Co-Prime Sampling Jitter Analysis

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***Abstract*—Co-prime arrays and samplers are popular sub-Nyquist schemes for estimating second order statistics at the Nyquist rate. This paper focuses on the perturbations in the array locations or sampling times, and analyzes its effect on the difference set. Based on this analysis we propose a method to estimate the autocorrelation which makes best use of the sampled data in order to improve the estimation accuracy of the autocorrelation and hence the spectral estimate. Our analysis indicates that such an advantage is limited only to samplers, and does not carry over to the antenna arrays. In addition, we obtain expressions for the computational complexity of the autocorrelation estimation and provide an upper bound on the number of multiplications and additions required for its hardware implementation.**

I. Introduction

Sampling jitter has been studied in the literature for the case when the samples are acquired at the Nyquist rate. The procedures for spectral estimation in the case of independent jitter is considered in [1]. The authors also develop procedures for autocovariance estimation under independent and dependent jitter conditions. The properties and relative efficiencies of these estimators are also discussed. The work in [2], analyses the effect of timing jitter on the spatio-frequency covariance matrix which contains delay and direction information. Methods for estimating the jitter variance and for compensating it have also been addressed. Covariance estimation from discrete time observations under jitter and delay conditions is studied in [3], and consistency and asymptotic normality of the estimators are established. System identification under the influence of stochastic sampling jitter noise is considered in [4]. It also provides ways to mitigate this effect for the case when the jitter is unknown and not measurable.

There is little work in the literature on sampling jitter analysis for sub-Nyquist arrays and sampling schemes. A jitter reconstruction system model for sub-Nyquist sampled signals using an annihilating filter and Slepian functions is proposed in [5]. Co-prime and nested samplers are analyzed in spatial and temporal domains under perturbed conditions in [6], [7]. It includes additive perturbations and sampling jitters. It is shown that the errors in the estimated autocorrelation due to non-ideal co-prime sampling is bounded under certain assumptions. Nearly all the work in this domain has analysed the perturbations in a statistical sense. We propose to analyze the effect of sampling jitter on the difference set and describe ways to efficiently estimate the second order statistics under these conditions.

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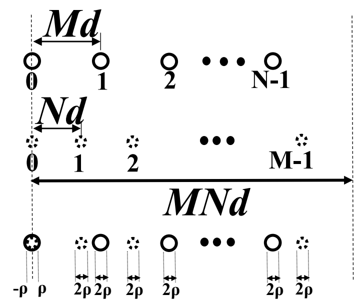

Fig. 1. Co-prime array structure without and with jitter

Section II describes the sampling or array structure under perturbations. The difference set is analyzed in Section III, while the weight function for a blind as well as a non-blind system is described in Section IV. The computational complexity of the proposed scheme is studied in Section V followed by concluding remarks.

II. Co-prime Structure under perturbation

The prototype co-prime array has two sub-arrays with inter-element spacing of Nd and Md, where N and M are co-prime numbers. The extended co-prime array has one of its sub-array extended over an extra period. This gives a filled difference set in the range $-(MN-1)$ to $(MN-1)$, which is not possible with the prototype co-prime array.

Practically, the locations of the array elements could differ from the ideal or design positions. This could be due to manufacturing and positioning errors, turbulence that the system encounters, or clock offsets in the case of sampling. The array structures without and with this perturbation or jitter are shown in Fig. 1. The discussion in this paper will focus on samplers and jitters in sampling times. We will assume that this random jitter, represented by ρ, is less than $1/4^{\text{th}}$ of the Nyquist period d, (i.e., $|\rho| < \frac{d}{4}$), to ensure that the difference set of the perturbed array has values within a tolerance band of $\pm\frac{1}{2}$ with respect to the unperturbed difference values. This is a reasonable assumption as the jitter in practice is not expected to be large and is also made for the ease of tractability in assigning the estimates in the range $l \pm \frac{1}{2}$ unambiguously to difference value l in the unperturbed structure.

