Quasi-Newton Least-Mean Fourth Adaptive Algorithm

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***Abstract*—This paper proposes a new Newton-based adaptive filtering algorithm, namely the Quasi-Newton Least-Mean Fourth (QNLMF) algorithm. The main goal is to have a higher order adaptive filter that usually fits the non-Gaussian signals with an improved performance behavior, which is achieved using the Newton numerical method. Both the convergence analysis and the steady-state performance analysis are derived. More importantly, unlike other stochastic based algorithms, the step size parameter that controls the convergence of the QNLMF is independent of the statistics of the input signal, and consequently, the analytical assessments show that the proposed algorithm enjoys an independent performance from the input signal eigenvalue spread. Finally, a number of simulation experiments are carried out to corroborate the theoretical findings.**

***Index Terms*—Newton Method, LMF, Adaptive filtering.**

I. Introduction

The adaptive Least-Mean Square (LMS) algorithm [1], [2] has received a great deal of attention and is now widely used in a variety of applications due to its simplicity and relative ease of analysis. Meanwhile, there exist a limited number of adaptive filtering algorithms that are based on higher order error criteria (non-mean-square error criteria). Despite of the potential advantages that these algorithms enjoy, especially in the non-Gaussian environments, they are still confiened in use and development when compared to the LMS algorithm, and this due to the fact that their analysis is quite involved [3].

The Least-Mean Fourth (LMF) algorithm is among these algorithms and has been under study for quite some time now [2]- [22]. Since the Walach and Widrow's work [3] which has first presented the convergence analysis of the LMF algorithm, many works have then been done on the LMF and its variants [2]- [22]. Most of the LMF variants have solved different issues in the LMF algorithm, however, none of them has seriously considered to improve its speed of convergence.

In this paper, we propose a new FIR channel identification algorithm using the Newton's method. The newly devised LMF variant algorithm has a notable improved convergence behaviour, which makes it more appealing especially in the practical use. More interestingly, its behavior is fully independent from the input data statistics, i.e., the eigenvalue spread, and so does its step size. In addition to the derivation of the QNLMF algorithm in this paper, it is convergence behaviour is analyzed, and its performance is assessed in different noise scenarios against the LMF algorithm. Moreover, the provided rigorous analysis is conducted based on the energy conservation relation approach which is adopted in [22].

II. QNLMF Algorithm

Starting with the general weight update equation which is given by [22]

$$\mathbf{w}_i = \mathbf{w}_{i-1} - \mu \mathbf{B} \left[\frac{\partial J(\mathbf{w}_i)}{\partial \mathbf{w}_i} \right]^T , \tag{1}$$

where $\mathbf{B}$ is the inverse of the Hessian matrix and is given by

$$\mathbf{B} = \left[\frac{\partial^2 J(\mathbf{w}_i)}{\partial \mathbf{w}_i^2} \right]^{-1} , \tag{2}$$

Note that setting $\mathbf{B} = 1$ reduces the subsequent derivation to the regular LMF algorithm.

The cost function J to be minimized, in our case is the least mean forth criteria, defined as

$$J(\mathbf{w}_i) = \mathsf{E}\left[e^4(i) \right] , \tag{3}$$

where

$$e(i) = d(i) - \mathbf{u}_i \mathbf{w}_i^T , \tag{4}$$

$\mathbf{w}_i$ is the system's unknown coefficient vector and $\mathbf{u}_i$ is the adaptive filter input vector. Considering the introduced value of $\mathbf{B}$, one can write the new weight update equation for the QNLMF algorithm as

$$\mathbf{w}_i = \mathbf{w}_{i-1} - \mu \left[\frac{\partial^2 J(\mathbf{w}_i)}{\partial \mathbf{w}_i^2} \right]^{-1} \left[\frac{\partial J(\mathbf{w}_i)}{\partial \mathbf{w}_i} \right]^T . \tag{5}$$

This recursion form is well-known as the Newton's method. Figure 1 depicts the block diagram of a system identification. In this figure, $d(i)$, $n(i)$ and $e(i)$ represent the desired, the noise and the estimation error signals, respectively. The unknown system to be identified is assumed to be linear and time-invariant.

