A Multimodal Asymmetric Exponential Power Distribution: Application to Risk Measurement for Financial High-Frequency Data

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***Abstract*—Interest in risk measurement for high-frequency data has increased since the volume of high-frequency trading stepped up over the two last decades. This paper proposes a multimodal extension of the Exponential Power Distribution (EPD), called the Multimodal Asymmetric Exponential Power Distribution (MAEPD). We derive moments and we propose a convenient stochastic representation of the MAEPD. We establish consistency, asymptotic normality and efficiency of the maximum likelihood estimators (MLE). An application to risk measurement for high-frequency data is presented. An autoregressive moving average multiplicative component generalized autoregressive conditional heteroskedastic (ARMA-mcsGARCH) model is fitted to Financial Times Stock Exchange (FTSE) 100 intraday returns. Performances for Value-at-Risk (VaR) and Expected Shortfall (ES) estimation are evaluated. We show that the MAEPD outperforms commonly used distributions in risk measurement.**

***Index Terms*—Multimodality, Asymmetric distributions, Expected shortfall, Value-at-Risk, Risk measurement**

I. Introduction

Over the two last decades, the growth of high-frequency trading and the availability of transaction data for financial assets led market participants to focus on intraday volatility. Modelling and forecasting high-frequency volatility is a subject of great importance in estimation of risk measures such as VaR and ES. Conventional GARCH models [1], [2] were run on high-frequency data and the resulting model parameters were not consistent between different intraday frequencies, mostly due to the noticeable diurnal patterns of volatility [3]. Recently, [3] developed the mcsGARCH model based on [4]. It decomposes the volatility of price returns into three multiplicative components, namely daily, diurnal and stochastic. Here, we are especially interested in the estimation of the daily component which is a daily determined forecast volatility. To do so, a large literature exists and GARCH-based models are efficient for daily data [5]. For daily data, a special attention shall be paid to the innovations of the GARCH models: their distribution exhibit high kurtosis and left-skewness highly impacting estimation of risk measures and therefore portfolio optimization.

Modelling these innovations has received a great interest among practitioners and parametric estimation of their distribution is conducted with distributions such as the Generalized Hyperbolic (GHYP) [6] and Skewed Exponential Power (SEPD) [7]-[8] distributions. Recently, Polynomial-Normal [9] and Polynomial-t-Student distributions [10] were used to fit the innovation of GARCH models for financial series. It is shown that such polynomial-distributions improve performance in risk measurement compared to their non-polynomial counterparts. In this paper, we propose a MAEPD which presents two different shape parameters controlling the semi-heavy tails away and apart a location parameter. A polynomial-like component is multiplied to manage multimodality.

The paper is organized as follows. Section II proposes a new MAEPD and its main characteristics are exposed. Section III is dedicated to an application of the MAEPD distribution to risk measurement for the FTSE 100 index over a period of two months covering "Brexit" referendum in June 2016. An ARMA-GARCH model with MAEPD innovations is derived for mcsGARCH daily component estimation. Then, an ARMA-mcsGARCH model with Student-t innovations is used to forecast risk for high-frequency data. Backtesting and comparison of performances for VaR and ES estimation under different innovation distributions are presented. Concluding remarks can be found in Section IV.

II. A multimodal asymmetric exponential power distribution

The generalized error distribution class, initially proposed by [11], is called Exponential Power Distribution (EPD) in [12]. The corresponding density function is

$$f_{\mathrm{EPD}}(x|\alpha,\sigma)=\frac{1}{\sigma}f_\alpha\left[\frac{x-\mu}{\sigma}\right]$$

where $f_\alpha(x)=c_\alpha\exp(-|x|^\alpha)$, $\alpha>0$ is the shape parameter, $c_\alpha^{-1}=2\Gamma(1+1/\alpha)$ and Γ is the Gamma function, $\mu\in\mathbb{R}$ is the location parameter and $\sigma>0$ is the scale parameter.

Extension of the EPD class was first considered by [13]. Then, [14] extended the EPD class by adding a skew parameter, giving the Skewed EPD (SEPD) class. Recently, [8] presented an Asymmetric EPD (AEPD) class in which two kinds of asymmetry is captured. Besides of reparametrizing the skew parameter in [14], heavy-tailedness asymmetry is managed by two different tail exponents on different sides of the location parameter μ. From another perspective, an interest has risen for bimodal skewed distributions [15].

