Characterization of Discrete Linear Shift-Invariant Systems

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***Abstract*—Linear time-invariant (LTI) systems are of fundamental importance in classical digital signal processing. LTI systems are linear operators commuting with the time-shift operator. For N-periodic discrete time series the time-shift operator is a circulant $N \times N$ permutation matrix. Sandryhaila and Moura developed a linear discrete signal processing framework and corresponding tools for datasets arising from social, biological, and physical networks. In their framework, the circulant permutation matrix is replaced by a network-specific $N \times N$ matrix $\boldsymbol{A}$, called a *shift matrix*, and the linear shift-invariant (LSI) systems are all $N \times N$ matrices $\boldsymbol{H}$ over $\mathbb{C}$ commuting with the shift matrix: $\boldsymbol{HA} = \boldsymbol{AH}$. Sandryhaila and Moura described all those $\boldsymbol{H}$ for the non-degenerate case, in which all eigenspaces of $\boldsymbol{A}$ are one-dimensional. Then the authors reduced the degenerate case to the non-degenerate one. As we show in this paper this reduction does, however, not generally hold, leaving open one gap in the proposed argument. In this paper we are able to close this gap and propose a complete characterization of all (i.e., degenerate and non-degenerate) LSI systems. Finally, we describe the corresponding spectral decompositions.**

I. INTRODUCTION

Linear time-invariant (LTI) systems are linear operators commuting with the time-shift operator. Such systems are of fundamental importance in classical digital signal processing. For N-periodic discrete time series the time-shift operator is the circulant permutation matrix corresponding to the cyclic permutation $(0, 1, \ldots, N-1)$. In [1], Sandryhaila and Moura developed a linear discrete signal processing framework and corresponding tools for datasets arising from social, biological, and physical networks. In their framework, the circulant permutation matrix is replaced by a network-specific matrix $\boldsymbol{A} \in \mathbb{C}^{N \times N}$, the *shift matrix*, and the linear shift-invariant (LSI) systems are all matrices in the *centralizer* of $\boldsymbol{A}$:

$$\mathcal{C}(\boldsymbol{A}) := \{\boldsymbol{H} \in \mathbb{C}^{N \times N} \mid \boldsymbol{HA} = \boldsymbol{AH}\}.$$

This centralizer has an additional algebraic structure, briefly reviewed in Subsection III-A: $\mathcal{C}(\boldsymbol{A})$ is a subalgebra of $\mathbb{C}^{N \times N}$. Moreover, $\mathcal{C}(\boldsymbol{A})$ contains the algebra $\mathbb{C}[\boldsymbol{A}]$ of all polynomial expressions in $\boldsymbol{A}$. Thus $\mathbb{C}[\boldsymbol{A}] \subseteq \mathcal{C}(\boldsymbol{A}) \subseteq \mathbb{C}^{N \times N}$.

In [1], the authors provide an explicit description of the centralizer of $\boldsymbol{A}$ in the non-degenerate case in which the eigenspace of every eigenvalue of $\boldsymbol{A}$ is one-dimensional. More precisely, they showed that in this case the centralizer of $\boldsymbol{A}$ coincides with the algebra of all polynomial expressions in $\boldsymbol{A}$, i.e., $\mathcal{C}(\boldsymbol{A}) = \mathbb{C}[\boldsymbol{A}]$. Note that in this case, the centralizer is a commutative algebra. If however $\boldsymbol{A}$ is the $N \times N$ unit matrix, then $\mathcal{C}(\boldsymbol{A}) = \mathbb{C}^{N \times N}$, which is non-commutative for $N \geq 2$. Thus centralizers can be non-commutative. In fact, we prove in Section IV, that the centralizer of $\boldsymbol{A}$ is commutative if and only if all eigenspaces of $\boldsymbol{A}$ are one-dimensional. Thus if we are in the degenerate case, then not all LSI systems w.r.t. $\boldsymbol{A}$ are of the form $h(\boldsymbol{A})$, for a polynomial h. Hence the argument [1] (p. 1647, following Theorem 2) reducing graph filters for the degenerate case to filters in the non-degenerate case does not hold.

In this paper, we give a complete description of LSI systems by describing the centralizer of arbitrary complex-valued $N \times N$ matrices $\boldsymbol{A}$. We start in Section II with two examples illustrating the degenerate case. To keep this paper to some extent self-contained, we introduce in Section III notation required in the following and discuss basic tools: associative algebras and their morphisms, minimal and characteristic polynomials, upper triangular Toeplitz matrices, Jordan canonical form, and Hermite interpolation. After compiling these tools, we present in Section IV the characterization of centralizers. Finally, in Section V we describe the spectral decompositions of the signal space $\mathbb{C}^N$ w.r.t. $\mathbb{C}[\boldsymbol{A}]$ and $\mathcal{C}(\boldsymbol{A})$.

II. ILLUSTRATING THE DEGENERATE CASE

There are a number of network-specific matrices $\boldsymbol{A}$ with repeated eigenvalues. Here are two examples.

Example 1. Figure 1 shows the Petersen graph.

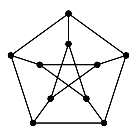

Fig. 1. Peterson graph

The characteristic polynomial $p_{\boldsymbol{A}}(X) := \det(X\mathbf{I} - \boldsymbol{A})$ of its adjacency matrix $\boldsymbol{A}$ equals $(X-1)^5 \cdot (X+2)^4 \cdot (X-3)$. Thus $\lambda_1 = 1$ is a 5-fold eigenvalue, whereas $\lambda_2 = -2$ is a 4-fold eigenvalue of $\boldsymbol{A}$. $\boldsymbol{A}$ is symmetric, hence diagonalizable. Thus the dimension of $\mathbb{C}[\boldsymbol{A}]$ equals the number of distinct eigenvalues of $\boldsymbol{A}$, which is 3, whereas the centralizer $\mathcal{C}(\boldsymbol{A})$ is isomorphic to the non-commutative algebra $\mathbb{C}^{5 \times 5} \oplus \mathbb{C}^{4 \times 4} \oplus \mathbb{C}^{1 \times 1}$, which is of dimension 42. $\diamondsuit$

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Example 2. As a second example we consider the adjacency matrix $\boldsymbol{A} = (a_{i,j})$ of a clique with N nodes. Here, $a_{i,i} = 0$ and $a_{i,j} = 1$, for all i and all $j \neq i$. Figure 2 illustrates the case $N = 6$.