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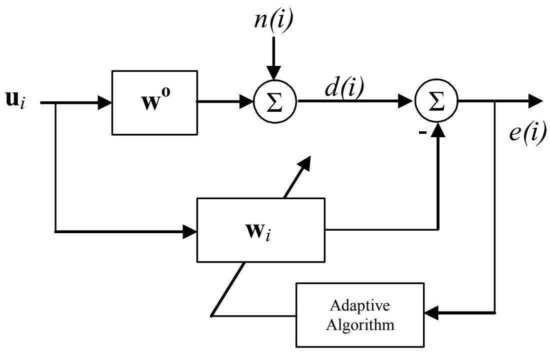

Fig. 1: Adaptive system identification model.

The update recursion for the QNLMF can be derived by evaluating the gradient vector and the Hessian matrix of the cost function defined in (3). These are, respectively, given by

$$\frac{\partial J(\mathbf{w}_i)}{\partial \mathbf{w}_i} = -\mathsf{E}\left[e^3(i)\mathbf{u}_i\right], \tag{6}$$

$$\frac{\partial^2 J(\mathbf{w}_i)}{\partial \mathbf{w}_i^2} = \mathsf{E}\left[e^2(i)\mathbf{u}_i\mathbf{u}_i^T\right]. \tag{7}$$

An approximation for the Hessian in (7), similar to the one in [22], is given by:

$$\hat{\mathbf{R}}_i = \alpha \sum_{j=0}^{i} (1-\alpha)^{i-j} e^2(j)\mathbf{u}_j\mathbf{u}_j^T. \tag{8}$$

Using (6) and (8), one can write the Quasi-Newton least-mean-fourth (QNLMF) algorithm (5) as follows:

$$\mathbf{w}_i = \mathbf{w}_{i-1} + \mu \mathbf{\Phi}_i^{-1}\mathbf{u}_i e^3(i), \tag{9}$$

where $\epsilon(i) = (1-\alpha)^{(i+1)}\kappa$ for a small positive scalar κ, and $\mathbf{\Phi}_i = \epsilon(i)\mathbf{I} + \hat{\mathbf{R}}_i$. After some algebraic manipulations, $\mathbf{\Phi}_i$ can be shown to be

$$\mathbf{\Phi}_i = (1-\alpha)\mathbf{\Phi}_{i-1} + \alpha e^2(i)\mathbf{u}_i\mathbf{u}_i^T, \tag{10}$$

where $(0 < \alpha \leq 0.1)$ is a defined parameter used for weighting the current and the previous values. Figure 2 shows the sensitivity analysis of α on the convergence of the QNLMF algorithm.

III. Convergence Analysis of the QNLMF Algorithm

Let the weight error vector be defined as $\mathbf{v}_i = \mathbf{w}^o - \mathbf{w}_i$, where $\mathbf{w}^o$ is the Wiener optimum weight solution. Consequently (9) becomes

$$\mathbf{v}_i = \mathbf{v}_{i-1} - \mu \mathbf{\Phi}_i^{-1}\mathbf{u}_i e^3(i). \tag{11}$$

Multiplying (11) by $\mathbf{u}_i^T$ from left, one can represent it in the form of *a-priori* $e_a(i) = \mathbf{u}_i^T\mathbf{v}_{i-1}$ and *a-posteriori* $e_p(i) = \mathbf{u}_i^T\mathbf{v}_i$ estimation errors as

$$e_p(i) = e_a(i) - \mu \|\mathbf{u}_i\|^2_{\mathbf{P}_i} e^3(i), \tag{12}$$

where $\|\mathbf{u}_i\|^2_{\mathbf{P}_i} = \mathbf{u}_i^T\mathbf{P}_i\mathbf{u}_i$ stands for the squared-weighted Euclidean norm of $\mathbf{u}_i$ and $\mathbf{P}_i = \mathbf{\Phi}_i^{-1}$ which, after applying the matrix inversion lemma [22] on (10), comes down to the following expression:

$$\mathbf{P}_i = \frac{1}{1-\alpha}\left[\mathbf{P}_{i-1} - \frac{\mathbf{P}_{i-1}^T\mathbf{u}_i\mathbf{u}_i^T\mathbf{P}_{i-1}}{\frac{1-\alpha}{\alpha e^2(i)} + \mathbf{u}_i^T\mathbf{P}_{i-1}\mathbf{u}_i}\right]. \tag{13}$$

Let us define a new term $\bar{\mu}_i = \frac{1}{\|\mathbf{u}_i\|^2_{\mathbf{P}_i}}$ and substitute expression (12) in equation (11) gives

$$\mathbf{v}_i = \mathbf{v}_{i-1} - \bar{\mu}_i\mathbf{P}_i\mathbf{u}_i\left[e_a(i) - e_p(i)\right]. \tag{14}$$

By evaluating the energies on both sides of (14), we obtain

$$\|\mathbf{v}_i\|^2_{\mathbf{P}_i^{-1}} + \bar{\mu}_i\left|e_a(i)\right|^2 = \|\mathbf{v}_{i-1}\|^2_{\mathbf{P}_i^{-1}} + \bar{\mu}_i\left|e_p(i)\right|^2. \tag{15}$$

This important fundamental energy relation developed previously in [22] will now be used to evaluate the Excess MSE (EMSE) of the proposed QNLMF at steady state. As is well-known, an adaptive filter is said to operate in steady-state iff:

$$\lim_{i\longrightarrow\infty}\mathsf{E}\left[\|\mathbf{v}_i\|^2\right] = \lim_{i\longrightarrow\infty}\mathsf{E}\left[\|\mathbf{v}_{i-1}\|^2\right]. \tag{16}$$

Taking the expected value on both sides of (15), we get

$$\mathsf{E}\left[\bar{\mu}_i\left|e_a(i)\right|^2\right] = \mathsf{E}\left[\bar{\mu}_i\left|e_a(i) - \frac{\mu}{\bar{\mu}_i}e^3(i)\right|^2\right]. \tag{17}$$

Assuming that the a-priori estimation error $e_a(i)$ and the noise process $n(i)$ are independent and are related through the relation: $e(i) = e_a(i) + n(i)$, it is then can be shown that, the EMSE of the adaptive filter can be found by evaluating the steady-state mean-square value of the a-priori estimation error as [22]

$$\mathrm{EMSE} = \lim_{\mathrm{i}\longrightarrow\infty}\mathsf{E}\left[\left|\mathrm{e_a(i)}\right|^2\right] = \zeta_*. \tag{18}$$

In doing so, at steady-state, the third and the higher powers of $e_a(i)$ become very small and therefore can be ignored. Let $\mathsf{E}\left[n^k(i)\right] = \delta_n^k$, and after some straightforward algebraic manipulations, (17) can be re-written as

$$6\sigma_n^2\zeta_i = \mu\mathsf{tr}\left\{\mathbf{R_u}\mathbf{P}\right\}\left[15\delta_n^4\zeta_i + \delta_n^6\right], \tag{19}$$

where $\mathbf{P} = \mathsf{E}(\mathbf{P}_i)$ and $\mathbf{R_u} = \mathsf{E}(\mathbf{u}_i\mathbf{u}_i^T)$. At steady-state, equation (19) will look like the following:

$$6\sigma_n^2\zeta_* = \frac{\mu M\left[15\delta_n^4\zeta_* + \delta_n^6\right]}{\zeta_* + \sigma_n^2} \tag{20}$$

where $\mathsf{tr}\left\{\mathbf{R_u}\mathbf{P}\right\} = \frac{M}{\zeta_*+\sigma_n^2}$ which is derived in Appendix-A.