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III. Difference Set in the presence of Jitter

A detailed discussion on the difference set of the prototype co-prime array has been presented in [8]. The definitions for the self and cross differences given there also holds in the presence of sampling jitter, except for an additional jitter term. The two perturbed co-prime samplers acquire data which are now represented as $x(Mn + \epsilon_1(n))$ and $x(Nm + \epsilon_2(m))$, where $\epsilon_1(n)$ and $\epsilon_2(m)$ represent the normalized instantaneous displacement or jitter from the true sampling instants with $n \in [0, N-1]$ and $m \in [0, M-1]$. The self difference set for these two samplers are denoted by $\mathcal{L}_{SM}^+$ and $\mathcal{L}_{SN}^+$ respectively, with $\mathcal{L}_{SM}^-$ and $\mathcal{L}_{SN}^-$ representing the negative self differences. $\mathcal{L}_S^+$ and $\mathcal{L}_S^-$ are the union of the positive self differences and the negative self differences respectively, while $\mathcal{L}_S$ represents the union of all the self differences. The cross difference set is denoted by $\mathcal{L}_C^+$, while $\mathcal{L}_C^-$ represents the negative of the values in $\mathcal{L}_C^+$. $\mathcal{L}_C$ represents the union of the positive and the negative cross difference sets.

The union of the positive and negative self differences of the two samplers is given by:

$$\begin{aligned}\mathcal{L}_{SM}^+ \cup \mathcal{L}_{SM}^- &= (Mn_1 + \epsilon_1(n_1)) - (Mn_2 + \epsilon_1(n_2)) \\ &= M(n_1 - n_2) + \Delta_1(n_1, n_2) \qquad (1)\end{aligned}$$

and similarly,

$$\mathcal{L}_{SN}^+ \cup \mathcal{L}_{SN}^- = N(m_1 - m_2) + \Delta_2(m_1, m_2) \qquad (2)$$

where $\Delta_1(n_1, n_2) = \epsilon_1(n_1) - \epsilon_1(n_2)$ and $\Delta_2(m_1, m_2) = \epsilon_2(m_1) - \epsilon_2(m_2)$ with $\Delta_1(n_1, n_2) = 0$ and $\Delta_2(m_1, m_2) = 0$ when $n_1 = n_2$ and $m_1 = m_2$, respectively. The self differences are shown in Fig. 2(a) and 2(b) for $M = 4$ and $N = 3$.

The cross difference set is given by:

$$\begin{aligned}\mathcal{L}_C^+ &= (Mn + \epsilon_1(n)) - (Nm + \epsilon_2(m)) \\ &= Mn - Nm - \Delta_{12}(n, m) \qquad (3)\end{aligned}$$

and similarly,

$$\mathcal{L}_C^- = Nm - Mn + \Delta_{12}(n, m) \qquad (4)$$

where $\Delta_{12}(n, m) = \epsilon_2(m) - \epsilon_1(n)$. The cross differences under perturbation are shown in Fig. 2(c) and 2(d).

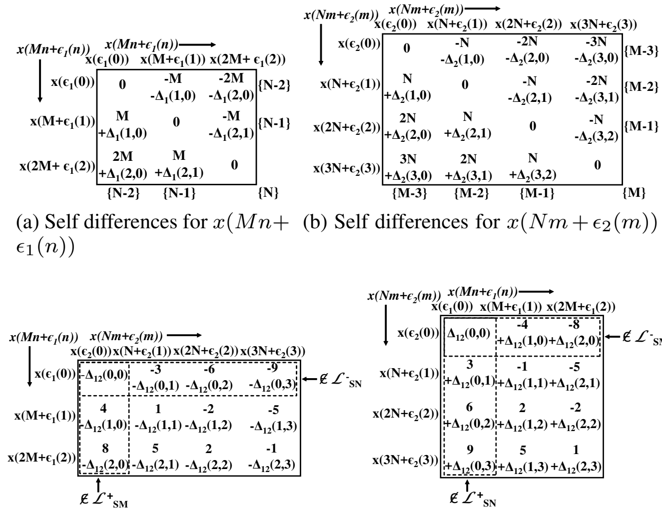
Let us consider an example for a better understanding of the difference set in the presence of sampling jitters. The actual sampling times vary about the ideal sampling times by $\rho \in \left(-\frac{d}{4}, \frac{d}{4}\right)$, which implies that the normalized instantaneous jitter $\epsilon_1(n), \epsilon_2(n) \in (-0.25, 0.25)$. Next, we assume that the jitters for the two samplers, with $M = 4$ and $N = 3$, are:

$$\epsilon_1(n) = \{0.1, -0.1, 0.2\} \text{ and } \epsilon_2(m) = \{0.01, 0.1, -0.02, -0.2\} \qquad (5)$$

The self and cross differences in such a scenario are shown in Fig. 3 and interesting insights can be drawn from these which are presented as Proposition I below. We will assume that the jitter variables; $\Delta_1(n_1, n_2)$, $\Delta_2(m_1, m_2)$, and $\Delta_{12}(n, m)$ modify the ideal difference locations creating new unique locations around the ideal location.

Proposition I:

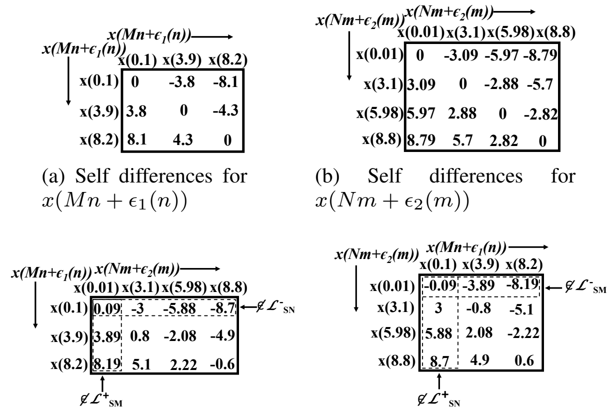
1) There are a maximum of $\frac{N(N-1)}{2} + 1$ distinct values in $\mathcal{L}_{SM}^+$ and $\mathcal{L}_{SM}^-$.



(a) Self differences for $x(Mn+\epsilon_1(n))$ (b) Self differences for $x(Nm+\epsilon_2(m))$

(c) Cross differences $\mathcal{L}_C^+$ (d) Cross differences $\mathcal{L}_C^-$

Fig. 2. Difference set in the presence of sampling jitters



(a) Self differences for $x(Mn + \epsilon_1(n))$ (b) Self differences for $x(Nm + \epsilon_2(m))$

(c) Cross differences $\mathcal{L}_C^+$ (d) Cross differences $\mathcal{L}_C^-$

Fig. 3. Difference set for jitter values given in (5)

2) There are a maximum of $\frac{M(M-1)}{2} + 1$ distinct values in $\mathcal{L}_{SN}^+$ and $\mathcal{L}_{SN}^-$.
3) There are a maximum of $\frac{M(M-1)}{2} + \frac{N(N-1)}{2} + 1$ distinct values in $\mathcal{L}_S^+$ and $\mathcal{L}_S^-$.
4) There are a maximum of $M(M-1) + N(N-1) + 1$ distinct values in $\mathcal{L}_S$.
5) There are a maximum of *MN* distinct values in $\mathcal{L}_C^+$ and $\mathcal{L}_C^-$.
6) There are a maximum of *2MN* distinct values in $\mathcal{L}_C$.
7) The self differences may not form a subset of the cross differences, i.e. $\mathcal{L}_S \not\subseteq \mathcal{L}_C$.
8) The maximum number of unique differences possible in set $\mathcal{L} = \mathcal{L}_C \cup \mathcal{L}_S$ is given by $(M+N)(M+N-1)+1$.