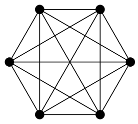

Fig. 2. Clique for $\boldsymbol{N = 6}$.

$\boldsymbol{A}$ is a circulant matrix. (Recall that an $N \times N$ matrix of the form $(a_{j-i \bmod N})_{i,j}$ is called *circulant*.) It is well-known that the $N \times N$ DFT matrix $\boldsymbol{F} = (\omega^{pq})_{0 \le p,q < N}$, $\omega := \exp(2\pi \mathrm{i}/N)$, performs a simultaneous diagonalization of all circulant $N \times N$ matrices $\boldsymbol{A}$. More precisely, if $\boldsymbol{a}$ denotes the leftmost column of $\boldsymbol{A}$, then

$$\boldsymbol{F}\boldsymbol{A}\boldsymbol{F}^{-1} = \mathrm{diag}(\boldsymbol{F}\boldsymbol{a}) =: \boldsymbol{\Delta}.$$

In our case, $\boldsymbol{a} = (0, 1, \ldots, 1)^\top \in \mathbb{C}^N$. A straightforward calculation shows that $\boldsymbol{F}\boldsymbol{a} = (N-1, -1, \ldots, -1)^\top$. In particular, $N - 1$ is a simple eigenvalue, whereas -1 is an $(N-1)$-fold eigenvalue of $\boldsymbol{A}$. Let's compare $\mathbb{C}[\boldsymbol{\Delta}] \simeq \mathbb{C}[\boldsymbol{A}]$ and $\boldsymbol{\mathcal{C}}(\boldsymbol{\Delta}) \simeq \boldsymbol{\mathcal{C}}(\boldsymbol{A})$ for $\boldsymbol{\Delta} := \mathrm{diag}(N-1, -1, \ldots, -1)$. If $h \in \mathbb{C}[X]$, then $h(\boldsymbol{\Delta}) = \mathrm{diag}(h(N-1), h(-1), \ldots, h(-1))$. This shows that $\mathbb{C}[\boldsymbol{\Delta}]$ and hence $\mathbb{C}[\boldsymbol{A}]$ is 2-dimensional, in contrast to the centralizer $\boldsymbol{\mathcal{C}}(\boldsymbol{\Delta})$ which is equal to the space $\mathbb{C}^{1\times 1} \oplus \mathbb{C}^{(N-1)\times(N-1)}$. Thus the centralizer $\boldsymbol{\mathcal{C}}(\boldsymbol{A})$ is of dimension $1 + (N-1)^2$. ♢

Both examples show that in the degenerate case the centralizer of $\boldsymbol{A}$ can be much larger than the space of all polynomial expressions in $\boldsymbol{A}$.

III. Preliminaries

A. Review of Associative Algebras

An associative $\mathbb{C}$-*algebra* $\boldsymbol{\mathcal{A}}$ is a vector space over $\mathbb{C}$ with an associative multiplication and a unit element $1_{\boldsymbol{\mathcal{A}}}$. Moreover, addition, scalar multiplication, and multiplication must be compatible. The space

$$\mathbb{C}[X] := \Big\{ \sum_{j=0}^{n} h_j X^j \mid n \ge 0; h_0, \ldots, h_n \in \mathbb{C} \Big\}$$

of all univariate polynomials or the space $\mathbb{C}^{N\times N}$ of all $N \times N$ matrices are examples of $\mathbb{C}$-algebras. A *subalgebra* $\boldsymbol{\mathcal{B}}$ of $\boldsymbol{\mathcal{A}}$ is a linear subspace of $\boldsymbol{\mathcal{A}}$ closed under multiplication and containing $1_{\boldsymbol{\mathcal{A}}}$. For example, the space of all upper triangular $N \times N$ matrices is a subalgebra of $\mathbb{C}^{N\times N}$. An *algebra morphism* $T : \boldsymbol{\mathcal{A}} \to \boldsymbol{\mathcal{B}}$ is a $\mathbb{C}$-linear map that is also multiplicative and maps $1_{\boldsymbol{\mathcal{A}}}$ to $1_{\boldsymbol{\mathcal{B}}}$. For example, for a fixed matrix $\boldsymbol{A} \in \mathbb{C}^{N\times N}$, the evaluation map $h(X) \mapsto h(\boldsymbol{A})$ defines an algebra morphism $\mathbb{C}[X] \to \mathbb{C}[\boldsymbol{A}]$. The *kernel* of the evaluation map T is defined by $T^{-1}[\boldsymbol{0}] := \{h \in \mathbb{C}[X] \mid h(\boldsymbol{A}) = \boldsymbol{0}\}$. It consists of all multiples of a uniquely determined monic polynomial of smallest degree in $T^{-1}[\boldsymbol{0}]$, the so-called *minimal polynomial* $m_{\boldsymbol{A}}(X)$ of $\boldsymbol{A}$. By the Cayley-Hamilton Theorem, the characteristic polynomial $p_{\boldsymbol{A}}(X) := \det(X\mathbf{I} - \boldsymbol{A})$ of $\boldsymbol{A}$ is a multiple of the minimal polynomial.

