Graph Adjacency Matrix Learning for Irregularly Sampled Markovian Natural Images

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***Abstract*—The boost of signal processing on graph has recently solicited research on the problem of identifying (learning) the graph underlying the observed signal values according to given criteria, such as graph smoothness or graph sparsity. This paper proposes a procedure for learning the adjacency matrix of a graph providing support to a set of irregularly sampled image values. Our approach to the graph adjacency matrix learning takes into account both the image luminance and the spatial samples' distances, and leads to a flexible and computationally light parametric procedure. We show that, under mild conditions, the proposed procedure identifies a near optimal graph for Markovian fields; specifically, the links identified by the learning procedure minimize the potential energy of the Markov random field for the signal samples under concern. We also show, by numerical simulations, that the learned adjacency matrix leads to a higly compact spectral wavelet graph transform of the so obtained signal on graph and favourably compares to state-of-the-art graph learning procedures, definetly matching the intrinsic signal structure.**

I. Introduction

The boost of signal processing on graph (SoG) has recently solicited research on the problem of graph learning, i.e. of identifying the graph underlying the observed data values according to given criteria, such as graph smoothness or graph sparsity [1], [2]. In fact, except for specific SoG applications where the data are univocally associated to an underlying graph structure -e.g. in sensor networks where the nodes correspond to spatial sensor locations and the links to actual communication channels between sensors-, the graph structure must be designed in accordance to the observed signal and different graph learning procedures can be envisaged. In random geometric graphs, links are established based on nodes distances, in a deterministic or probabilistic fashion [3]. In mesh-based image representation, the graph is straightforwardly induced by the topology of the mesh [4], since the nodes and the signal samples at the nodes are extracted at the vertices of the mesh and the node links reproduce the mesh grid; this is exemplified in Figs.1,2(a)-(c), where two different images, a depth map and a nevus image, are displayed, together with the meshes obtained by Delaunay triangulation and the corresponding graphs. Model-based graph learning has been recently analyzed in [1], where the authors resort to a parametric smooth signal model and develop a procedure for learning the graph Laplacian matrix via an alternating minimization algorithm jointly enforcing the SoG smoothness and the Laplacian properties of sparsity and semi-definite positiveness. In [2], the authors propose an iterative Laplacian matrix learning algorithm that, stemming on the knowledge of which edges are active, updates one row/column of the precision matrix at a time by solving a non-negative quadratic program and superimposing prior constraints on the Laplacian matrix structure.

This paper addresses the problem of building a suitable graph given a set of irregularly sampled image values. Our approach to adjacency matrix construction takes into account both the image luminance and the spatial samples' distances, and leads to a parametric procedure. We show that, in the limit as the weight of the luminance values overcomes that of the inter-node distances in the adjacency matrix learning, the procedure provides the optimal graph for Markovian fields; specifically, the generated signal on graph achieves a minimum of the Markovian field potential energy for the signal samples under concern. Finally, we analyze the impact of the graph selection on the spectral wavelet SoG representation and we show, by numerical simulation results, that the spectral wavelet transform associated to the graph leads to a compact signal representation, and the learning algroithm favourably compares to state-of-the-art procedures.

II. ID-LD adjacency matrix learning

Let us consider a bi-dimensional signal sampled at N spatial locations $x_i, y_i, i = 0, \cdots N-1$, where it assumes the values $f_i, i = 0, \cdots N-1$, and let us denote as $\mathbf{f} = [f_0 \cdots f_{N-1}]^T$ the vector collecting the signal samples. We are interested in building the undirected graph underlying the signal $\mathbf{f}$. The graph $\mathcal{G}$ is defined as the triplet $\mathcal{G} = (\mathcal{N}, \mathcal{E}, A)$ where $\mathcal{N}$ represents the set of graph nodes corresponding to the spatial sampling locations, $\mathcal{E}$ the set of edges (links) between nodes, and A is the unknown adjacency matrix, whose elements a_{ij} are real or binary values representing the weight of the graph links.

Here, we propose a procedure for learning the adjacency matrix A given the set of N samples $\mathbf{f}$ acquired in known spatial locations. The procedure accounts both for geometric constraints as well as for graph smoothness. Specifically, the adjacency matrix is built based on the spatial Inter-nodes Distances and Luminance Differences (ID-LD).