As shown in Fig. 2(a), the lower and upper triangle represent the self difference sets $\mathcal{L}_{SM}^+$ and $\mathcal{L}_{SM}^-$, respectively with the diagonal being common to both. The number of unique values is given by: $1 + \sum_{n=1}^{N-1} n = 1 + \frac{N(N-1)}{2}$. Similar analysis

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holds for the sets $\mathcal{L}_{SN}^{+}$ and $\mathcal{L}_{SN}^{-}$. $\mathcal{L}_{SM}^{+}$ and $\mathcal{L}_{SN}^{+}$ ($\mathcal{L}_{SM}^{-}$ and $\mathcal{L}_{SN}^{-}$) have '0' as a common value, hence $\mathcal{L}_{S}^{+}$ ($\mathcal{L}_{S}^{-}$) has $\frac{M(M-1)}{2}+\frac{N(N-1)}{2}+1$ unique values. The only overlapping self difference value between sampler $x(Mn+\epsilon_1(n))$ and $x(Nm+\epsilon_2(m))$ i.e. ($\mathcal{L}_{SM}^{+}\cup\mathcal{L}_{SM}^{-}$) and ($\mathcal{L}_{SN}^{+}\cup\mathcal{L}_{SN}^{-}$) is '0'. Hence the number of unique values in $\mathcal{L}_S$ is given by: $2\left(\frac{M(M-1)}{2}+\frac{N(N-1)}{2}+1\right)-1=M(M-1)+N(N-1)+1$. This proves Proposition I-1 to I-4.

Proof of Proposition I-5: Let $l_{c_1}=Mn_1+\epsilon_1(n_1)-(Nm_1+\epsilon_2(m_1))$ and $l_{c_2}=Mn_2+\epsilon_1(n_2)-(Nm_2+\epsilon_2(m_2))$ be the elements in set $\mathcal{L}_{C}^{+}$. Let us assume that $l_{c_1}=l_{c_2}$ for some $0\le n_1,n_2\le N-1$ and $0\le m_1,m_2\le M-1$, then:

$$\frac{M}{N}=\frac{(m_1-m_2)+\frac{\Delta_2(m_1,m_2)-\Delta_1(n_1,n_2)}{N}}{n_1-n_2} \tag{6}$$

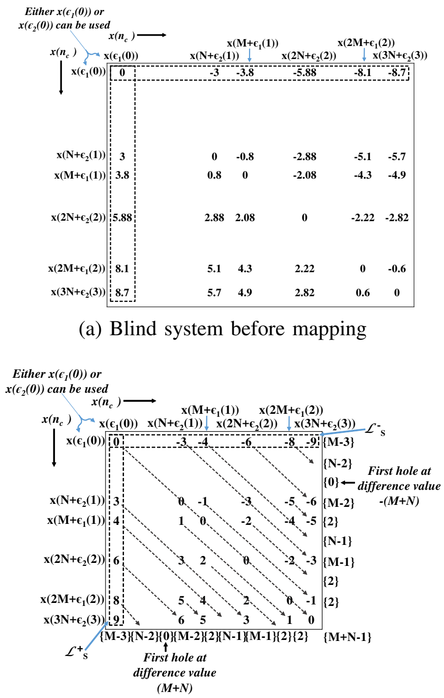
Since $\epsilon_1(n_1)$, $\epsilon_1(n_2)$, $\epsilon_2(m_1)$, and $\epsilon_2(m_2)$ take values in the range $(-\frac{1}{4},\frac{1}{4})$, we have $\Delta_1(n_1,n_2)$ and $\Delta_2(m_1,m_2)$ in the range $(-\frac{1}{2},\frac{1}{2})$. Using the extreme values of this range, we obtain the range for $\Delta_2(m_1,m_2)-\Delta_1(n_1,n_2)$ as $(-1,1)$. When $\Delta_2(m_1,m_2)-\Delta_1(n_1,n_2)=0$, we have $\frac{M}{N}=\frac{m_1-m_2}{n_1-n_2}$, which can never hold since M and N are co-prime, and $m_1-m_2<M$ and $n_1-n_2<N$. Similarly, when $\Delta_2(m_1,m_2)-\Delta_1(n_1,n_2)=\pm1$, we have $\frac{M}{N}=\frac{(m_1-m_2)\pm\frac{1}{N}}{n_1-n_2}$. In general, $\frac{1}{N}<1$ and the right hand side will never equal the co-prime ratio. Hence, set $\mathcal{L}_{C}^{+}$ has MN unique differences. A similar argument holds for $\mathcal{L}_{C}^{-}$, thus proving Proposition I-5.