B. Upper Triangular Toeplitz Matrices

Recall that a matrix $\boldsymbol{T} = (t_{i,j}) \in \mathbb{C}^{m\times n}$ is a *Toeplitz matrix*, if each descending diagonal from left to right is constant, i.e., $t_{i,j} = t_{i+1,j+1}$, whenever both sides are defined. Thus every $m \times n$ Toeplitz matrix is completely specified by the entries in its leftmost column and its topmost row. We are mainly interested in upper triangular Toeplitz matrices. Such matrices are specified by its topmost row. For $\boldsymbol{a} = (a_0, \ldots, a_{d-1}) \in \mathbb{C}^d$ define $\boldsymbol{T}_d(\boldsymbol{a}) = \boldsymbol{T}_d(a_0, \ldots, a_{d-1}) = (t_{i,j}) \in \mathbb{C}^{d\times d}$ by $t_{i,j} := 0$, if $i > j$, and $t_{i,j} := a_{j-i}$, if $i \le j$. Thus

$$\boldsymbol{T}_d(\boldsymbol{a}) = \begin{bmatrix} a_0 & a_1 & a_2 & \ldots & a_{d-2} & a_{d-1} \\ & a_0 & a_1 & a_2 & & a_{d-2} \\ & & \ddots & \ddots & \ddots & \vdots \\ & & & a_0 & a_1 & a_2 \\ & & & & a_0 & a_1 \\ & & & & & a_0 \end{bmatrix}.$$

$\mathcal{T}_d$ denotes the set of all $d \times d$ upper triangular Toeplitz matrices.

Lemma 3. $\mathcal{T}_d$ is a commutative $\mathbb{C}$-algebra. More precisely, if $\boldsymbol{e}_j$ denotes the jth unit vector in $\mathbb{C}^d$, then the matrices $\boldsymbol{T}_d(\boldsymbol{e}_0), \ldots, \boldsymbol{T}_d(\boldsymbol{e}_{d-1})$ are a basis of $\mathcal{T}_d$ and for $0 \le i, j < d$,

$$\boldsymbol{T}_d(\boldsymbol{e}_i) \cdot \boldsymbol{T}_d(\boldsymbol{e}_j) = \begin{cases} \boldsymbol{T}_d(\boldsymbol{e}_{i+j}) & \text{if } i + j < d, \\ \boldsymbol{0} & \text{otherwise.} \end{cases}$$

In general, if $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} \in \mathbb{C}^d$ and $\boldsymbol{T}_d(\boldsymbol{a}) \cdot \boldsymbol{T}_d(\boldsymbol{b}) = \boldsymbol{T}_d(\boldsymbol{c})$, then $c_k = \sum_{j=0}^{k} a_j b_{k-j}$, for $k \in [0, d-1]$.

Proof. Follows by a straightforward computation. □

Thus the multiplication of upper triangular Toeplitz matrices in $\mathcal{T}_d$ boils down to truncated polynomial convolution. Via FFT, this multiplication can be performed with $O(d \log d)$ arithmetic operations. Below, we need the following generalization of $\mathcal{T}_d$.

Definition 4. For positive integers d, e let $\mathcal{T}_{d,e}$ denote the set of all $d \times e$ matrices $\boldsymbol{T} = (t_{i,j})_{0 \le i < d, 0 \le j < e}$, such that $t_{i,j} = t_{i+1,j+1}$, whenever both sides are defined. Moreover, $t_{i,j}$ is zero if $j - i < \max(0, e-d)$.

Note that $\mathcal{T}_{d,d} = \mathcal{T}_d$.

Example 5. Let $a_0, \ldots, a_4 \in \mathbb{C}$. Then typical elements in $\mathcal{T}_{3,5}$ resp. $\mathcal{T}_{5,3}$ look as follows:

$$\begin{array}{|cc|ccc|} \hline 0 & 0 & a_2 & a_3 & a_4 \\ 0 & 0 & 0 & a_2 & a_3 \\ 0 & 0 & 0 & 0 & a_2 \\ \hline \end{array} \quad \text{resp.} \quad \begin{array}{|ccc|} \hline a_0 & a_1 & a_2 \\ 0 & a_0 & a_1 \\ 0 & 0 & a_0 \\ \hline 0 & 0 & 0 \\ 0 & 0 & 0 \\ \hline \end{array}.$$

An element $\boldsymbol{T} = (t_{i,j}) \in \mathcal{T}_{d,e}$ is completely specified by the entries $a_0, \ldots, a_{e-1}$ in its first row. We indicate this by writing

$$\boldsymbol{T} = \boldsymbol{T}_{d,e}(a_0, \ldots, a_{e-1}).$$

Thus for $i \in [0, d-1]$ and $j \in [0, e-1]$ the following holds:

$$t_{i,j} = \begin{cases} a_{j-i}, & \text{if } j - i \ge \max(0, e-d) \\ 0 & \text{otherwise.} \end{cases} \qquad (1)$$

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The following observation is obvious but crucial.

Lemma 6. Let d, e, f be positive integers. Then $\mathcal{T}_{d,e} \cdot \mathcal{T}_{e,f} \subseteq \mathcal{T}_{d,f}$. More precisely, if $a_i, b_j \in \mathbb{C}$ then

$$\boldsymbol{T}_{d,e}(a_0, \ldots, a_{e-1}) \cdot \boldsymbol{T}_{e,f}(b_0, \ldots, b_{f-1})$$

equals $\boldsymbol{T}_{d,f}(c_0, \ldots, c_{f-1})$, where $c_k = \sum_{j=0}^{k} a_j \cdot b_{k-j}$. In particular, if $d < e$ or $e < f$, then $c_k = 0$, for all $k \in [0, \max(e-d-1, f-e-1)]$.

C. Review of Jordan Canonical Form

Special upper triangular Toeplitz matrices are involved in the Jordan canonical form of a matrix. The *Jordan matrix* corresponding to $\lambda \in \mathbb{C}$ and $d \in \mathbb{N}$ is defined by

$$\boldsymbol{J}_{\lambda,d} := \boldsymbol{T}_d(\lambda, 1, 0, \ldots, 0) \in \mathcal{T}_d.$$

Note that $\boldsymbol{J}_{\lambda,1} = (\lambda) \in \mathbb{C}^{1\times 1}$.