As $(i \longrightarrow \infty)$ and after some algebraic manipulations, we obtain a second order equation of the EMSE ζ_* as

$$6\sigma_n^2\zeta_*^2 + \left(6\sigma_n^4 - \mu 15\delta_n^4 M\right)\zeta_* - \mu\delta_n^6 M = 0. \tag{21}$$

Since $\zeta_* << 1$, then an asymptotic approximation of the EMSE is given by

$$\zeta_* \approx \frac{\mu\delta_n^6 M}{6\sigma_n^4 - \mu 15\delta_n^4 M}. \tag{22}$$

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In case of sufficiently small step-size value μ, the value of ζ_* can be further simplified and approximated as

$$\zeta_* \approx \frac{\mu \delta_n^6 M}{6\sigma_n^4}. \tag{23}$$

The above expression reveals that the EMSE is independent of the eigenvalue spread of the input auto-correlation matrix, hence the QNLMF algorithm is not sensitive to the regression data statistics. In a similar way to the other adaptive filters, the EMSE of the QNLMF algorithm varies in accordance with the FIR channel's length M.

TABLE I: Computational Cost for the QLMF and LMF algorithms.

Algorithm	$\times$	$+$
LMF	$2M+3$	$2M$
QNLMF	$2M^2+2M+1$	$2M^2-2M$

1) Convergence in the mean: To find the condition on the step size μ which enables the proposed QNLMF to converge in the mean sense, using (6)-(7), subtract $\mathbf{w}^o$ from both side of (5) and then take the expected value of the both sides of the resulting equation to get:

$$\mathsf{E}[\mathbf{v}_i] = \mathsf{E}[\mathbf{v}_{i-1}] + \mu\left\{\mathsf{E}\left[e^2(i)\mathbf{u}_i\mathbf{u}_i^T\right]\right\}^{-1}\mathsf{E}\left[e^3(i)\mathbf{u}_i\right] \tag{24}$$

The term $\mathsf{E}\left[e^2(i)\mathbf{u}_i\mathbf{u}_i^T\right]$ can be expanded as

$$\begin{aligned}\mathsf{E}\left[e^2(i)\mathbf{u}_i\mathbf{u}_i^T\right] &= \mathsf{E}\left[n^2(i)\mathbf{u}_i\mathbf{u}_i^T\right] + 2\mathsf{E}\left[n(i)e_a(i)\mathbf{u}_i\mathbf{u}_i^T\right]\\ &\quad + \mathsf{E}\left[e_a^2(i)\mathbf{u}_i\mathbf{u}_i^T\right],\end{aligned} \tag{25}$$

which at steady-state can be approximated to

$$\mathsf{E}\left[e^2(i)\mathbf{u}_i\mathbf{u}_i^T\right] = \sigma_e^2(i)\mathbf{R}_\mathbf{u} \tag{26}$$

Also, from [8], it can be shown that

$$\mathsf{E}\left[e^3(i)\mathbf{u}_i\right] = -3\sigma_e^2(i)\mathbf{R}_\mathbf{u}\mathsf{E}[\mathbf{v}_{i-1}] \tag{27}$$

Ultimately, (24) can be setup into

$$\begin{aligned}\mathsf{E}[\mathbf{v}_i] &= \mathsf{E}[\mathbf{v}_{i-1}] - 3\mu\left\{\sigma_e^2(i)\mathbf{R}_\mathbf{u}\right\}^{-1}\sigma_e^2(i)\mathbf{R}_\mathbf{u}\mathsf{E}[\mathbf{v}_{i-1}],\\ &= [1-3\mu]\,\mathsf{E}[\mathbf{v}_{i-1}]\end{aligned} \tag{28}$$

From (28), it is easy to show that the mean behavior of the weight error vector $\mathsf{E}[\mathbf{v}_i]$ converges to the zero vector if the step size μ is selected such that

$$0 < \mu < \frac{2}{3}, \tag{29}$$

and hence the QNLMF algorithm converges in the mean sense. Unlike the LMF algorithm, the step size of the QNLMF algorithm does not depend upon the energy of the input signal. This feature, as assessed in the simulation results section, makes the QNLMF algorithm a very potential candidate to other stochastic gradient based algorithms which suffers from the appropriate choice of the step size.

