GREEDY PURSUITS ASSISTED BASIS PURSUIT FOR COMPRESSIVE SENSING

Sathiya Narayanan, Sujit Kumar Sahoo and Anamitra Makur

School of Electrical and Electronic Engineering, Nanyang Technological University, Singapore
Email: sathiya3@e.ntu.edu.sg

ABSTRACT

Fusion based Compressive Sensing (CS) reconstruction algorithms combine multiple CS reconstruction algorithms, which worked with different principles, to obtain a better signal estimate. Examples include Fusion of Algorithms for Compressed Sensing (FACS) and Committee Machine Approach for Compressed Sensing (CoMACS). However, these algorithms involve solving a least squares problem which may be ill-conditioned. Modified CS algorithms such as Modified Basis Pursuit (Mod-BP) ensured a sparse signal can efficiently be reconstructed when a part of its support is known. Since Mod-BP makes use of available signal knowledge to improve upon BP, we propose to employ multiple Greedy Pursuits (GPs) to derive a partial support for Mod-BP. As Mod-BP makes use of signal knowledge derived using GPs, we term our proposed algorithm as Greedy Pursuits Assisted Basis Pursuit (GPABP). Experimental results show that our proposed algorithm performs better than the state-of-the-art algorithms - FACS and its variants.

Index Terms— Fusion of Algorithms, Basis Pursuit, Greedy Pursuit, Modified Basis Pursuit.

1. INTRODUCTION

Compressive Sensing (CS) ensures the recovery of a sparse signal $x \in \mathbb{R}^n$ using a small number of linear observations of the form $y = \Phi x + w \in \mathbb{R}^m$, where $\Phi \in \mathbb{R}^{m \times n}$ is a known matrix with $m \ll n$ and w is the observation noise of variance σ^2. CS reconstruction algorithms can be broadly classified as convex relaxation methods and Greedy Pursuits (GPs). For a K-sparse signal x, exact recovery is possible using a convex relaxation method such as Basis Pursuit (BP) provided the number of measurements $m = O(K \log(n/K))$ [1] [2]. The problem is formulated as

$$\hat{x} = \arg\min_{\tilde{x}} \|\tilde{x}\|_1 \text{ s.t. } \|\Phi\tilde{x} - y\|_2^2 \leq \epsilon \tag{1}$$

where $\epsilon = \sigma^2$ (if the noise variance is known) or $\epsilon = \delta$ (δ is the error tolerance). GPs are iterative algorithms selecting one or more non-zero locations in each iteration. Popular examples of a GP include Orthogonal Matching Pursuit (OMP) [3], Subspace Pursuit (SP) [4] and Compressive Sampling Matching Pursuit [5]. Robustness of CS algorithms can be studied using the Restricted Isometry Property (RIP) of the sensing matrix. For all K-sparse x, a sensing matrix Φ is said to follow RIP if there exist a restricted isometry constant δ_K satisfying

$$(1 - \delta_K)\|x\|_2^2 \leq \|\Phi x\|_2^2 \leq (1 + \delta_K)\|x\|_2^2. \tag{2}$$

Motivation and Relation to Prior Work: In CS, different reconstruction algorithms can be combined to improve sparse recovery. A fusion framework, Fusion of Algorithms for Compressed Sensing (FACS) proposed in [6], employed two or more CS reconstruction algorithms and fused the individual estimates to obtain a better sparse signal estimate. FACS performed better than the best participating algorithm. Assume BP, OMP and SP are involved in the fusion framework, and BP is the best performing algorithm. Then, the performance of FACS will be better than that of the BP. Later, two variants of FACS were proposed namely Modified FACS (M-FACS) and Committee Machine Approach for Compressed Sensing (CoMACS) [7, 8]. However, these algorithms involve solving a least squares problem which may be ill-conditioned. Though M-FACS attempted to ensure the least squares problem is well-conditioned, M-FACS could outperform FACS in the low measurement regime only. Our goal is to obtain a fusion framework that could perform better than that of FACS (and its variants).