Theorem 7 (Jordan canonical form)**.** For each $\boldsymbol{A} \in \mathbb{C}^{N\times N}$ there exist an invertible $N \times N$ matrix $\boldsymbol{V}$ as well as pairs $(\lambda_1, d_1), \ldots, (\lambda_I, d_I)$ such that

$$\boldsymbol{V}^{-1} \cdot \boldsymbol{A} \cdot \boldsymbol{V} = \bigoplus_{i=1}^{I} \boldsymbol{J}_{\lambda_i,d_i}.$$

Up to reordering, the so-called *Jordan blocks* $\boldsymbol{J}_{\lambda_i,d_i}$ are uniquely determined by $\boldsymbol{A}$. $\bigoplus_{i=1}^{I} \boldsymbol{J}_{\lambda_i,d_i}$ is called the *Jordan decomposition* of $\boldsymbol{A}$. Moreover, each λ_i is an eigenvalue of $\boldsymbol{A}$.

Proof. See, e.g., [2]. □

The next result describes the characteristic polynomial $p_{\boldsymbol{A}}(X)$ and the minimal polynomial $m_{\boldsymbol{A}}(X)$ of $\boldsymbol{A}$ in terms of the Jordan canonical form of $\boldsymbol{A}$.

Theorem 8 (Characteristic versus minimal polynomial)**.** If $\bigoplus_{i=1}^{I} \boldsymbol{J}_{\lambda_i,d_i}$ is the Jordan decomposition of $\boldsymbol{A}$, then

$$p_{\boldsymbol{A}}(X) = \prod_{i=1}^{I}(X - \lambda_i)^{d_i} \quad \text{and} \quad m_{\boldsymbol{A}}(X) = \prod_{\lambda}(X - \lambda)^{r_\lambda},$$

where the product runs through all distinct eigenvalues λ of $\boldsymbol{A}$ and $r_\lambda := \max\{d_i \mid \lambda_i = \lambda\}$.

Proof. See, e.g., [2]. □

In particular, the minimal polynomial divides the characteristic polynomial. Moreover, both polynomials coincide if and only if $\lambda_1, \ldots, \lambda_I$ are pairwise different, i.e., the eigenspace of each eigenvalue λ_i of $\boldsymbol{A}$ is one-dimensional.

D. Review of Hermite Interpolation

Theorem 9 (Hermite Interpolation)**.** Let $x_0, \ldots, x_n \in \mathbb{C}$ be pairwise different, and let $d_0, \ldots, d_n \in \mathbb{N}$. Then for every family of complex numbers, $y_{ik} \in \mathbb{C}$, where $i \in [0, n]$ and $k \in [0, d_i]$, there exists a unique polynomial p of degree at most $N := n + d_0 + \ldots + d_n$ such that for all (i, k)

$$p^{(k)}(x_i) = y_{ik}, \tag{2}$$

where $p^{(k)}$ denotes the k-fold derivative of the polynomial p.

Proof. Although this is a well-known result, see, e.g., [3], we recall its short proof for the reader's convenience.

Uniqueness: Let p and q be polynomials of degree at most N satisfying (2). Then $r := p - q$ is a polynomial of degree at most N, having each x_i as root of multiplicity $d_i + 1$. Thus r has at least $(d_0 + 1) + (d_1 + 1) + \ldots + (d_n + 1) > N$ roots, hence $r = 0$, i.e., $p = q$.

Existence: With the ansatz $p(x) = \sum_{j=0}^{N} a_j x^j$, Equation (2) is equivalent to the following system of linear equations:

$$\boldsymbol{X}\boldsymbol{a} = \boldsymbol{y},$$

where $\boldsymbol{a} = (a_0, \ldots, a_N)^\top \in \mathbb{C}^{N+1}$ and

$$\boldsymbol{y} = (y_{0,0}, \ldots, y_{0,d_0}, \ldots, y_{n,0}, \ldots, y_{n,d_n})^\top \in \mathbb{C}^{N+1}.$$

With $\boldsymbol{y}$ in mind, we index the rows of $\boldsymbol{X} \in \mathbb{C}^{(N+1)\times(N+1)}$ by the elements in $\bigcup_{i=0}^{n} \{i\} \times [0, d_i]$ and the columns by $j \in [0, N]$. Then the entry at position $((i, k), j)$ in $\boldsymbol{X}$ satisfies $\boldsymbol{X}_{(i,k),j} = j(j-1)\cdots(j-k+1) \cdot x_i^{j-k}$.

Consider the special case, where $\boldsymbol{y} = \boldsymbol{0}$. With the same reasoning as in the proof of uniqueness we see that $\boldsymbol{a} = \boldsymbol{0}$ is the unique solution of the homogeneous system $\boldsymbol{X}\boldsymbol{a} = \boldsymbol{0}$. Thus $\boldsymbol{X}$ is invertible. Hence $\boldsymbol{X}\boldsymbol{a} = \boldsymbol{y}$ has the unique solution $\boldsymbol{a} = \boldsymbol{X}^{-1}\boldsymbol{y}$. □

Fast (parallel) algorithms for Hermite interpolation via divided differences are presented in [4].

IV. Linear Shift-Invariant Systems

In this section we describe the centralizer of an arbitrary matrix $\boldsymbol{A} \in \mathbb{C}^{N\times N}$. Conjugate matrices $\boldsymbol{A}$ and $\boldsymbol{B}$, i.e., $\boldsymbol{B} = \boldsymbol{V}\boldsymbol{A}\boldsymbol{V}^{-1}$ for some invertible matrix $\boldsymbol{V}$, have conjugate centralizers:

$$\mathcal{C}(\boldsymbol{B}) = \boldsymbol{V} \cdot \mathcal{C}(\boldsymbol{A}) \cdot \boldsymbol{V}^{-1}.$$

Thus we can assume that $\boldsymbol{A}$ is already in Jordan canonical form: $\boldsymbol{A} = \bigoplus_{i=1}^{I} \boldsymbol{J}_{\lambda_i,d_i}$. Suppose $\boldsymbol{H}\boldsymbol{A} = \boldsymbol{A}\boldsymbol{H}$. Writing $\boldsymbol{H} = (\boldsymbol{H}_{i,j})$ with blocks $\boldsymbol{H}_{i,j} \in \mathbb{C}^{d_i\times d_j}$, for all i, j we obtain the conditions

$$\boldsymbol{J}_{\lambda_i,d_i}\boldsymbol{H}_{i,j} = \boldsymbol{H}_{i,j}\boldsymbol{J}_{\lambda_j,d_j}.$$

To study such conditions more closely, we introduce *intertwining spaces*: for $\lambda, \mu \in \mathbb{C}$ and $d, e \in \mathbb{N}$ let

$$\mathbf{Int}(\boldsymbol{J}_{\lambda,d}, \boldsymbol{J}_{\mu,e}) := \{\boldsymbol{H} \in \mathbb{C}^{d\times e} \mid \boldsymbol{J}_{\lambda,d}\boldsymbol{H} = \boldsymbol{H}\boldsymbol{J}_{\mu,e}\}.$$

Lemma 10. Let $\lambda, \mu \in \mathbb{C}$ and $d, e \in \mathbb{N}$.