2) Adaptation time constant: Using (28), and from the definition of the time constant, we can shown that the time constant for the QNLMF algorithm, at a sufficiently chosen small step size, is given by

$$\tau \approx \frac{1}{6\mu}. \tag{30}$$

In comparison with other Newton-based derived algorithms, it is evident that the learning curves derived from Newton's method are identical [23], in the sense that all of them have the same time constant. Hence, its rate of convergence is predictable and does not depend on the initial conditions. Moreover, in comparison with the LMF algorithm, again the time constant is affected by the input regression statistics, while the proposed QNLMF is not.

Finally, Table I reports the computational complexity of both algorithms, namely, the LMF and the QNLMF. As can be seen from this table, the QNLMF algorithm has a much higher computational cost than its counterpart the LMF algorithm.

IV. Simulation Results

In this section, the performance analysis of the proposed QNLMF algorithm is evaluated and compared to that of the conventional LMF counterpart. Considering a system identification scenario, the unknown system is $\mathbf{w}^o = [0.227, 0.460, 0.688, 0.460, 0.227]^T$. Different scenarios and noise environments have been investigated. Unless otherwise stated the signal-to-noise ratio (SNR) is 20 dB, $\mu = 0.01$ and $\epsilon = 10^{-6}$.

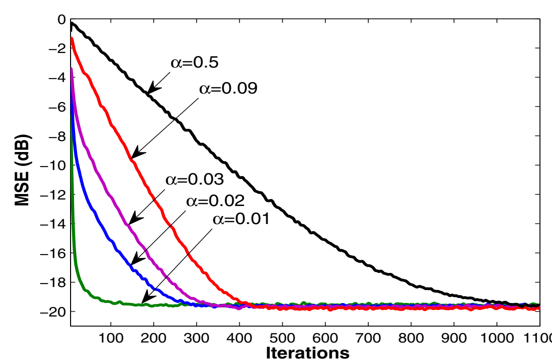

Fig. 2: The MSE for the proposed QNLMF algorithm for different α values.

Figure 3 shows the variation of the MSE of the proposed algorithm against different values of μ. It is clear that the proposed algorithm diverges for $\mu \geq 2/3$ in agreement with (29). Indeed, the algorithm doesn't depend on the signal eigenvalue spread of the input signal.

Figure 4 verifies the robustness of the proposed QNLMF algorithm against the input regressor's eigenvalue spread ρ (defined as $\lambda_{max}/\lambda_{min}$), whereby two scenarios have been considered, i.e., $\rho = 5$ and $\rho = 80$. Once more, the experimental findings coincide with the theoretical ones.

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Considering the former setup, Fig. 5 compares the performance of the proposed QNLMF algorithm against that of the LMF algorithm. Both algorithms reached the same steady-state level with a clear credit, in term of convergence speed, for the proposed algorithm.

Figure 6 tests the algorithms' response to an abrupt change in the coefficient values (all set to zero at iteration $12,500$). Once again, the QNLMF algorithm adapts much faster than the LMF algorithm.

The effect of the noise environment on the proposed QNLMF algorithm performance is investigated in Fig. 7. Three noise scenarios were tested, i.e., Laplacian, Gaussian and Uniform. The algorithm has shown a superiority in the later noise environment. Ultimately, Fig. 8 compares the experimental and theoretical matching in terms of the MSE by changing the applied step-size.