To elaborate, let us consider the following representation of the image samples in $\mathcal{R}^3$:

$$P_i^{(\alpha)} = (x_i, y_i, \alpha f_i),\ i = 0, \cdots N-1 \qquad (1)$$

being $\alpha > 0 \in \mathcal{R}$ a real positive parameter.

We associate the generic element a_{ij} of the ID-LD adjacency matrix A, i.e. the weight of the link between the i-th and j-th

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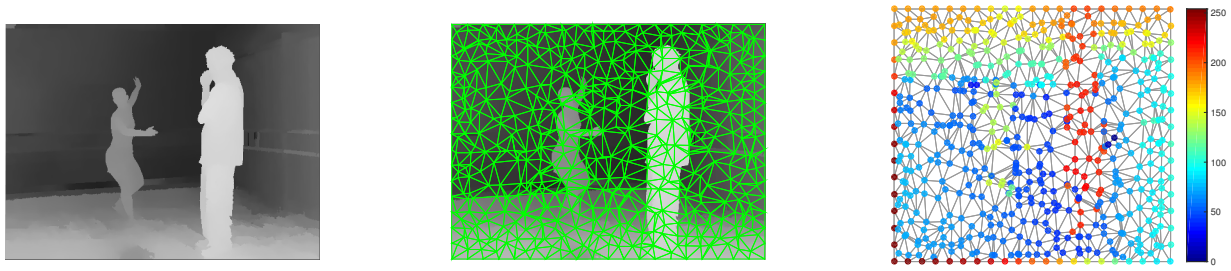
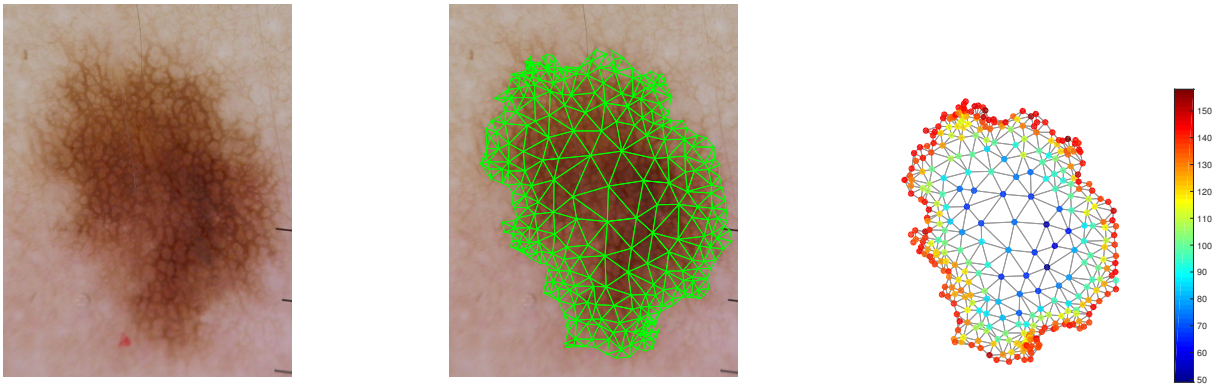


(a) (b) (c)

Fig. 1. Example of depth map, a mesh built on it and the corresponding Signal on Graph (SoG).



(a) (b) (c)

Fig. 2. Example of biomedical image (nevus), a mesh built on it and the corresponding Signal on Graph (SoG).

nodes, to the Euclidean distance between $P_i^{(\alpha)}$ and $P_j^{(\alpha)}$. To this aim, we define the function

$$w_{\sigma_w}(P_i^{(\alpha)}, P_j^{(\alpha)}) = e^{-\frac{(x_i-x_j)^2+(y_i-y_j)^2+\alpha^2(f_i-f_j)^2}{2\sigma_w^2}} \quad (2)$$

measuring the distance between the points as well as their difference in luminance. We then propose the following definition of the ID-LD adjacency matrix element a_{ij}:

$$a_{ij} = \begin{cases} w_{\sigma_w}(P_i^{(\alpha)}, P_j^{(\alpha)}) & i \neq j,\ w_{\sigma_w}(P_i^{(\alpha)}, P_j^{(\alpha)}) \geq \theta \\ 0 & \textit{otherwise} \end{cases} \quad (3)$$