(a) $\mathbf{Int}(\boldsymbol{J}_{\lambda,d}, \boldsymbol{J}_{\lambda,e}) = \mathcal{T}_{d,e}$.
(b) $\mathbf{Int}(\boldsymbol{J}_{\lambda,d}, \boldsymbol{J}_{\lambda,d}) = \mathcal{C}(\boldsymbol{J}_{\lambda,d}) = \mathcal{T}_d$.
(c) If $\lambda \neq \mu$, then $\mathbf{Int}(\boldsymbol{J}_{\lambda,d}, \boldsymbol{J}_{\mu,e}) = \{\boldsymbol{0}\}$.

Proof. (a) Let $\boldsymbol{H} = (h_{i,j}) \in \mathbb{C}^{d\times e}$. Then $\boldsymbol{H}\boldsymbol{J}_{\lambda,e} = \boldsymbol{J}_{\lambda,d}\boldsymbol{H}$ if and only if for all $a \in [0, d-1]$ and all $b \in [0, e-1]$ the entries at position (a, b) in $\boldsymbol{H}\boldsymbol{J}_{\lambda,e}$ and $\boldsymbol{J}_{\lambda,d}\boldsymbol{H}$ coincide. Putting $h_{a,-1} := 0$ and $h_{d,b} := 0$, for all a and b, a straightforward computation shows that the above condition is equivalent to

$$h_{a,b-1} + \lambda h_{a,b} = \lambda h_{a,b} + h_{a+1,b}.$$

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In particular, $\boldsymbol{H}$ is a Toeplitz matrix. Furthermore, for $a = d-1$ we obtain $h_{d-1,b-1} = 0$. Hence $h_{d-1,j} = 0 = h_{0,j-(d-1)}$, for all $j \in [0, e-2]$. Thus $\boldsymbol{H} \in \mathcal{T}_{d,e}$.

(b) is a special case of (a).

(c) Now let $\lambda \neq \mu$. Keeping the above notation and conventions we get

$$(\lambda - \mu) \cdot h_{a,b} = h_{a,b-1} - h_{a+1,b}, \tag{3}$$

for all $a \in [0, d-1]$ and $b \in [0, e-1]$. In particular, for $a = d-1$, this yields $(\lambda - \mu) \cdot h_{d-1,b} = h_{d-1,b-1}$. Thus $h_{d-1,0} = 0$ and hence $h_{d-1,b} = 0$ for all $b \in [0, e-1]$. The other extreme case, $b = 0$, yields $(\lambda - \mu) \cdot h_{a,0} = -h_{a+1,0}$. Thus $h_{d-2,0} = 0$ and hence $h_{a,0} = 0$ for all $a \in [0, d-1]$. Thus the leftmost column and the last row in the matrix $(\lambda - \mu)\boldsymbol{H}$ are zero. This vanishing of the border in combination with Equation (3) triggers a domino effect, proving $\boldsymbol{H} = \boldsymbol{0}$. □

Now we are prepared to describe the centralizer for matrices in Jordan canonical form.

Theorem 11. Let $\boldsymbol{A} = \bigoplus_\lambda \big(\boldsymbol{J}_{\lambda,d_1^\lambda} \oplus \ldots \oplus \boldsymbol{J}_{\lambda,d_{m_\lambda}^\lambda}\big)$ denote an $N \times N$ matrix in Jordan canonical form. Here λ runs through all distinct eigenvalues of $\boldsymbol{A}$. Then the centralizer of $\boldsymbol{A}$ satisfies $\boldsymbol{\mathcal{C}}(\boldsymbol{A}) = \bigoplus_\lambda \boldsymbol{\mathcal{C}}(\boldsymbol{A})_\lambda$, where

$$\boldsymbol{\mathcal{C}}(\boldsymbol{A})_\lambda = \boxed{\begin{array}{ccc} \mathcal{T}_{d_1^\lambda,d_1^\lambda} & \cdots & \mathcal{T}_{d_1^\lambda,d_{m_\lambda}^\lambda} \\ \vdots & \ddots & \vdots \\ \mathcal{T}_{d_{m_\lambda}^\lambda,d_1^\lambda} & \cdots & \mathcal{T}_{d_{m_\lambda}^\lambda,d_{m_\lambda}^\lambda} \end{array}}\,.$$

Proof. Apply Lemma 10. □

Example 12. Let $\boldsymbol{A} = \boldsymbol{J}_{\lambda,3} \oplus \boldsymbol{J}_{\lambda,3} \oplus \boldsymbol{J}_{\lambda,2} \oplus \boldsymbol{J}_{\mu,4}$ with $\lambda \neq \mu$. Then a typical element of $\boldsymbol{\mathcal{C}}(\boldsymbol{A})_\lambda$ reads as follows

$$\left[\begin{array}{ccc|ccc|cc} \boldsymbol{a_0} & a_1 & a_2 & \boldsymbol{b_0} & b_1 & b_2 & \boldsymbol{c_0} & c_1 \\ 0 & a_0 & a_1 & 0 & b_0 & b_1 & 0 & c_0 \\ 0 & 0 & a_0 & 0 & 0 & b_0 & 0 & 0 \\ \hline \boldsymbol{d_0} & d_1 & d_2 & \boldsymbol{e_0} & e_1 & e_2 & \boldsymbol{f_0} & f_1 \\ 0 & d_0 & d_1 & 0 & e_0 & e_1 & 0 & f_0 \\ 0 & 0 & d_0 & 0 & 0 & e_0 & 0 & 0 \\ \hline 0 & g_1 & g_2 & 0 & h_1 & h_2 & \boldsymbol{i_0} & i_1 \\ 0 & 0 & g_1 & 0 & 0 & h_1 & 0 & i_0 \end{array}\right].$$

Two extreme cases deserve special mention.