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A. Definition

This paper extends the EPD family by following the approach in [8]. The MAEPD presents two different shape parameters α_1 and α_2 controlling the semi-heavy tails away and apart a location parameter. In [7], the asymmetry parameter β is defined such that the location parameter μ is the β-quantile of the SEPD. This approach is here chosen. The singularity of the MAEPD is the presence of a multimodal component, which is described by the multimodality parameter δ and the mode shape parameter ν. The MAEPD density function f_{MAEPD} is given by:

$$f_{\text{MAEPD}}(x|\eta) = \tag{1}$$
$$\begin{cases} \left(\frac{C_n(\eta)}{\sigma}\right)\left(\delta^\nu + \frac{1}{\nu}\left|\frac{x-\mu}{2\sigma\beta^*}\right|^\nu\right)\exp\left(-\frac{1}{\alpha_1}\left|\frac{x-\mu}{2\sigma\beta^*}\right|^{\alpha_1}\right), \text{if } x < \mu \\ \left(\frac{C_n(\eta)}{\sigma}\right)\left(\delta^\nu + \frac{1}{\nu}\left|\frac{x-\mu}{2\sigma(1-\beta^*)}\right|^\nu\right)\exp\left(-\frac{1}{\alpha_2}\left|\frac{x-\mu}{2\sigma(1-\beta^*)}\right|^{\alpha_2}\right), \\ \text{if } x \geq \mu \end{cases}$$

where $\eta = (\alpha_1, \alpha_2, \mu, \sigma, \beta, \delta, \nu)$, $\alpha_1 > 0$ and $\alpha_2 > 0$ are the shape parameters, $\sigma > 0$ is the scale one, $\beta \in (0,1)$ is the skewness parameter, $\delta \geq 0$ and $\nu > 0$ are, respectively, the multimodality and mode shape parameters. $C_n(\eta)$ and β^* are defined as follows: $C_n(\eta) = \beta C_0^1 + (1-\beta)C_0^2(\alpha_2)$ and $\beta^* = \beta C_0^1 / C_n(\eta)$ where $C_0^1 = \frac{1}{2K_0(\alpha_1)}$, $C_0^2 = \frac{1}{2K_0(\alpha_2)}$ and $K_i(\alpha) = \delta^\nu \alpha^{(i+1)/\alpha - 1}\Gamma(\frac{1+i}{\alpha}) + \frac{1}{\nu}\alpha^{(\nu+1+i)/\alpha-1}\Gamma(\frac{1+\nu+i}{\alpha})$. The density form (1) can be rescaled as follows:

$$f_{\text{MAEPD}}(x|\eta) = \tag{2}$$
$$\begin{cases} \frac{1}{\sigma}\left(\delta^\nu + \frac{1}{\nu}\left|\frac{x-\mu}{2\sigma\beta C_0^1}\right|^\nu\right)\exp\left(-\frac{1}{\alpha_1}\left|\frac{x-\mu}{2\sigma\beta C_0^1}\right|^{\alpha_1}\right), \\ \text{if } x < \mu; \\ \frac{1}{\sigma}\left(\delta^\nu + \frac{1}{\nu}\left|\frac{x-\mu}{2\sigma(1-\beta)C_0^2}\right|^\nu\right)\exp\left(-\frac{1}{\alpha_2}\left|\frac{x-\mu}{2\sigma(1-\beta)C_0^2}\right|^{\alpha_2}\right), \\ \text{if } x \geq \mu. \end{cases}$$

The density (2) is convenient to compute the information matrix of the MLE. Figure 1 displays different plots of f_{MAEPD} for $(\mu, \sigma) = (0, 1)$ and demonstrates its potential in accommodating shapes in terms of multimodality, skewness and kurtosis.