A few remarks are in order. Firstly, the weights' definition in Eq.3 differs both from the definition adopted in random geometric graphs, accounting only for the distances, and the definition adopted Gaussian weighted graphs, accounting only for the signal value. Yet, the definition encompasses these two cases for particular values of α, namely $\alpha = 0$, $\alpha \to \infty$, respectively, and the amplifying factor α in Eq.1 allows to naturally and smoothly switch between considering the geometric distances of the points and the differences in the observed luminance. Secondly, the thresholding operated in Eq.3 allows to control the graph connectivity by suitably selecting the number of nonzero elements in A. Thirdly, the ID-LD adjacency matrix A in Eq.3 has an interesting interpretation. When the signal $\mathbf{f}$ is considered as a realization of a Markov Random Field over an irregular lattice, the neighboring system induced by the graph topology implied by the ID-LD adjacency matrix achieves the minimum potential energy. This is shown in the next subsection.

We exemplify the results of the ID-LD adjacency matrix learning in Fig.3(a)-(c), where we plot as red points the signal samples of Fig.2(c), corresponding to the luminance values at the vertices mesh of Fig.2(b), as in Eq. 1, for $\alpha = 1, 2.5, 5$ respectively; for better visualization, the spatial scale has been normalized so that the width of the visualization region is equal to 1 for the three cases. Fig.3(a)-(c) also shows as blue lines the graph links obtained by applying the adjacency model proposed in Eq.3, by setting σ equal to the 12.5% of the spatial coordinates range and θ equal to 0.85.

A. ID-LD Adjacency Matrix and Markovian SoGs

Let us us now recast the signal $\mathbf{f}$ associated to a set of graph nodes as a realization of a Markov Random Field (MRF), which is a powerful model frequently adopted both for natural images [5] and textures [6]. The MRF under concern is such that i) the graph nodes represent the sites of the (irregular)

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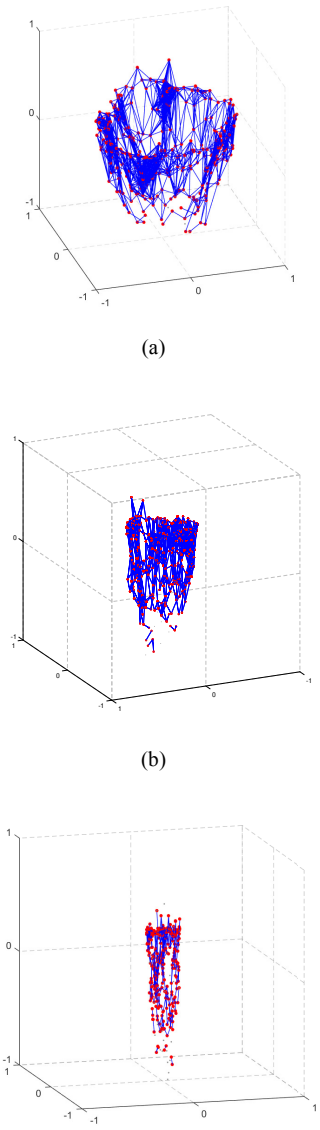

(a) (b) (c)

Fig. 3. Example of graphs built using a) $\alpha = 1.0$,$\sigma = 0.125$, $\theta = 0.85$, b) $\alpha = 2.5$,$\sigma = 0.125$, $\theta = 0.85$ c) $\alpha = 5.0$,$\sigma = 0.125$, $\theta = 0.85$.

lattice over which the field is defined, and ii) the graph adjacency matrix induces the neighborhood system of the MRF. Specifically, we define the neighborhood $\eta_i^{(A)}$ of the i-th node as the set of nodes j for which a link to i exists, i.e.

$$\eta_i^{(A)} = \{j \text{ s.t. } a_{ij} \neq 0\} \tag{4}$$

Besides, let us refer to clique systems[1] made up by two pixels-cliques, namely $c = \{(f_i, f_j) \text{ s.t. } a_{ij} \neq 0\}$, and clique potential functions $V(c)$ which are increasing with respect to the Euclidean distance between f_i, f_j, i.e. $V(c) = V(|f_i - f_j|)$. Examples of such clique potential function can be found in the literature [5], and include, but are not limited to, quadratic functions of the form $V(|f_i - f_j|) = \frac{(f_i - f_j)^2}{\sigma_v^2}$ as well as exponential functions of the form $V(|f_i - f_j|) = 1 - 2\exp^{-\frac{(f_i - f_j)^2}{\sigma_v^2}}$.