It is well known that a sparse signal can be reconstructed from a limited number of its linear projections when a part of its support is known. Modified CS algorithms such as Modified Basis Pursuit (Mod-BP) [9], Least-Squares-CS-Residual (LS-CS) [10], etc could serve this purpose. The "known" part of the support can be obtained from prior knowledge of the signal. Since Mod-BP or LS-CS makes use of available signal knowledge to improve upon BP, we propose Greedy Pursuits Assisted Basis Pursuit (GPABP) that employs multiple GPs to derive a partial support for Mod-BP or LS-CS. Experimental results show that our proposed algorithm performs better than the state-of-the-art reconstruction algorithms (FACS and its variants). The rest of this paper is organized as follows. In section 2, we discuss the existing modified BP algorithms, and FACS framework and

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its variants. We propose and analyze our GPABP framework in section 3. In section 4, we present the simulation results of GPABP comparing its performance to that of the state-of-the art algorithms. Section 5 concludes the paper.

2. LITERATURE REVIEW

In this section, we first describe the Mod-BP algorithm, and then discuss FACS and its variants.

Modified Basis Pursuit Algorithms for Compressive Sensing: Vaswani et al modified CS for problems where signal support is partially known. The support of x is denoted as T and it can be split as $T = \tilde{T} \cup \Delta \backslash \Delta_e$ where $\tilde{T}$ is the partially known support, Δ_e is the error in $\tilde{T}$ and Δ is the unknown part to be estimated. The known part is either available from prior knowledge (as in static problems) or an estimate of support obtained from the known signal in the ensemble (as in time sequence problems). $|\Delta|$ and $|\Delta_e|$ are assumed to be much smaller than $|\tilde{T}|$.

Mod-BP [9] aimed at estimating the signal that is sparsest outside $\tilde{T}$ and also satisfies the data constraint. The convex optimization problem is formulated as

$$\hat{x} = \arg\min_{\tilde{x}} \|\tilde{x}_{T^c}\|_1 \text{ s.t. } \|\Phi\tilde{x} - y\|_2^2 \leq \epsilon \tag{3}$$

where $\tilde{T}^c := [1 : n]\backslash\tilde{T}$ is the complement of $\tilde{T}$. [9] applied Mod-BP for time sequence reconstruction problem and showed that the RIP requirements for Mod-BP are much weaker compared to that of the BP. The key to this algorithm is the reliability of the partial support $\tilde{T}$. In [11], it is shown that a reliable partial support can improve the recovery performance further.

Another BP based modified CS algorithm, LS-CS, used signal knowledge in a different manner compared to that of Mod-BP. LS-CS replaced CS on the measurement vector by CS on the least squares measurement residual vector computed using $\tilde{T}$. Reason why LS-CS significantly outperformed regular CS was the signal minus its LS estimate contains much fewer non-zero elements than the signal itself. Though LS-CS is faster compared to Mod-BP, reconstruction accuracy of latter is better than that of the former. In [12], a weighted ℓ_1 minimization was proposed for reconstructing signals whose partial support information is available. It was shown that, if at least 50% of the (partial) support information is accurate, then weighted ℓ_1 minimization is stable and robust under weaker sufficient conditions than the analogous conditions for standard ℓ_1 minimization.

Fusion of Algorithms for Compressive Sensing: Fusion based algorithms employed multiple CS reconstruction algorithms, which worked with different principles, and fused their resultant estimates to obtain a better signal estimate. As T denotes the actual support, let $\hat{T}_i$ be the support estimated by i^{th} algorithm. Let us mention the union of the estimated support sets as *joint support* (denoted by Γ) and the intersection of the estimated support sets as *common support* (denoted by Λ). As $|\hat{T}_i| = K$, we have $K \leq |\Gamma| \leq LK$ and $0 \leq |\Lambda| \leq K$ where L is the number of CS reconstruction algorithms involved in the fusion framework. FACS showed that the probability of estimating more correct atoms from the union set is higher than that individually estimated by ingredient algorithms [6]. The main steps of the FACS involving three CS algorithms (a BP and two GPs) is summarized in Algorithm 1.