B. Properties of the MAEPD

Assume that the random variable X follows a MAEPD. We have

$$\mathrm{E}(X^k) = \beta\,\mathrm{E}(X^k|X<\mu) + (1-\beta)\,\mathrm{E}(X^k|X\geq\mu). \tag{3}$$

Using integral 3.478/1 in [16], we deduce from (2) that

$$\mathrm{E}(X^k|X\geq\mu) = \sum_{i=0}^{k}\binom{k}{i}\mu^{k-i}\left(\frac{(1-\beta)\sigma}{C_n(\eta)}\right)^i\frac{K_i(\alpha_2)}{K_0(\alpha_2)^{i+1}},$$
$$\mathrm{E}(X^k|X<\mu) = \sum_{i=0}^{k}\binom{k}{i}\mu^{k-i}\left(\frac{\beta\sigma}{C_n(\eta)}\right)^i\frac{K_i(\alpha_1)}{K_0(\alpha_1)^{i+1}},$$

and replacing in (3), we get $\mathrm{E}(X^k)$ for any $k \geq 0$. Setting $\mathrm{E}(X) = 0$ leads to an expression of μ in terms of η and setting $\mathrm{E}(X^2) = 1$ gives an expression of σ. Replacing μ and σ by these expressions in f_{MAEPD} provides a standardized version of the MAEPD. The resulting standardized MAEPD is used for modelling the innovations of an ARMA-GARCH model and forecasting risk measures in Section III.

There are three sets $\Theta_1, \Theta_2, \Theta_3$ such that: $\eta_0 \in \Theta_1$, $\eta_0 \in \Theta_2$ and $\eta_0 \in \Theta_3$ implies that $f_{\text{MAEPD}}(x|\eta_0)$ has respectively one, two and three modes.

The MAEPD has a stochastic representation, which is of importance for simulation and performance evaluation. For given values of parameters, generation of random numbers from the MAEPD can be done by the following algorithm. First, random numbers are drawn independently from the uniform distributions $U_1(0,1)$ and $U_2(0,1)$ and from the generalized gamma distribution $W_{\alpha,q}$ with density function $f_{GG}(x) = \frac{\alpha}{\sigma\Gamma(q/\alpha)}(\frac{x}{\sigma})^{q-1}\exp(-(\frac{x}{\sigma})^\alpha)$. Second, the random variable Y is expressed as:

$$\begin{aligned} &Y = N(U_2,\beta) \\ &\left[\mu - 2\beta^*\alpha_1^{1/\alpha_1}\left(W_{\alpha1,1}N(U_1,\omega_1) + W_{\alpha1,\nu+1}P(U_1,\omega_1)\right)\right] \\ &+ P(U_2,\beta) \\ &\left[\mu + 2(1-\beta^*)\alpha_2^{1/\alpha_2}(W_{\alpha2,1}N(U_1,\omega_2) + W_{\alpha2,\nu+1}P(U_1,\omega_2))\right] \end{aligned} \tag{4}$$

where $N(U,\omega) = \frac{\text{sign}(U-\omega)-1}{-2}$, $P(U,\omega) = \frac{\text{sign}(U-\omega)+1}{2}$, $\text{sign}(x) = +1$ if $x > 0$ and $\text{sign}(x) = -1$ if $x \leq 0$, $\omega_1 = \frac{\delta^\nu\alpha_1^{1/\alpha_1 - 1}\Gamma(1/\alpha_1)}{K_0(\alpha_1)}$ and $\omega_2 = \frac{\delta^\nu\alpha_2^{1/\alpha_2-1}\Gamma(1/\alpha_2)}{K_0(\alpha_2)}$.

C. Parameter Estimation

Let $x_t, t = 1, \ldots, T$ be a random sample from the MAEPD $f(x_t|\eta)$ defined in (2). Let

$$L = L(x_t|\eta) = \frac{|x_t - \mu|}{2\alpha_1^{1/\alpha_1}\sigma\beta C_0^1}\mathbb{1}_{\{x<\mu\}} \tag{5}$$

$$R = R(x_t|\eta) = \frac{|x_t-\mu|}{2\alpha_2^{1/\alpha_2}\sigma(1-\beta)C_0^2}\mathbb{1}_{\{x\geq\mu\}} \tag{6}$$

Then, the log-likelihood function is expressed as $\ln f(x_t|\eta) = -\ln\sigma - L^{\alpha_1} - R^{\alpha_2} + \ln(\delta^\nu + \frac{\alpha_1^{1/\alpha_1}L^\nu}{\nu}) + \ln(\delta^\nu + \frac{\alpha_2^{1/\alpha_2}R^\nu}{\nu})$

Theorem 1: Let (X_t) be a MAEPD r.v. with density function $f(x|\eta_0)$ defined in (2). We suppose that η lies in a parameter space $\Theta \subset (0,\infty)^3 \times (0,1) \times (0,\infty)^2$, where Θ is a compact set and η_0 is an interior point of Θ. Then, the MLE $\hat{\eta}$ of η_0 verifies the following properties when $n \to \infty$,

1) $\hat{\eta}_n \xrightarrow{a.s.} \eta_0$,
2) When $\alpha_1, \alpha_2, \nu > 1$, $n^{1/2}(\hat{\eta}_n - \eta_0) \xrightarrow{d} \mathcal{N}(0, \Sigma)$, where $\Sigma = I^{-1}(\eta_0)$ and $I(\eta_0)$ is the Fisher information matrix.