Corollary 13. Let $\boldsymbol{A} = \bigoplus_{i=1}^I \boldsymbol{J}_{\lambda_i,d_i}$.

(a) If all λ_i are pairwise different, then $\boldsymbol{\mathcal{C}}(\boldsymbol{A}) = \bigoplus_{i=1}^I \mathcal{T}_{d_i}$.
(b) If $\boldsymbol{J}_{\lambda_i,d_i} = \boldsymbol{J}_{\lambda,d}$ for all i, then $\boldsymbol{\mathcal{C}}(\boldsymbol{A}) = \mathcal{T}_d^{I \times I}$, i.e., $\boldsymbol{\mathcal{C}}(\boldsymbol{A})$ is the algebra of all $I \times I$ matrices with coefficients in $\mathcal{T}_d$.

If all λ_i are pairwise different, then all LSI systems w.r.t. $\boldsymbol{A}$ are given by *polynomial expressions* in $\boldsymbol{A}$:

Theorem 14 ([1])**.** Assume that the characteristic and minimal polynomials of the matrix $\boldsymbol{A}$ are equal. Then the centralizer of $\boldsymbol{A}$ consists of all polynomial expressions in the shift matrix: $\boldsymbol{\mathcal{C}}(\boldsymbol{A}) = \mathbb{C}[\boldsymbol{A}]$.

Proof. Obviously, if $h \in \mathbb{C}[X]$ and $\boldsymbol{H} = h(\boldsymbol{A})$, then $\boldsymbol{HA} = \boldsymbol{AH}$. Thus it remains to prove that $\boldsymbol{HA} = \boldsymbol{AH}$ implies $\boldsymbol{H} = h(\boldsymbol{A})$, for some polynomial h. We proceed in two steps.

Step 1: We prove this claim for the special case, where $\boldsymbol{A}$ equals its Jordan decomposition: $\boldsymbol{A} = \bigoplus_{i=1}^I \boldsymbol{J}_{\lambda_i,d_i}$.

Lemma 15. Evaluating a polynomial $h(X)$ at a Jordan block $\boldsymbol{J}_{\lambda,d}$ yields an upper triangular Toeplitz matrix:

$$h(\boldsymbol{J}_{\lambda,d}) = \boldsymbol{T}_d\big(h^{(0)}(\lambda)/0!, \ldots, h^{(d-1)}(\lambda)/(d-1)!\big).$$

Proof. Using Lemma 3 an easy induction on n proves the following formula for the nth power of a Jordan block:

$$\boldsymbol{J}_{\lambda,d}^n = \boldsymbol{T}_d(a_0, \ldots, a_{d-1}), \quad \text{where} \quad a_j = \tbinom{n}{j}\lambda^{n-j}. \tag{4}$$

If $h(X) = \sum_{n=0}^L h_n X^n$, then

$$\frac{h^{(j)}(\lambda)}{j!} = \sum_{n=0}^L h_n \binom{n}{j} \lambda^{n-j}. \tag{5}$$

Hence

$$\begin{aligned} h(\boldsymbol{J}_{\lambda,d}) &= \sum_{n=0}^m h_n \boldsymbol{J}_{\lambda,d}^n \\ &\overset{(4)}{=} \sum_{n=0}^L h_n \boldsymbol{T}_d\Big(\tbinom{n}{0}\lambda^n, \tbinom{n}{1}\lambda^{n-1}, \ldots\Big) \\ &= \boldsymbol{T}_d\Big(\sum_{n=0}^L h_n \tbinom{n}{0}\lambda^n, \sum_{n=0}^L h_n \tbinom{n}{1}\lambda^{n-1}, \ldots\Big) \\ &\overset{(5)}{=} \boldsymbol{T}_d\Big(h^{(0)}(\lambda)/0!, \ldots, h^{(d-1)}(\lambda)/(d-1)!\Big). \end{aligned}$$

This proves Lemma 15. □

According to Corollary 13 (a), $\boldsymbol{H} = \bigoplus_{i=1}^I \boldsymbol{H}_i$ with $\boldsymbol{H}_i \in \mathcal{T}_{d_i}$. Thus for suitable $y_{i,k}$ we can write $\boldsymbol{H}_i = \boldsymbol{T}_{d_i}(y_{i,0}, \ldots, y_{i,d_i-1})$. We are looking for a polynomial $h(X)$ of minimum degree, such that for all $i \in [1, I]$

$$\begin{aligned} \boldsymbol{H}_i &= \boldsymbol{T}_{d_i}(y_{i,0}, \ldots, y_{i,d_i-1}) \\ &= \boldsymbol{T}_{d_i}\big(h^{(0)}(\lambda_i)/0!, \ldots, h^{(d_i-1)}(\lambda_i)/(d_i-1)!\big). \end{aligned}$$

By Hermite interpolation, such a polynomial exists and is uniquely determined by these conditions. With Lemma 15 we get $\boldsymbol{H}_i = h(\boldsymbol{J}_{\lambda_i,d_i})$, thus $\boldsymbol{H} = h(\boldsymbol{J})$, thereby concluding the proof of Step 1.