Algorithm 1 FACS(GP$_1$, GP$_2$ and BP)

Input: Φ, y, K and $\epsilon = \sigma^2$ or δ

Procedure:
Step 1: $\hat{T}_1 =$ GP$_1$(Φ, y, K)
Step 2: $\hat{T}_2 =$ GP$_2$(Φ, y, K)
Step 3: $\hat{T}_3 =$ BP(Φ, y, ϵ)
Step 4: *Joint support* $\Gamma = \hat{T}_1 \cup \hat{T}_2 \cup \hat{T}_3$
Step 5: $\breve{x}$, such that $\breve{x}_\Gamma = \Phi_\Gamma^\dagger y$ and $\breve{x}_{\Gamma^c} = 0$.
Step 6: $\hat{T}$ = indices corresponding to the K largest magnitude entries in $\breve{x}$.
Step 7: $\hat{x}$, such that $\hat{x}_{\hat{T}} = \breve{x}_{\hat{T}}$ and $\hat{x}_{\hat{T}^c} = 0$.

Output: $\hat{x}$ and $\hat{T}$.

FACS involves solving a least squares problem (step 5 in algorithm 1) which may be ill-conditioned. A modification to FACS, M-FACS proposed in [7], included a step prior to least squares to ensure the least squares problem is well-conditioned. They kept removing the index of the smallest coefficient of $\Phi_\Gamma^\dagger y$ from the set Γ until the 2-norm condition number of Φ_Γ falls below a predefined condition number threshold. M-FACS gave better reconstruction performance that of FACS in the low dimension measurement regime. The CoMACS algorithm used both Γ and Λ to estimate $\hat{T}$ [8]. They had $\Lambda \subset \hat{T}$ where $\hat{T}$ denotes the support estimated support set. Then they estimated the remaining $K - |\Lambda|$ indices in a similar manner as in step 6 of FACS algorithm. The main assumption for all these algorithms is that $|\Gamma| \leq m$.

3. GREEDY PURSUITS ASSISTED BASIS PURSUIT (GPABP)

Among fusion based algorithms, FACS suffers ill-conditioning of LS problem. Though M-FACS tried to prune Γ, the modification works only in the low measurement regime. In CoMACS, since $\Lambda \subset \hat{T}$, a wrongly chosen atom in Λ will remain in the estimated support $\hat{T}$. For every wrong index in Λ, there will be a correct index undetected. On the other hand, recovery conditions and error bounds of Mod-BP revealed the fact that a good prior knowledge will give a much better recovery. Therefore, we propose to employ multiple GPs and derive Λ to apply Mod-BP.

Since a modified BP uses signal knowledge derived using GPs, we term our proposed algorithm as Greedy Pursuits Assisted Basis Pursuit. Therefore, we obtain a fusion framework (involving modified BP algorithm) that could perform better than the existing fusion based algorithms. The RIP requirement for GPABP is expected to be favourable compared to that of FACS and CoMACS. Note that the *joint support* Γ is not required for GPABP. The key steps of our proposed algorithm (involving Mod-BP and two GPs) are summarized in Algorithm 2.