Theorem 1 provides consistent estimates of VaR and ES in the next section.

III. Real data example

This section aims to evaluate the performance of risk measure estimation under unstable periods of time. Especially, we analyze the VaR and ES estimation with high-frequency FTSE data during the "Brexit" period. An ARMA-mcsGARCH model is fitted and this method requires two steps. First, we consider FTSE daily log-return (Y_t^d) from January 1, 2013 to May 14, 2016, totalling $T^d = 850$ daily observations. An

ISBN 978-0-9928626-7-1

ARMA-GARCH model with MAEPD innovations is fitted to the in-sample data and one-day-ahead daily volatility forecast $\hat{\sigma}_t^d$ are computed over the out-of-sample period from May 15, 2016 to July 15, 2016 ($F^d = 44$ forecasts). Second, 1-minute FTSE returns $(Y_{t,i}^{HF})$ follow an ARMA-mcsGARCH model for the in-sample period ($T^{HF} = 840$ observations for each of the $N^{HF} = 24$ days from May 15, 2016 to June 19, 2016). One minute-ahead VaR and ES are then estimated over the period from June 20, 2016 to July 15, 2016 covering the "Brexit" referendum and backtesting procedures are applied.

A. Model description

(Y_t^d) is assumed to follow an ARMA(p,q)-GARCH(r,s) model with MAEPD innovations,

$$\begin{aligned} Y_t^d &= \sum_{i=1}^{p} \phi_i^d Y_{t-i}^d + X_t^d + \sum_{i=1}^{q} \theta_i^d X_{t-i}^d, \\ X_t^d &= \sigma_t^d \varepsilon_t^d, \\ (\sigma_t^d)^2 &= a_0 + \sum_{i=1}^{r} a_i (X_{t-i}^d)^2 + \sum_{i=1}^{s} b_i (\sigma_{t-i}^d)^2, \end{aligned} \tag{7}$$

where the polynomials $\phi(z) = 1 - \phi_1^d z - \cdots - \phi_p^d z^p$ and $\theta(z) = 1 + \theta_1^d z + \cdots + \theta_q^d z^q$ have no common zeros and neither $\phi(z)$ nor $\theta(z)$ has zeros in the closed unit disk $\{z \in \mathbb{C} : |z| \leq 1\}$, $a_0 > 0$, all coefficients (a_i, b_i)'s are nonnegative, $\sum_{i=1}^{\max(r,s)} (a_i + b_i) < 1$, and (ε_t^d) is a sequence of independent and identically distributed (iid) random variables satisfying $\mathrm{E}(\varepsilon_t^d) = 0$ and $\mathrm{E}(\varepsilon_t^d)^2 = 1$. ε_t^d follows a standardized MAEPD distribution whose density is defined by (1). Performance of the MAEPD in modelling daily innovations ε_t^d is compared to other distributions.

We compute the Bayesian information criterion (BIC) for each ARMA(p,q) model with $0 \leq p,q \leq 6$, and the smallest value is -5547.277 and is obtained with $(p,q) = (2,2)$. In order to avoid over-parameterization, we fit a GARCH(1,1) model to (X_t^d) which is usual in the financial literature.

[3] proposes a GARCH model for high-frequency intraday financial returns called mcsGARCH. The conditional variance is specified to be a multiplicative product of daily, diurnal, and stochastic intraday volatility. Days are here indexed by $t = 1,...,N^{HF}$ and 1-minute intervals by $i = 1,...,T^{HF}$. Under this indexation, we note $(t,i)+j$ the jth data point following the one at indexed time (t,i). $(Y_{t,i}^{HF})$ is described by the following process:

$$\begin{aligned} Y_{t,i}^{HF} &= \sum_{j=1}^{l} \phi_j^{HF} Y_{(t,i)-j}^{HF} + X_{t,i}^{HF} + \sum_{j=1}^{m} \theta_j^{HF} X_{(t,i)-j}^{HF}, \\ X_{t,i}^{HF} &= \sigma_t^d s_i q_{t,i} \varepsilon_{t,i}^{HF}, \\ (q_{t,i})^2 &= a_0 + \sum_{j=1}^{u} a_j (X_{(t,i)-j}^{HF})^2 + \sum_{j=1}^{v} b_j (q_{(t,i)-j})^2 \end{aligned} \tag{8}$$

where σ_t^d is the daily volatility and is estimated by its forecast $\hat{\sigma}_t^d$ from model (7), s_i is the diurnal volatility pattern , $q_{t,i}$ is the intraday volatility component with $E(q_{t,i}^2) = 1$ and $\varepsilon_{t,i}^{HF}$ is an error term. $\varepsilon_{t,i}^{HF}$ follows a Student-t distribution. High-frequency FTSE data exhibit symmetry and its size leads us to choose a distribution with a small number of parameters. The model is estimated into two steps. First, the diurnal component s_i is estimated by

$$\hat{s_i}^2 = \frac{1}{N^{HF}} \sum_{t=1}^{N^{HF}} \frac{(X_{t,i}^{HF})^2}{(\hat{\sigma}_t^d)^2} \tag{9}$$

Second, $z_{t,i} = \frac{X_{t,i}^{HF}}{\hat{\sigma}_t^d s_i}$ is modelled as a GARCH(u,v) process where $(u,v) = (1,1)$. We compute the Bayesian information criterion (BIC) for each ARMA(l,m) model with $0 \leq l,m \leq 6$, and the smallest value is 208476.1 and is obtained with $(l,m) = (3,1)$.The MLE of both model parameters is conducted using the source code of the R package rugarch [17].

B. Backtesting

Backtesting risk measure models consists in designing statistical tests to compare actual losses and VaR or ES calculations. For each trading minute (t,i), the value at risk $\mathrm{VaR}_\alpha^{t,i}$ at the probability level α, $0 < \alpha < 1$, is the α-quantile of $(z_{t,i})$, with a negative value corresponding to a loss. Equivalently,

$$\mathsf{Pr}(z_{t,i} \leq \mathrm{VaR}_\alpha^{t,i}) = \alpha. \tag{10}$$

The null hypothesis in the unconditional coverage test of Kupiec *(UC)* is that the exception rate $\hat{\alpha}$ obtained from model (8) is equal to the true probability level α, see [18]. Replacing the unknown parameters η by their MLE $\hat{\eta}_T$ obtained from $(z_{t,i})$ with $(t,i) = (1,1),\ldots,(N^{HF},T^{HF})$ in the expression of VaR_α, we get $\mathrm{VaR}_\alpha(\hat{\eta}_T)$ and $\hat{\alpha}$ is estimated by e/n where $e = \sum_{(t,i)=(N^{HF},T^{HF})+1}^{(F^d,T^{HF})} \mathbb{1}_{z_{t,i} < \mathrm{VaR}_\alpha(\hat{\eta}_T)}$ and $n = (F^d - N^{HF})T^{HF}$ is the number of out-of-sample observations. The likelihood ratio $\mathrm{LR}_{\mathrm{UC}}$ is

$$\mathrm{LR}_{\mathrm{UC}} = -2\ln\left(\frac{(1-\alpha)^{n-e}\,\alpha^e}{(1-e/n)^{n-e}\,(e/n)^e}\right).$$

Under the null hypothesis, $\mathrm{LR}_{\mathrm{UC}}$ is asymptotically distributed with a χ_1^2 distribution.

The conditional coverage test of Christoffersen *(CC)* adds a test for independence *(ind)* to the unconditional coverage one, see [19]. The null hypothesis in the test for independence is then $\pi_{01} = \pi_{11}$ where π_{ij} denotes the conditional probability of condition j assuming that condition i occurred on the previous day. Condition "1" means that an exception occurs and condition "0", no exception. π_{01} and π_{11} are estimated by

$$\hat{\pi}_{01} = \frac{n_{01}}{n_{00}+n_{01}}, \qquad \hat{\pi}_{11} = \frac{n_{11}}{n_{10}+n_{11}}.$$

where n_{ij} denotes the number of minutes when the sequence ij occurs. The likelihood ratio $\mathrm{LR}_{\mathrm{ind}}$ is

$$\mathrm{LR}_{\mathrm{ind}} = -2\ln\left(\frac{(1-\frac{e}{n})^{n_{00}+n_{10}}\,\frac{e}{n}^{n_{01}+n_{11}}}{(1-\pi_0)^{n_{00}}\pi_0^{n_{01}}(1-\pi_1)^{n_{10}}\pi_1^{n_{11}}}\right)$$

The quantity $\mathrm{LR}_{\mathrm{CC}} = \mathrm{LR}_{\mathrm{UC}} + \mathrm{LR}_{\mathrm{ind}}$ is asymptotically distributed with a χ_2^2 distribution and is used to measure the conditional coverage performance of the model.