Proof of Proposition I-6: Let $l_{c_1}=Mn_1+\epsilon_1(n_1)-(Nm_1+\epsilon_2(m_1))$ and $l_{c_2}=Nm_2+\epsilon_2(m_2)-(Mn_2+\epsilon_1(n_2))$ be the elements in the sets $\mathcal{L}_{C}^{+}$ and $\mathcal{L}_{C}^{-}$, respectively. Let us assume that $l_{c_1}=l_{c_2}$ for some $0\le n_1,n_2\le N-1$ and $0\le m_1,m_2\le M-1$, then:

$$\frac{M}{N}=\frac{(m_1+m_2)+\frac{\Delta_{12}(n_1,m_1)+\Delta_{12}(n_2,m_2)}{N}}{n_1+n_2} \tag{7}$$

$\Delta_{12}(n_1,m_1)$ and $\Delta_{12}(n_2,m_2)$ also have values in the range $(-\frac{1}{2},\frac{1}{2})$, hence $\Delta_{12}(n_1,m_1)+\Delta_{12}(n_2,m_2)$ will be in the range $(-1,1)$. When $\Delta_{12}(n_1,m_1)+\Delta_{12}(n_2,m_2)=0$, we have $\frac{M}{N}=\frac{m_1+m_2}{n_1+n_2}$. Since $m_1+m_2<2M$ and $n_1+n_2<2N$, there is a possibility of the right hand side being equal to the co-prime ratio. However, under the assumption that $\Delta_{12}(n_1,m_1)+\Delta_{12}(n_2,m_2)\neq0$, Proposition I-6 holds. Let us assume that $\Delta_{12}(n_1,m_1)+\Delta_{12}(n_2,m_2)$ takes the extreme values of ±1 i.e., $\frac{M}{N}=\frac{(m_1+m_2)\pm\frac{1}{N}}{n_1+n_2}$. Then $m_1+m_2+\frac{1}{N}$ is not an integer and hence we cannot obtain the co-prime ratio. Since $\mathcal{L}_{C}^{+}$ and $\mathcal{L}_{C}^{-}$ have MN unique differences, it can safely be concluded that $\mathcal{L}_C=\mathcal{L}_{C}^{+}\cup\mathcal{L}_{C}^{-}$ has $2MN$ unique values.

Proof of Proposition I-7: Let $l_c=Mn+\epsilon_1(n)-(Nm+\epsilon_2(m))$ be an element in the set $\mathcal{L}_{C}^{+}$. Substituting $m=0$ in this equation gives $l_c=Mn-\Delta_{12}(n,0)$. Substituting $n_2=0$ and $n_1=n$ in the self difference equation (1) gives $l_s=Mn+\Delta_1(n,0)$. l_c and l_s are not equal under the assumption that $-\Delta_{12}(n,0)\neq\Delta_1(n,0)$. Next, we substitute $n=0$ in the cross difference equation giving $l_c=-Nm-\Delta_{12}(0,m)$. Substituting $m_1=0$ and $m_2=m$ in (2) gives $l_s=-Nm+\Delta_2(0,m)$. Then $l_c\neq l_s$ under the assumption that $-\Delta_{12}(0,m)\neq\Delta_1(n,0)$. A similar argument



(a) Blind system before mapping

(b) Blind system after mapping

Fig. 4. Combined set of a blind system

holds true for $l_c\in\mathcal{L}_{C}^{-}$. This proves the claim that $\mathcal{L}_S\not\subseteq\mathcal{L}_C$.