V. Conclusion

In this work, a new Newton-based LMF algorithm, namely QNLMF, is proposed. The proposed algorithm is analytically and numerically tested and evaluated. The analytic evaluation includes the EMSE, convergence in the mean and the rate of convergence. The proposed algorithm numerical performance is compared to the well established LMF algorithm in different noise environment. Interestingly, the proposed algorithm's behavior is fully independent from the input data statistics, i.e., the eigenvalue spread, and so does its step size.

Appendix A

From (9), we can write

$$\begin{aligned}\mathbf{P}_i^{-1} &= (1-\alpha)^{(i+1)}\epsilon + \hat{\mathbf{R}}_i &(31)\\ \mathbf{P}_i^{-1} &= (1-\alpha)^{(i+1)}\epsilon + \alpha[e^2(i)\mathbf{u}_i\mathbf{u}_i^T \\ &\quad +(1-\alpha)e^2(i-1)\mathbf{u}_{i-1}\mathbf{u}_{i-1}^T &(32)\\ &\quad +\cdots+(1-\alpha)^i e^2(0)\mathbf{u}_0\mathbf{u}_0^T],\end{aligned}$$

since, $\mathbf{P}_i^{-1} = \mathbf{\Phi}_i$ and $(1-\alpha) < 1$, therefore, as $i \longrightarrow \infty$, the steady-state mean value of $\mathbf{P}_i^{-1}$ is given by

$$\lim_{i\longrightarrow\infty} \mathsf{E}\left(\mathbf{P}_i^{-1}\right) = \frac{\alpha\mathsf{E}\left[e^2(i)\mathbf{u}_i\mathbf{u}_i^T\right]}{1-(1-\alpha)} = \mathsf{E}\left[e^2(i)\mathbf{u}_i\mathbf{u}_i^T\right] = \mathbf{P}^{-1}, \tag{33}$$

$$\mathsf{E}\left(\mathbf{P}_i\right) \approx \left[\mathsf{E}\left(\mathbf{P}_i^{-1}\right)\right]^{-1} = \left\{\mathsf{E}\left[e^2(i)\mathbf{u}_i\mathbf{u}_i^T\right]\right\}^{-1} = \mathbf{P}, \tag{34}$$

which, from (26) , reduces to

$$\mathsf{E}\left(\mathbf{P}_i\right) \approx \mathbf{P} = \left[\sigma_e^2\mathbf{R}_{\mathbf{u}}\right]^{-1} \tag{35}$$

Therefore,

$$\begin{aligned}\mathsf{tr}\left\{\mathbf{R}_{\mathbf{u}}\mathbf{P}\right\} &= \mathsf{tr}\left\{\mathbf{R}_{\mathbf{u}}\left(\sigma_e^2\mathbf{R}_{\mathbf{u}}\right)^{-1}\right\}\\ &= \frac{\mathsf{tr}\left\{\mathbf{I}\right\}}{\sigma_e^2}\\ &= \frac{M}{\zeta_* + \sigma_n^2} &(36)\end{aligned}$$

where, from [8], $\sigma_e^2 = \zeta_* + \sigma_n^2$.

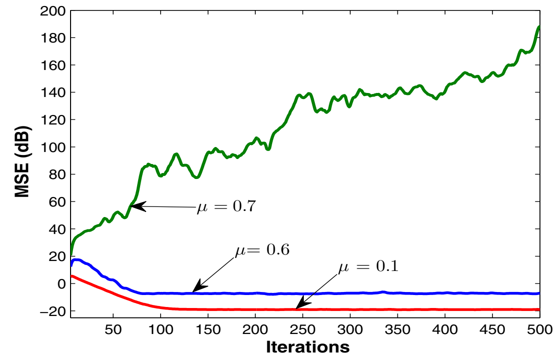

Fig. 3: The MSE for the proposed QNLMF algorithm for different values of μ, $\alpha = 0.4$.