The ES at the probability level α is

$$\mathrm{ES}_{\alpha}^{t,i} = \frac{1}{\alpha}\int_0^{\alpha} \mathrm{VaR}_p^{t,i}\, dp \tag{11}$$

and coincides with the Conditional Value-at-Risk $\mathrm{CVaR}_{\alpha}^{t,i}$ defined by

$$\mathrm{CVaR}_{\alpha}^{t,i} = \mathrm{E}[z_{t,i} \mid z_{t,i} < \mathrm{VaR}_{\alpha}^{t}], \tag{12}$$

since the MAEPD distribution is continuous, see [20].

The ES backtesting procedure is the one presented in [21]. This procedure presents a coverage test for any spectral risk measure such as ES. The null hypothesis for the Spectral Risk Measure Coverage Test is that the exception rate $\hat{\alpha}$ obtained from model (8) is equal to the true probability level α. The Z-score $Z^n_{\mathrm{ES}_\alpha}$ is

$$Z^n_{\mathrm{ES}_\alpha} = \sqrt{3n}\frac{2\overline{X}^n_{\mathrm{ES}_\alpha} - \alpha}{\sqrt{\alpha(4-3\alpha)}}.$$

where

$$\overline{X}^n_{\mathrm{ES}_\alpha} = \frac{1}{n}\sum_{(t,i)=(N^{HF},T^{HF})+1}^{(F^d,T^{HF})} \frac{1}{\alpha}\int_0^{\alpha} \mathbb{1}_{\{z_{t,i}\leq \mathrm{VaR}_p(\hat{\eta}_T)\}}\, dp.$$

Under the null hypothesis, $Z^n_{\mathrm{ES}_\alpha}$ is asymptotically distributed with a normal distribution.

The three above-mentioned backtests are conducted to evaluate the performance of the MAEPD in estimating the daily volatility component in model (8). To do so, both model (7) and (8) are refitted every day. The likelihood ratios $\mathrm{LR}_{\mathrm{UC}}$ and $\mathrm{LR}_{\mathrm{CC}}$ and the Z-score $Z^n_{\mathrm{ES}_\alpha}$ are displayed in Table I. Other distributions commonly used in finance -namely GHYP, Skewed Student-t (SSTD) and Skewed Generalized Error (SGED) Distributions- can describe ε_t^d in (7) and enable to compare the performance of the model depending on the innovations. We rank the performance of the models according to their Z-score/Likelihood Ratio and the closest Z-score/Likelihood Ratio to zero corresponds to the best model.

TABLE I: Backtesting VaR and ES for the MAEPD, GHYP, SSTD and SGED distributions.

Dist	Score	$\mathrm{VaR}_{1\%}$	$\mathrm{VaR}_{2.5\%}$	$\mathrm{ES}_{1\%}$	$\mathrm{ES}_{2.5\%}$
MAEP	$\mathrm{LR}_{\mathrm{UC}}$	**0.479**	**0.409**	-	-
	$\mathrm{LR}_{\mathrm{CC}}$	**2.366**	**5.245**	-	-
	$Z^n_{\mathrm{ES}_\alpha}$	-	-	**2.005***	**1.191**
GHYP	$\mathrm{LR}_{\mathrm{UC}}$	0.846	0.543	-	-
	$\mathrm{LR}_{\mathrm{CC}}$	6.511*	6.403*	-	-
	$Z^n_{\mathrm{ES}_\alpha}$	-	-	2.106*	1.245
SSTD	$\mathrm{LR}_{\mathrm{UC}}$	**0.479**	0.543	-	-
	$\mathrm{LR}_{\mathrm{CC}}$	4.184	6.403*	-	-
	$Z^n_{\mathrm{ES}_\alpha}$	-	-	2.08*	1.252
SGED	$\mathrm{LR}_{\mathrm{UC}}$	0.991	**0.409**	-	-
	$\mathrm{LR}_{\mathrm{CC}}$	4.412	**5.245**	-	-
	$Z^n_{\mathrm{ES}_\alpha}$	-	-	2.149*	1.29

* Rejection at the 5% significance level.