By definition of MRF, the local conditional probability of a sample satisfies the property $p(f_i|f_j, \forall j \neq i) = p(f_i|f_j, \forall j \in \eta_i^{(A)})$. According to the Hammersley-Clifford theorem, this local property reflects into the following global property: the *a priori* probability measure of the MRF realization $\mathbf{f}$ is computed in accordance to the Gibbs distribution:

$$p(\mathbf{f}; \eta^{(A)}) = \frac{1}{Z} \exp\left\{-U(\mathbf{f}; \eta^{(A)})\right\}$$

where Z is a normalization factor, and $U(\mathbf{f};; \eta^{(A)})$ is the potential energy function defined as

$$U(\mathbf{f}; \eta^{(A)}) = \sum_i \sum_{j \in \eta_i^{(A)}} V(|f_i - f_j|)$$

where the first sum spans the set of nodes, the second sum over the i-th node neighbors, i.e. the nodes forming a clique with the i-th node.

We show here that, under mild hypotheses, when the ID-LD adjacency matrix A in Eq.3 is computed, it induces an associated neighborhood system $\eta^{(A)}$ which minimizes the potential energy function $U(\mathbf{f}; \eta^{(A)})$ of the signal $\mathbf{f}$ under the constraint of an assigned number of links.

Theorem: Let us consider the graph configuration achieved by considering the ID-LD adjacency matrix in Eq.3, for a given θ. Let us denote by N_θ the resulting number of links. In the limit for $\alpha \gg 1$, the ID-LD adjacency matrix A in Eq.3 leads to a neighborhood system $\eta_i^{(A)}$ which minimizes the potential energy function $U(\mathbf{f}; \eta^{(A)})$ relative to the signal $\mathbf{f}$.

Proof.: With the afore presented positions, the potential energy function specifies as

$$U(\mathbf{f}; \eta^{(A)}) = \sum_i \sum_{i,j;\, a_{ij} \neq 0} V(|f_i - f_j|)$$

According to Eq.3, by construction any f_i, f_j pair such that $a_{ij} \neq 0$ satisfies the inequality

$$(f_i - f_j)^2 \leq \ln(\theta)\ 2\sigma^2/\alpha^2,$$

whereas $(f_i - f_j)^2 > \ln(\theta)\ 2\sigma^2/\alpha^2$ for any $a_{ij} = 0$. Under the constraint of constant number of links N_θ, a new graph link can be added *iff* a link is removed. We recognize that

$$\sup\{(f_i - f_j)^2,\ a_{ij} \neq 0\} < \inf\{(f_i - f_j)^2,\ a_{ij} = 0\}$$

i.e., due to the ID-LD definition, all the pixel pairs corresponding to existing graph links have uniformly lower distances than those not connected by graph links. Thereby, no link can be removed from the graph to add a novel one without increasing the energy associated to the graph configuration.

This leads to an interesting interpretation of the ID-LD matrix A in Eq.3 as the adjacency matrix minimizing the potential energy of the actual signal configuration:

$$A_{ID\text{–}LD} = \arg\min_C U(\mathbf{f}; \eta^{(C)})$$

[1]A clique is a set of pixels that belong to each other neighborhood.

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This result suggests [7] that the ID-LD matrix has the capability of yielding a compact spectral representation of the SoG, as we experimentally assess in the next Section.

III. Experimental results

In this Section we presents numerical results in order to assess the ability of the graph represented by the ID-LD adjacency matrix A to capture the graph structure underlying the visual data $\mathbf{f}$.