Algorithm 2 Proposed GPABP(GP_1, GP_2 and Mod-BP)

Input: Φ, y, K and $\epsilon = \sigma^2$ or δ

Procedure:
Step 1: $\hat{T}_1 = \text{GP}_1(\Phi, \text{y}, \text{K})$
Step 2: $\hat{T}_2 = \text{GP}_2(\Phi, \text{y}, \text{K})$
Step 3: *Common support* $\Lambda = \hat{T}_1 \cap \hat{T}_2$
Step 4: Obtain $\breve{x}$ using Mod-BP:

$$\breve{x} = \arg\min_{\tilde{x}} \|\tilde{x}_{T^c}\|_1 \text{ s.t. } \|\Phi\tilde{x} - y\|_2^2 \leq \epsilon$$

Step 5: $\hat{T}$ = indices corresponding to the K largest magnitude entries in $\breve{x}$.
Step 6: $\hat{x}$, such that $\hat{x}_{\hat{T}}$=$\breve{x}_{\hat{T}}$ and $\hat{x}_{\hat{T}^c}$ = 0.

Output: $\hat{x}$ and $\hat{T}$.

The prior knowledge of the partial signal supports may not be available in many scenarios, and we cannot apply Mod-BP directly. Also the general CS theory do not assume any such prior knowledge. For reconstructing such signals using Mod-BP, one might use $\Phi^{\dagger}y$ to obtain a support estimate. But the reliability of such estimate is crucial to satisfy the conditions in Theorem 1. The correct locations in the estimated support will improve recovery performance and at the same time, the wrong ones will degrade it as well. Eventually, GPABP shows how to use a prior knowledge based algorithm in scenarios where prior knowledge is not available. The *common support* Λ of GPABP is highly reliable as each of its entry is chosen by every ingredient GP.

Theorem 1 (Exact Reconstruction): If Λ is the derived *common support* such that $T = \Lambda \cup \Delta \backslash \Delta_e$, $\breve{x}$ is the unique minimizer of Mod-BP (in Algorithm 2) if

$$2\delta_{2|\Delta|} + \delta_{3|\Delta|} + \delta_{|\Lambda|} + \delta^2_{|\Lambda|+|\Delta|} + 2\delta^2_{|\Lambda|+2|\Delta|} < 1. \quad (4)$$

Proof of the above theorem can be drawn in similar lines as that of the proof of Theorem 1 in [9]. Only difference is that the partial knowledge $\tilde{T}$ is replaced by a derived knowledge Λ. Assuming $|\Delta| \leq |\Lambda|$, the worst case RIP requirement of GPABP may be obtained as follows. The condition in (4) is satisfied when $\delta_{|\Lambda|+2|\Delta|} < \frac{1}{5}$, which simplifies to $\delta_{3|\Lambda|} < \frac{1}{5}$. Since $|\Lambda| \leq K$, the RIP requirement becomes $\delta_{3K} < \frac{1}{5}$.

Theorem 2 (GPABP error bound): If $\|w\| \leq \epsilon$ and $\delta_{max(3|\Delta|, K+|\Delta|+|\Delta_e|)} < \sqrt{2} - 1$, then

$$\|x - \hat{x}\| \leq B(max(3|\Delta|, K + |\Delta| + |\Delta_e|))\epsilon, \quad \text{where}$$

$$B(S) \triangleq \frac{4\sqrt{1 + \delta_S}}{1 - (\sqrt{2} - 1)\delta_S}. \quad (5)$$

As stated in [8], Λ has at least the 'higher accuracy' as of the intersection of the subsets of both $\hat{T}_1$ and $\hat{T}_2$ with same cardinality. Therefore, $|\Delta_e|$ will be much smaller compared to K and the second condition in Theorem 2 can be simplified to $\delta_{max(3|\Delta|, K+|\Delta|)} < \sqrt{2} - 1$.

Our proposed algorithm can be applied not only for the sparse signals whose prior knowledge is unavailable but also those signals whose partially known support is unreliable. For example, correlated sparse signals (joint sparsity model-1 in [13]) with a smaller common support compared to its innovation support. In such cases, fusion technique will give a better partial support compared to that obtained from prior knowledge. Though we discuss only Mod-BP based GPABP, our proposed algorithm can handle any modified CS reconstruction algorithm (e.g. LS-CS) that can make use of signal knowledge to improve sparse recovery.