From Proposition I-7 we can conclude that the number of unique values in the set $\mathcal{L}=\mathcal{L}_C\cup\mathcal{L}_S$ is the sum of the distinct values in $\mathcal{L}_S$ and $\mathcal{L}_C$ given by Proposition I-4 and I-6 i.e., $2MN+M(M-1)+N(N-1)+1=(M+N)(M+N-1)+1$.

IV. Weight function in the presence of Jitter

We have obtained the expression for the weight function of the prototype co-prime array in [8]. Here we will obtain the weight function for the prototype co-prime array in the presence of sampling jitter. We describe two types of systems: a blind system and a non-blind system. A blind system is one in which the presence of jitters in the sampling instants is unknown, and the autocorrelation estimation follows the same procedure as in the ideal scenario. While a non-blind system tries to improve the estimation by efficiently utilizing the available data under the assumption that the sampling instants are perturbed.

A blind system assumes that $x(Mn)=x(Nm)$ for $n=m=0$, and the combined set is shown in Fig. 4(a). Such a system would assume a mapping of $[l-\frac{1}{2},l+\frac{1}{2})\to l$, refer Fig. 4(b), under the false assumption that $\epsilon_1(n)=\epsilon_2(m)=0$ which is the case for practical implementation. The number of sample pairs contributing to the estimate prior to mapping is given by Proposition II, while the weight function of the blind system after mapping turns out to be same as that of the prototype co-prime array, as evident from Fig. 4(b).

We refer to this as a blind system since both in the presence of jitter as well as under ideal conditions we follow the same

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estimation procedure, resulting in the same weight function as that for the prototype co-prime array.

Proposition II: Let the number of elements contributing to the estimation at difference value l be denoted by $z(l)$, where l represents the unmapped location and need not be an integer.

1) $z(l) = 1, \{l \in \mathcal{L}_C^+\}$
2) $z(l) = 1, \{l \in \mathcal{L}_C^-\}$
3) $z(l) = 1, \{l \in \mathcal{L}_C^+ \cup \mathcal{L}_C^-\}$
4) $z(l) = M + N,$ for $l = 0$.
5) $z(l) = 1, l \in \mathcal{L}_{SM}^+ \cup \mathcal{L}_{SM}^- - \{0\}$
6) $z(l) = 1, l \in \mathcal{L}_{SN}^+ \cup \mathcal{L}_{SN}^- - \{0\}$

Proposition II is pictorially depicted in Fig. 5 for $M = 4$ and $N = 3$. The example with specific jitter values, as described in the previous section has also been shown in this figure in shaded boxes. Proposition II follows from the previous discussion and Proposition I.

We seek to improve the number of unique sample pairs for estimation in the presence of sampling jitter by efficiently using the available data. The non-blind system maps the differences $l \pm \frac{1}{2} \rightarrow l$, thus leading to Proposition III.

Proposition III: Let the number of elements contributing to the estimation at difference value l for the non-blind system be denoted by $z_{nb}(l)$, where l is an integer that represents the mapped location.

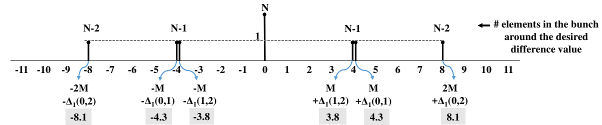
1) For $l \in \mathcal{L}_{SM}^+ \cup \mathcal{L}_{SM}^- - \{0\}$,
   $z_{nb}(l) = (N - i) + 1,$ for $l = \pm Mi, 1 \leq i \leq N - 1$
2) For $l \in \mathcal{L}_{SN}^+ \cup \mathcal{L}_{SN}^- - \{0\}$,
   $z_{nb}(l) = (M - i) + 1,$ for $l = \pm Ni, 1 \leq i \leq M - 1$
3) For $l = 0$, $z_{nb}(l) = M + N + 1$
4) For $l \in \mathcal{L}_C - \mathcal{L}_S$, $z_{nb}(l) = 2$.