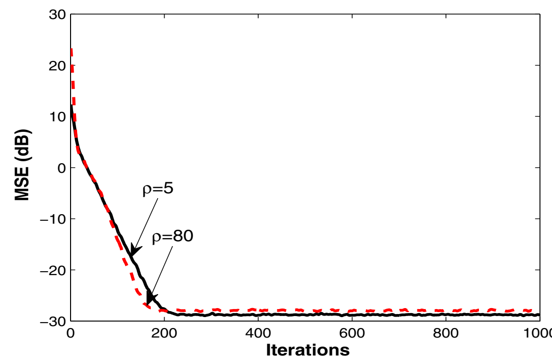

Fig. 4: The MSE for the QNLMF algorithm for $\rho = 5$ ($\mu = 0.05$) and $\rho = 80$ ($\mu = 0.08$). $\alpha = 0.09$, SNR = 30 dB.

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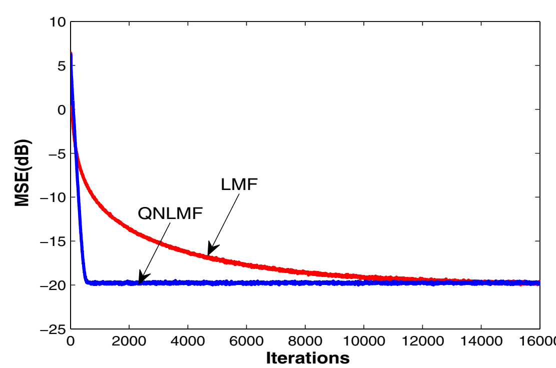

Fig. 5: The MSE for the proposed algorithm and the LMF algorithm in AWGN environment. QNLMF ($\mu = 0.1, \alpha = 0.07$) and LMF ($\mu = 0.004$).

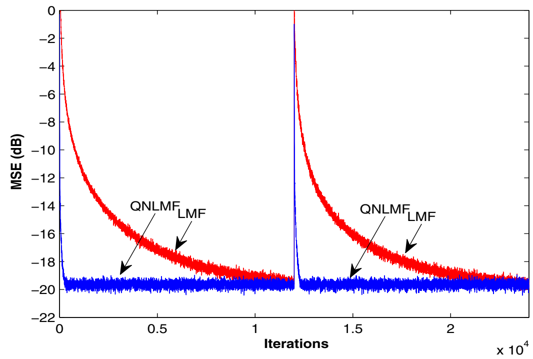

Fig. 6: The MSE for the proposed algorithm and the LMF algorithm under an abrupt change scenario.

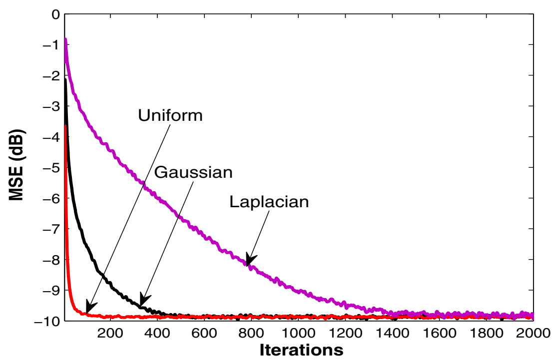

Fig. 7: Convergence behavior of the QNLMF algorithm in presence of Gaussian, Uniform and Laplacian environments with SNR = 10 dB.

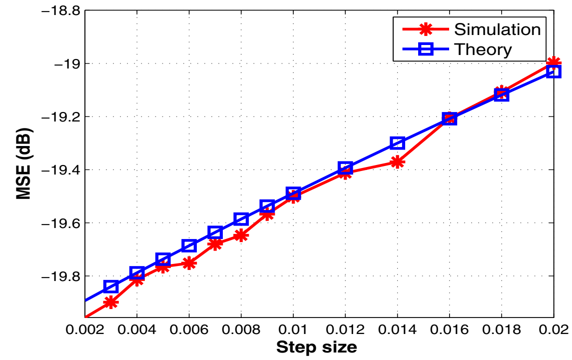

Fig. 8: Theoretical and experimental MSE for the proposed QNLMF algorithm.

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