Complexity Analysis: If the ingredient greedy pursuit algorithm has a complexity say $C(m, n, K)$ then the fusion framework has a complexity $\approx L \times C(m, n, K)$. However, the computational complexity of GP will be much lower compared to that of a convex relaxation algorithm (BP, Mod-BP or LS-CS) [14]. For example, OMP and SP algorithms require $O(mnK)$ and $O(mn \log K)$ computations respectively, and on the other hand, convex relation algorithm requires $O(m^2 n^{\frac{3}{2}})$ computations. Therefore, the computational complexity of GPABP will be in the same order as that of the modified BP algorithm used. Low computational complexity of GP allows us to include more number of GPs to derive Λ. However, the accuracy of Λ do not vary much when L is increased beyond 2.

4. SIMULATION RESULTS

In this section, we present the experimental results for synthetic signals and real compressible signals such as ECG signals. We performed signal recovery using four methods: FACS, M-FACS, CoMACS and our GPABP. Except for GPABP, in all other methods, fusion involved three algorithms OMP, SP and BP. In the case of GPABP, fusion step involved two algorithms: OMP and SP for GP_1 and GP_2. For M-FACS, the condition number threshold is fixed as 10 (as in [7]). In the case of noisy measurements, if σ^2 is known, ϵ can be fixed as σ^2. However, we do blind (in terms of knowledge about w) reconstruction by fixing $\epsilon = \delta = 10^{-3}$. The *Sparselab* solver (available at http://sparselab.stanford.edu) is used for the implementation of BP in FACS, M-FACS and CoMACS.

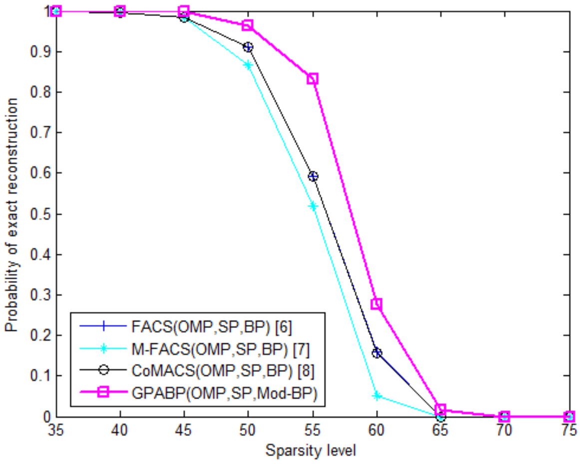

Fig. 1. Noiseless measurement case: Exact reconstruction versus sparsity level

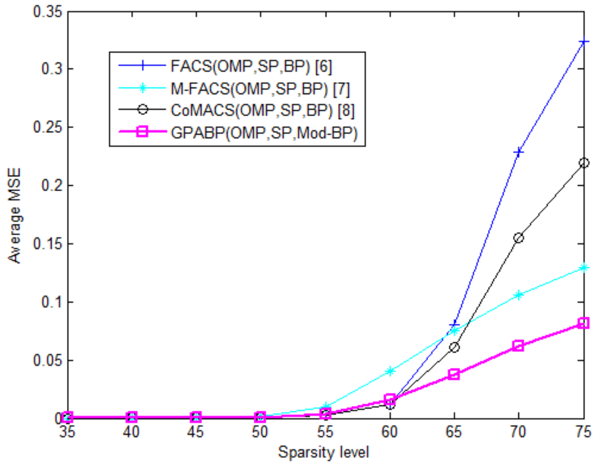

Fig. 2. Noiseless measurement case: Average MSE versus sparsity level

The *cvx* solver (available at http://cvxr.com/cvx/) is used for the implementation of Mod-BP in GPABP.