The difference between the weight function for the non-blind system and the ideal prototype co-prime array is an additional unique sample pair mapped to the self differences (except for difference value '0') from sets $\mathcal{L}_C^+$ and $\mathcal{L}_C^-$ as the self differences are not a subset of the cross differences.

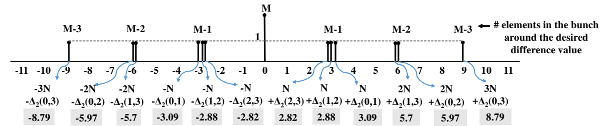
For the case when $l = 0$, the self differences have $M + N$ unique sample pairs plus an additional pair from the set $\mathcal{L}_C^+$ i.e. $(x(\epsilon_1(0)), x(\epsilon_2(0)))$. Thus leading to $M + N + 1$ contributors. Note that the set $\mathcal{L}_C^-$ also gives an estimate at difference value '0' but is generated by the same pair $(x(\epsilon_1(0)), x(\epsilon_2(0)))$ and hence does not provide additional sample pair.

The weight function of a non-blind system based on Proposition III is shown in Fig. 6(a), while Fig. 6(b) displays the weight function for a blind system after mapping, which as described previously, is the same as the prototype co-prime array weight function. Clearly, the proposed non-blind system has more number of contributors for autocorrelation estimation.

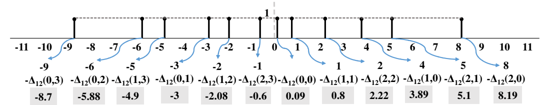
Therefore, for practical estimation of the second order statistics in the presence of sampling jitter we should first compute the estimate using the self differences obtained by the individual samplers (Fig. 2(a) and 2(b)), having contributors as shown in Fig. 5(a) and 5(b). Next, we estimate the cross differences (Fig. 2(c) and 2(d)), having contributors as shown



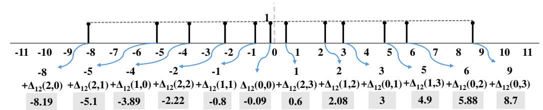
(a) Jitter perturbed weight function of sampler $x(Mn)$



(b) Jitter perturbed weight function of sampler $x(Nm)$

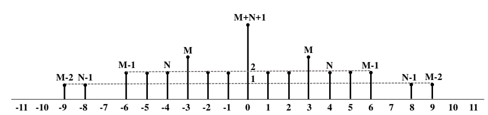


(c) Jitter perturbed cross differences $\mathcal{L}_C^+$

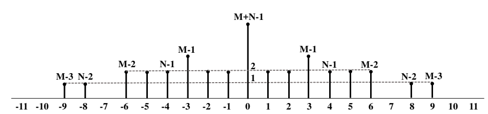


(d) Jitter perturbed cross differences $\mathcal{L}_C^-$

Fig. 5. Jitter perturbed weight function prior to mapping for $M = 4$ and $N = 3$.



(a) Jitter non-blind system



(b) Jitter blind system

Fig. 6. Jitter perturbed weight function post-mapping for $M = 4$ and $N = 3$.

in Fig. 5(c) and 5(d). The advantages obtained for a co-prime sampler are not applicable to an antenna array, since the zeroth location has only one antenna element.

V. Computational Complexity

Given the input samples over one co-prime period, we determine the cost for hardware implementation of autocorrelation estimation in terms of the number of multiplications and additions. This is directly related to the weight function of the co-prime sampler. The proposed non-blind system is expected to require a slightly higher number of multipliers and adders as compared to the blind system.