Step 2: We prove the claim for the general case, where $\boldsymbol{A} = \boldsymbol{VJV}^{-1}$ and $\boldsymbol{J} = \bigoplus_{i=1}^I \boldsymbol{J}_{\lambda_i,\underset{\sim}{d_i}}$ with pairwise different λ_i. Suppose that $\boldsymbol{HA} = \boldsymbol{AH}$. Put $\widetilde{\boldsymbol{H}} := \boldsymbol{V}^{-1}\boldsymbol{HV}$. Then

$$\begin{aligned} \widetilde{\boldsymbol{H}}\boldsymbol{J} &= \boldsymbol{V}^{-1}\boldsymbol{HVJV}^{-1}\boldsymbol{V} = \boldsymbol{V}^{-1}\boldsymbol{HAV} \\ &= \boldsymbol{V}^{-1}\boldsymbol{AHV} = \boldsymbol{V}^{-1}\boldsymbol{VJV}^{-1}\boldsymbol{HV} = \boldsymbol{J}\widetilde{\boldsymbol{H}}. \end{aligned}$$

According to Step 1, there exists a polynomial $h(X) = \sum_{n=0}^L h_n X^n$ such that $\widetilde{\boldsymbol{H}} = h(\boldsymbol{J})$. Then

$$\begin{aligned} \boldsymbol{H} &= \boldsymbol{V}\widetilde{\boldsymbol{H}}\boldsymbol{V}^{-1} = \boldsymbol{V}\Big(\sum_n h_n \boldsymbol{J}^n\Big)\boldsymbol{V}^{-1} \\ &= \sum_n h_n (\boldsymbol{VJV}^{-1})^n = \sum_n h_n \boldsymbol{A}^n = h(\boldsymbol{A}). \end{aligned}$$

This proves Theorem 14. □

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Our next goal is to characterize the non-degenerate case.

Theorem 16. For $\boldsymbol{A} \in \mathbb{C}^{N\times N}$ the following statements are equivalent:

(a) All eigenspaces of $\boldsymbol{A}$ are one-dimensional.
(b) $p_{\boldsymbol{A}}(X) = m_{\boldsymbol{A}}(X)$.
(c) $\mathcal{C}(\boldsymbol{A}) = \mathbb{C}[\boldsymbol{A}]$.
(d) The centralizer of $\boldsymbol{A}$ is commutative.

Proof. (a) $\to$ (b) follows from Theorem 8. (b) $\to$ (c) follows from Theorem 14. (c) $\to$ (d): trivial.

(d) $\to$ (a): We show that $\neg$ (a) implies $\neg$ (d). W.l.o.g. let $\boldsymbol{A} = \boldsymbol{J}_{\lambda,d} \oplus \boldsymbol{J}_{\lambda,e} \oplus \dots$ Then, by Theorem 11,

$$\mathcal{C}(\boldsymbol{A})_\lambda = \boxed{\begin{matrix} \mathcal{T}_{d,d} & \mathcal{T}_{d,e} & \cdots \\ \mathcal{T}_{e,d} & \mathcal{T}_{e,e} & \cdots \\ \vdots & \ddots & \vdots \end{matrix}}.$$

Now a straightforward computation shows that the upper left 2×2 block $\boxed{\begin{matrix} \mathcal{T}_{d,d} & \mathcal{T}_{d,e} \\ \mathcal{T}_{e,d} & \mathcal{T}_{e,e} \end{matrix}}$ is not commutative. $\square$

The results of this section yield a complete picture of all possible LSI systems. Moreover, Theorem 16 yields a precise separation of the non-degenerate and the degenerate cases.

V. Spectral Decompositions

Let $\boldsymbol{A} \in \mathbb{C}^{N\times N}$. Then both $\mathbb{C}[\boldsymbol{A}]$ and $\mathcal{C}(\boldsymbol{A})$ are subalgebras of $\mathbb{C}^{N\times N}$. By the Krull-Remak-Schmidt Theorem (see, e.g., [2]), the signal space $\mathbb{C}^N$ can be written (in an essentially unique way) as a direct sum of indecomposable $\mathbb{C}[\boldsymbol{A}]$-invariant and $\mathcal{C}(\boldsymbol{A})$-invariant subspaces, respectively. As $\mathbb{C}[\boldsymbol{A}]$ is a subalgebra of $\mathcal{C}(\boldsymbol{A})$, the indecomposable $\mathcal{C}(\boldsymbol{A})$-invariant subspaces further decompose into the direct sum of indecomposable $\mathbb{C}[\boldsymbol{A}]$-invariant subspaces. We are going to describe these spectral decompositions. Let $\boldsymbol{A} = \boldsymbol{V}\boldsymbol{J}\boldsymbol{V}^{-1} \in \mathbb{C}^{N\times N}$ with Jordan canonical form $\boldsymbol{J} = \bigoplus_\lambda \boldsymbol{J}_\lambda$, where, for each eigenvalue λ of $\boldsymbol{A}$, $\boldsymbol{J}_\lambda = \boldsymbol{J}_{\lambda,d_1^\lambda} \oplus \dots \oplus \boldsymbol{J}_{\lambda,d_{m_\lambda}^\lambda}$. In view of the structure of $\boldsymbol{J}$, the columns of $\boldsymbol{V}$ can be parametrized as $\boldsymbol{V}_{\lambda,i,j}$, where $i \in [1, m_\lambda]$ and $j \in [1, d_i^\lambda]$. W.l.o.g. we can assume that the columns of $\boldsymbol{V}$ are a canonical basis of $\mathbb{C}^N$ (also called a *graph Fourier basis* in [1]), i.e., these columns are composed entirely of Jordan chains, see, e.g., [2]. This means that for all λ, i, j, and all $p \leq 0$

$$\boldsymbol{V}_{\lambda,i,j-1} = (\boldsymbol{A} - \lambda\boldsymbol{I})\boldsymbol{V}_{\lambda,i,j}, \text{ and } \boldsymbol{V}_{\lambda,i,1} \neq \boldsymbol{0} =: \boldsymbol{V}_{\lambda,i,p}. \quad (6)$$

Let $\mathcal{S}_{\lambda,i} := \operatorname{span}\{\boldsymbol{V}_{\lambda,i,j} \mid j \in [1, d_i^\lambda]\}$ and $\mathcal{S}_\lambda := \bigoplus_{i=1}^{m_\lambda} \mathcal{S}_{\lambda,i}$.

Theorem 17 (Spectral decompositions)**.** With the above notation, the following holds.

