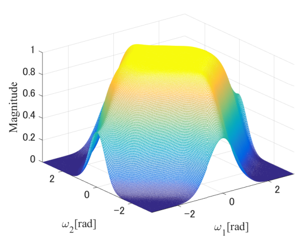



(a) The perspective plot

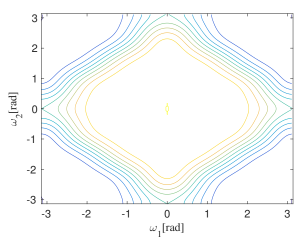



(b) The contour plot

Fig. 5. The frequency responses of $(10 \times 18)$ order MFDSH-BLP FIR digital filters with $s_{12} = 4$




(a) The perspective plot




(b) The contour plot

Fig. 6. The frequency responses of $(10 \times 18)$ order MFDSH-BLP FIR digital filters with $s_{12} = 5$

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