Let $m_b(l)$ and $m_{nb}(l)$ represent the number of multiplications required for autocorrelation estimation at each difference value l, while $a_b(l)$ and $a_{nb}(l)$ represent the corresponding number of additions. The subscripts 'b' and 'nb' refers to the blind and non-blind system, respectively. The number of multiplications and additions required per difference value are

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given in (8) and (9) respectively. $z_b(l)$ represents the weight function of the blind system and is given in [8].

$$m_b(l) = z_b(l) \text{ and } m_{nb}(l) = z_{nb}(l) \tag{8}$$

$$\begin{aligned} a_b(l) &= z_b(l) - 1; \{l | z_b(l) > 1\} \\ a_{nb}(l) &= z_{nb}(l) - 1; \{l | z_{nb}(l) > 1\} \end{aligned} \tag{9}$$

Let C_{M_b} denote the total number of multiplications required to estimate the autocorrelation over one co-prime period for a blind system and is obtained as the cumulative sum of $m_b(l)$ for $l \in [0, MN-1]$ as given below:

$$\begin{aligned} C_{M_b} &= \sum_{l=0}^{MN-1} m_b(l) \\ &= \sum_{n=0}^{N-1}(N-n) + \sum_{m=0}^{M-1}(M-m) + (N-1)(M-1) - 1 \\ &= \frac{(M+N)(M+N-1)}{2} \end{aligned} \tag{10}$$

Let $C_{M_{nb}}$ denote the total number of multiplications for a non-blind system and is obtained as the cumulative sum of $m_{nb}(l)$:

$$\begin{aligned} C_{M_{nb}} &= \sum_{l=0}^{MN-1} m_{nb}(l) \\ &= \sum_{n=0}^{N-1}(N-n+1) \\ &\quad + \sum_{m=0}^{M-1}(M-m+1) + (N-1)(M-1) - 1 \\ &= \frac{(M+N)(M+N+1)}{2} \end{aligned} \tag{11}$$

Let C_{A_b} denote the total number of additions required to estimate the autocorrelation over one co-prime period for a blind system and is obtained as the cumulative sum of $a_b(l)$ for $l \in [0, MN-1]$ as given below:

$$\begin{aligned} C_{A_b} &= \sum_{\{l|m_b(l)>1\}} a_b(l) = \sum_{l=0}^{MN-1} m_b(l) - \sum_{\{l|m_b(l)>1\}} 1 \\ &= \frac{(M+N)(M+N-1)}{2} \\ &\quad - \left(M+N-1+\frac{(N-1)(M-1)}{2}\right) \\ &= \frac{M(M-2)+N(N-2)+MN+1}{2} \end{aligned} \tag{12}$$

Let $C_{A_{nb}}$ denote the total number of additions for a non-blind system and is obtained as the cumulative sum of $a_{nb}(l)$:

$$\begin{aligned} C_{A_{nb}} &= \sum_{\{l|m_{nb}(l)>1\}} a_{nb}(l) = \sum_{l=0}^{MN-1} m_{nb}(l) - \sum_{\{l|m_{nb}(l)>1\}} 1 \\ &= \frac{(M+N)(M+N+1)}{2} \\ &\quad - \left(M+N-1+\frac{(N-1)(M-1)}{2}\right) \\ &= \frac{M^2+N^2+MN+1}{2} \end{aligned} \tag{13}$$

Therefore, the proposed non-blind system requires $(M+N)$ additional additions and multiplications as compared to the ideal prototype co-prime array as well as the non-ideal blind system. It is important to note that the overall cost will also depend on the total number of snapshots (or co-prime periods, L) being considered for the estimation; and will be L times the number of additions and multiplications derived above (plus some overheads to combine the snapshots). The computational complexity of the ideal prototype co-prime array derived here is contrary to the claims made in (14) and (15) of [9].

VI. Conclusion

The effect of sampling jitter on the difference set was studied and analyzed for co-prime arrays and samplers. We described two systems for autocorrelation estimation: a blind system and a non-blind system. The blind system under utilizes the information present in the acquired data and has a mapped weight function similar to the ideal prototype co-prime array. On the other hand the proposed non-blind system has a larger number of contributors for autocorrelation estimation and hence the potential for high fidelity estimation in the presence of jitter. We also derive expressions for the computational complexity of the two systems. The advantages obtained from a sampling perspective cannot be emulated for the antenna arrays since the zeroth location has only one antenna element.

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