Firstly, we present a toy example illustrating the relation between the adjacency matrix in Eq.3 of a graph and the potential energy of the associated signal. Specifically, in the first row of Fig.4 we show the adjacency matrix coefficients corresponding to each and every (f_i, f_j) pair ($\alpha = 5,\ \sigma_w = 10,\ \theta = 0.5$); for clarity sake, on the left we highlight (solid green line) the graph edges and the corresponding a_{ij}, while on the right we highlight (red dotted lines) the pairs not connected by edges, having $a_{ij} = 0$. In the second row we show the corresponding clique potential functions computed as $V(c) = 1 - 2\exp^{-(f_i - f_j)^2/\sigma_v^2}$. In this toy case we recognize that the ID-LD adjacency matrix identifies the links, and hence the neighborhood system, that minimizes the potential energy relative to the assigned signal configuration. In fact, no link can be substituted to an existing one without increasing the energy $U(\mathbf{f}; \eta^{(A)}) = \sum_i \sum_{j \in \eta_i^{(A)}} V(f_i, f_j)$ associated to the graph configuration.

Secondly, in order to support the claim that the ID-LD adjacency matrix computed as in Eq.3 captures the underlying signal structure, we show that the spectral wavelet transform (SGWT) [8] of the associated graph is extremely compact. The SGWT is built by i) performing the spectral decomposition of the discrete graph Laplacian L to identify the eigenvalues and eigenvectors $\lambda_l, \mathbf{u}^{(l)},\ l = 0 \cdots N-1$, ii) selecting the scales of interest by suitably windowing the Laplacian eigenvalues λ, and iii) finally localizing the frequency content around different graph nodes by means of an indicator function. In formulas, the SGWT basis element referring at scale j and node n is a signal on graph assuming on the m-th node the value

$$\psi_m^{(j,n)} = \sum_{l=0}^{N-1} g(j\ \lambda_l) u_n^{(l)} u_m^{(l)}$$

where $g(\cdot)$ represents a suitable windowing function. Following the SGWT theoretical analysis in [8] and its implementation in [11], we consider here SGWT reconstruction of the graphs in Fig.3 by truncating the spectral graph wavelet coefficients to a restrained number of k scales out of the K leading to perfect reconstruction. In Tables I,II we report the SGWT mean square reconstruction error obtained when using the proposed adjacency matrix for $K = 5$ and k from 1 to 4. For comparison sake, we also consider the mean square error for the graphs in Figs.1 , 2. We recognize that adopting the proposed approach significantly improves the compression efficiency of the SGWT. The reason why this occurs is because the graph tightly matches the actual signal $\mathbf{f}$ smoothness, since it induces links among similar signal values, while controlling the geometrical graph complexity.

TABLE I
Mean square Value of the SGWT Reconstruction Error using wavelet coefficients up to scale k (Nevus image, $\sigma = 0.125,\ \theta = 0.85$)

SGWT scale	$k=1$	$k=2$	$k=3$	$k=4$
Proposed $\alpha = 5$	3887	1537	200	6
Mesh-based	54845	21842	9966	242

TABLE II
Mean square Value of the SGWT Reconstruction Error using wavelet coefficients up to scale k (Depth Map, $\sigma = 0.125,\ \theta = 0.85$)

SGWT scale	$k=1$	$k=2$	$k=3$	$k=4$
Proposed $\alpha = 5$	78958	37648	14548	319
Mesh-based	1279404	824045	414240	7911

TABLE III
Fmeasure for the ID-LD matrix and the method in [1].

Number of communities	$P=2$	$P=3$	$P=4$	$P=5$
Proposed $\alpha = 5$	0.74	0.66	0.62	0.57
Method in [1]	0.67	0.58	0.51	0.49

Furthermore, in Fig.5 we show the SoGs obtained using the adjacency matrix A as in Eq.3 on the images in Figs.1, 2. We recognize that the ID-LD matrix clearly induces links between homogeneous regions and inhibits links across image discontinuities.

Finally, we test the proposed graph learning algorithm on a random graph, built by $N = 25$ nodes belonging to P communities, spatially distributed in equiangularly spaced circular regions with centers on the unit circle and unitary radius; we set the intra-community edge probability 0.8 and inter-community edge probability 0.2 and nonzero weights uniformly distributed in $[0, 1]$; the signal samples on nodes within communities are independently uniformly distributed in the community dependent ranges $[p, p+1]\ \ p = 1, \cdots P$. We compare the graph learning results with those obtained by using the method *GL-SigRep*[2] in [1]. Specifically, Table III reports the F-measure, defined as the harmonic mean of the recall and precision performances of the proposed method and that in [1], averaged over 20 runs and for different values of P. The ID-LD adjacency matrix well captures the signal smoothness, well performing w.r.t. [1].