The MAEPD distribution shows the best performance in terms of ES and VaR estimation compared to commonly used distributions in risk measurement in this unstable context. For the MAEPD, only the hypothesis for $\mathrm{ES}_{1\%}$ estimation is rejected at the 5% significance level.

IV. Conclusion

The paper aimed to define a multimodal extension of the EPD, the MAEPD. We derive moments and we propose a convenient stochastic representation of the MAEPD. We establish consistency, asymptotic normality and efficiency of the MLE. An application to risk measurement for high-frequency data is given and time series analysis through the mcsGARCH model exhibit good performance.

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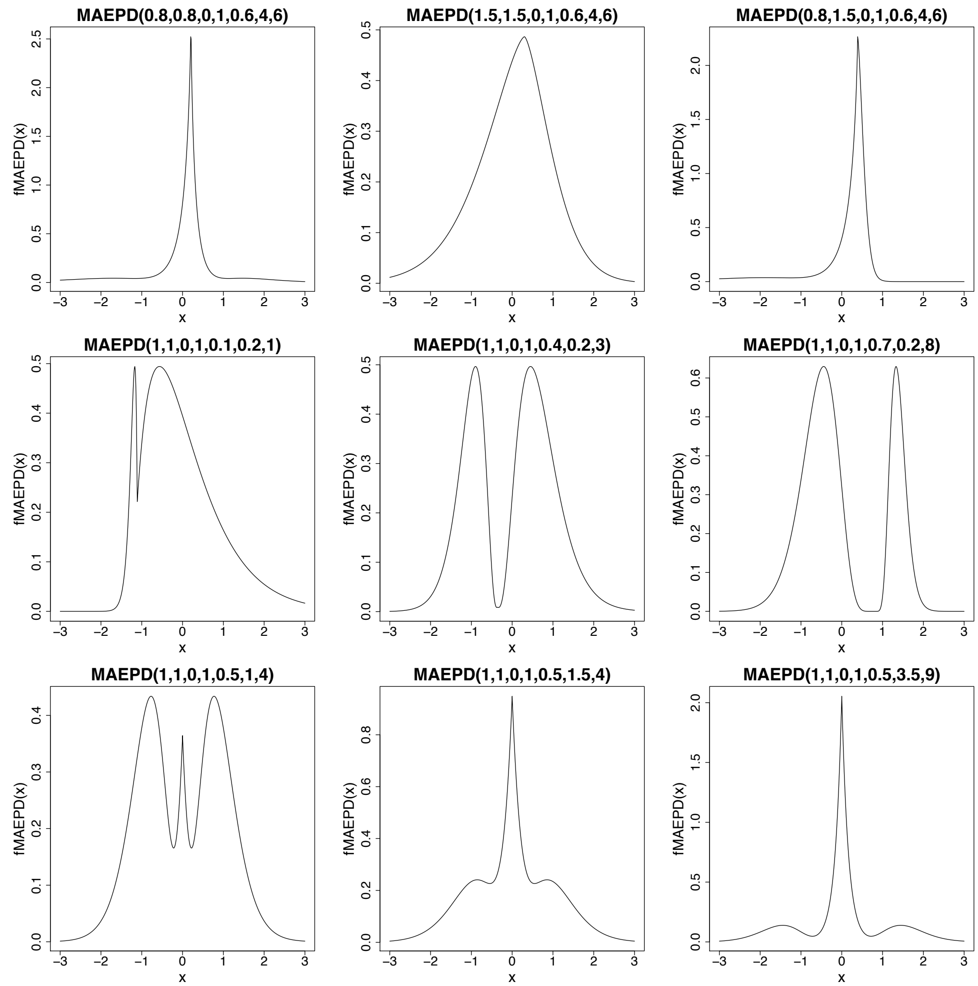

Fig. 1: Density function of $\mathrm{MAEPD}(\alpha1, \alpha2, \mu, \sigma, \beta, \delta, \nu)$

ISBN 978-0-9928626-7-1 © EURASIP 2017