(a) $\mathbb{C}^N = \bigoplus_\lambda \bigoplus_{i=1}^{m_\lambda} \mathcal{S}_{\lambda,i}$ is the decomposition of $\mathbb{C}^N$ into indecomposable $\mathbb{C}[\boldsymbol{A}]$-invariant subspaces.
(b) $\mathbb{C}^N = \bigoplus_\lambda \mathcal{S}_\lambda$ is the decomposition of $\mathbb{C}^N$ into indecomposable $\mathcal{C}(\boldsymbol{A})$-invariant subspaces.

Proof. W.l.o.g. it suffices to prove our claims for the special case when $\boldsymbol{A}$ is already in Jordan canonical form: $\boldsymbol{A} = \boldsymbol{J}$. Thus $\boldsymbol{V} = \boldsymbol{I}$ is the unit matrix and the unit vectors form a canonical basis of $\mathbb{C}^N$. We will write $\boldsymbol{e}_{\lambda,i,j}$ instead of $\boldsymbol{V}_{\lambda,i,j}$.

(a) Obviously, $\mathbb{C}^N = \bigoplus_\lambda \bigoplus_{i=1}^{m_\lambda} \mathcal{S}_{\lambda,i}$ is a decomposition of $\mathbb{C}^N$ into $\mathbb{C}[\boldsymbol{J}]$-invariant subspaces. Thus it remains to show that every $\mathcal{S}_{\lambda,i}$ is indecomposable. Suppose, $\boldsymbol{X}$ is a non-zero $\mathbb{C}[\boldsymbol{J}]$-invariant subspace of $\mathcal{S}_{\lambda,i}$ and let $\boldsymbol{0} \neq \boldsymbol{x} \in \boldsymbol{X}$. Then $\boldsymbol{x} = \sum_{j=1}^{d_i^\lambda} \xi_j \boldsymbol{e}_{\lambda,i,j}$. Let $\xi_k \neq 0$ but $\xi_j = 0$, for all $j > k$. Then, by Equation (6), $(\boldsymbol{J} - \lambda\boldsymbol{I})^{k-1}\boldsymbol{x} = \xi_k \boldsymbol{e}_{\lambda,i,1} \in \boldsymbol{X}$. Thus $\boldsymbol{e}_{\lambda,i,1}$ is contained in every non-zero $\mathbb{C}[\boldsymbol{J}]$-invariant subspace of $\mathcal{S}_{\lambda,i}$. This proves the indecomposability of $\mathcal{S}_{\lambda,i}$ with respect to the algebra $\mathbb{C}[\boldsymbol{A}]$.

(b) Obviously, $\mathbb{C}^N = \bigoplus_\lambda \mathcal{S}_\lambda$ is a decomposition of $\mathbb{C}^N$ into $\mathcal{C}(\boldsymbol{J})$-invariant subspaces. Thus it remains to show that every $\mathcal{S}_\lambda$ is indecomposable. Suppose, $\boldsymbol{X}$ is a non-zero $\mathcal{C}(\boldsymbol{J})$-invariant subspace of $\mathcal{S}_\lambda$ and let $\boldsymbol{0} \neq \boldsymbol{x} \in \boldsymbol{X}$. Write $\boldsymbol{x} = \sum_i \boldsymbol{x}_i$, with $\boldsymbol{x}_i \in \mathcal{S}_{\lambda,i}$. Suppose, $\boldsymbol{x}_i \neq \boldsymbol{0}$. Project $\boldsymbol{x}$ via a suitable element in $\mathcal{C}(\boldsymbol{J})$ onto $\boldsymbol{x}_i$. Then $\boldsymbol{x}_i \in \boldsymbol{X}$, and as $\mathbb{C}[\boldsymbol{J}] \subseteq \mathcal{C}(\boldsymbol{J})$ we know by (a), that also $\boldsymbol{e}_{\lambda,i,1} \in \boldsymbol{X}$. Combining Theorem 11 with the constellation of the bold entries in Example 12, it follows that $\boldsymbol{e}_{\lambda,k,1} \in \boldsymbol{X}$, for all k corresponding to Jordan blocks $\boldsymbol{J}_{\lambda,d_k^\lambda}$ of maximal size, i.e., $d_k^\lambda = \max\{d_i^\lambda \mid i \in [1, m_\lambda]\}$. As those $\boldsymbol{e}_{\lambda,k,1}$ are independent of $\boldsymbol{x}$, we see that those vectors are contained in every non-zero $\mathcal{C}(\boldsymbol{J})$-invariant subspace of $\mathcal{S}_\lambda$. This proves the indecomposability of $\mathcal{S}_\lambda$ w.r.t. the algebra $\mathcal{C}(\boldsymbol{J})$. $\square$

Note that in the non-degenerate case (all $m_\lambda = 1$) both spectral decompositions coincide.

VI. Conclusions

We have characterized all linear shift invariant systems corresponding to a given shift matrix. Our result extends the result of Sandryhaila and Moura, who described the non-degenerate case. It turned out that in general the centralizer of a matrix $\boldsymbol{A}$ in Jordan canonical form has a nested block structure: the macro blocks correspond to the distinct eigenvalues λ of $\boldsymbol{A}$. The micro blocks within the λ-block correspond to pairs of Jordan blocks of $\boldsymbol{A}$ corresponding to the same λ. Each micro block is a space of rectangular upper Toeplitz matrices. The examples in Section II indicate that in the degenerate case the algebra $\mathcal{C}(\boldsymbol{A})$ of all LSI systems can be of much higher dimension than the subalgebra of all polynomial expressions in $\boldsymbol{A}$. This opens new and versatile possibilities for the design of linear shift invariant systems in Digital Signal Processing.

The spectral decomposition in Theorem 17 (b) unifies the non-degenerate and the degenerate case. In both cases there is only one spectral component of frequency λ, for each eigenvalue λ of $\boldsymbol{A}$. The spectral decomposition in Theorem 17 (a) seems to be problematic for the degenerate case, for there are several spectral components corresponding to the same frequency.

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