To sum up, the ID-LD adjacency matrix provides a theoretically sound tool for describing the graph structure underlying irregularly sampled image values. Besides, the ID-LD approach nicely induces a compact SGWT representation, which in turns can be adopted in many signal on graph processing procedures.

IV. Conclusion

In this paper we have presented a procedure for graph learning given a set of irregularly sampled image values. Specifi-

[2]The GL-SigRep parameters have been selected for best performance at $\alpha \approx 10^{-4}$, $\beta \approx 10^{-5}$.

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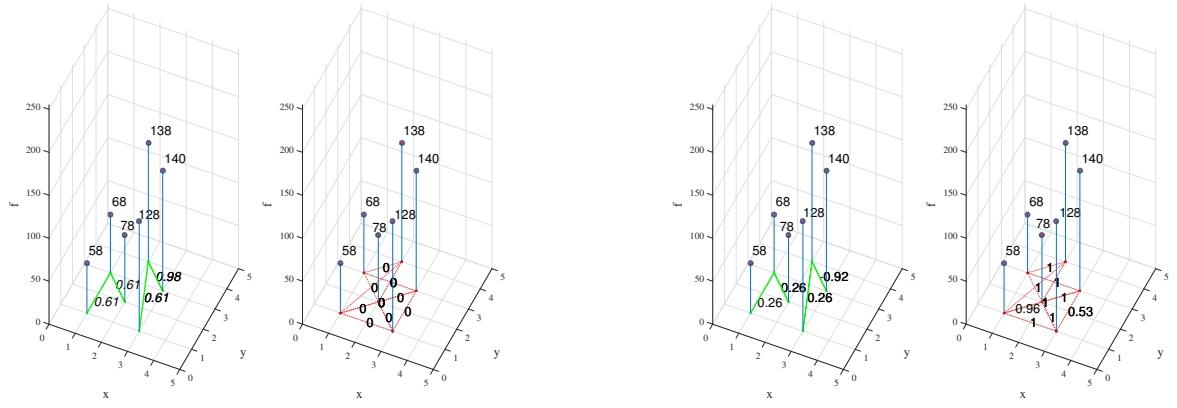
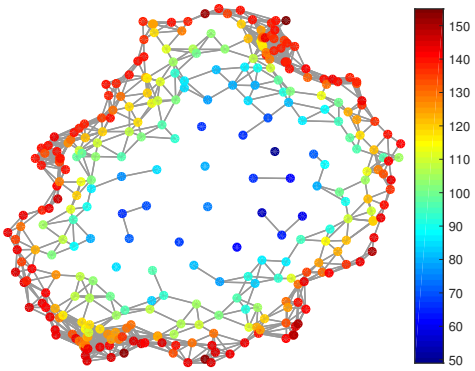
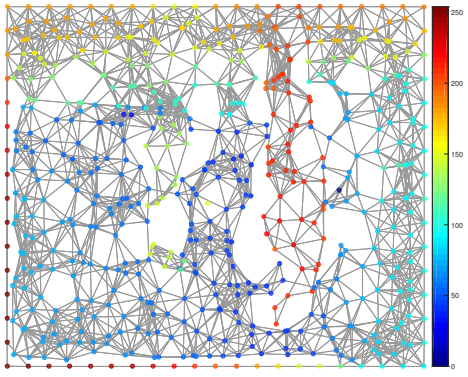


(a) (b)

Fig. 4. Toy example of graph built using the ID-LD adjacency matrix A in Eq.3: (a) adjacency matrix coefficients a_{ij} and (b) potential functions $Vc(|f_i - f_j|)$ on each pair (i, j).



(a)



(b)

Fig. 5. Example of graphs built using the proposed graph construction algorithm $\alpha = 5.0$,$\sigma = 0.125$, $\theta = 0.9$.

cally, we propose a light and flexible parametric approach to computing the adjacency matrix by taking into account both the spatial sampling location distance and the difference in luminance. We also show that the learned adjacency matrix minimizes the potential energy of the actual signal configuration in accordance to a Markovian signal model. Finally, we experimentally show that the adjacency matrix yields a compact spectral representation of the SoG and favourably compares to state-of-the-art graph learning procedures.

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