Synthetic sparse signal: For our experiments, we generated Gaussian sparse signal of length n=256. First, we present the probability of exact reconstruction as a function of the sparsity level K. For each value of K, 250 independent trials are performed to obtain the average results. In each trial, an $m \times n$ Gaussian random measurement matrix is generated. Number of measurements m is fixed to be 128 and the sparsity levels were chosen from K=35 to K=75 in steps of 5. If the maximum magnitude difference between the original signal and the reconstructed signal is smaller than 10^{-3}, the reconstruction is considered to be perfect. Fig. 1 shows that our proposed GPABP has the best probability of exact reconstruction among all four methods. Surprisingly, the performances of FACS and CoMACS are indistinguishable (as can be seen in fig. 1). Also, the performance of M-FACS is worse compared to that of FACS. For the same parameters, average Mean Square Error (MSE) plot is shown in fig. 2. Average MSE stands for MSE averaged over 250 trials. The MSE is computed as follows,

$$\text{MSE} = \frac{\|x - \hat{x}\|_2^2}{n}. \quad (6)$$

It can be noticed that GPABP has the least MSE among all the methods and is stable even for high K values. For $K > 60$, CoMACS performs slightly better than FACS. For $K > 65$, M-FACS gives lesser error compared to that of FACS. This indicates that the modification to FACS is effective only in the high K regime. We repeated the same experiment for noisy measurements. Measurement vector y is corrupted by a noise such that its SMNR is 15 dB. It can be seen from fig. 3 that GPABP is least affected by measurement noise particularly when $K \geq 55$. Also, M-FACS has the best performance among the rest of the methods. Though all four methods used the same number (and type) of ingredient algorithms, proposed GPABP gives the best performance.

Real world signal - Compressible ECG signal: Next experiment illustrates the performance for real compressible signals. The leads (ECG signals) are extracted from records 100, 101, 102 and 103 from the MIT-BIH Arrhythmia database [15]. They are processed in chunks of 256 samples (with amplitudes ranging from 0 to 255). These signals will serve as the ground truth for MSE computation. Discrete cosine transform is used to obtain sparse representations of the signals. The sparsity level K is fixed as $\lfloor \frac{m}{\log n} \rfloor$. For each chunk, the measurement vector y is corrupted by a noise such that its SMNR is 15 dB. Fig. 4 shows the average MSE (MSE per chunk) as a function of measurement ratio $\frac{m}{n}$ (varied between 0.2 and 0.4). It is evident that GPABP gives better recovery performance (compared to FACS and its variants) in the case of real world compressible signals too. Again, the performances of FACS and CoMACS are indistinguishable.

5. CONCLUSION

In this paper, we proposed an efficient fusion approach involving GPs and Mod-BP. Multiple GPs were employed to derive a *common support* Λ, which is used for Mod-BP. Experimental results show that our proposed GPABP scheme gives better reconstruction accuracy compared to that of the existing fusion based algorithms. Our proposed GPABP is a prior based recovery technique for scenarios where prior knowledge is not available. GPABP can also be extended to recover signals whose prior knowledge of the partial support is not reliable.

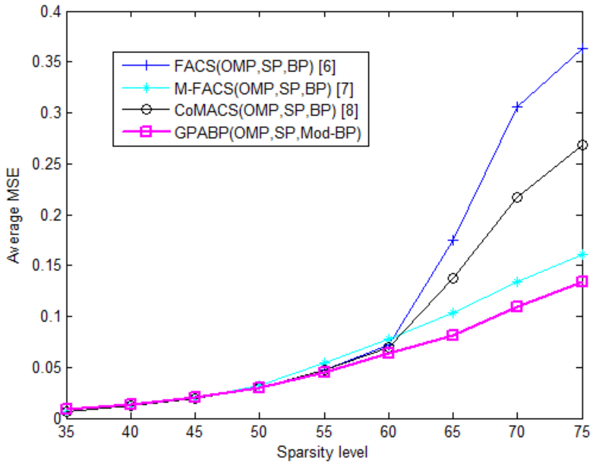

Fig. 3. Noisy measurement case (SMNR = 15 dB): Average MSE versus sparsity level

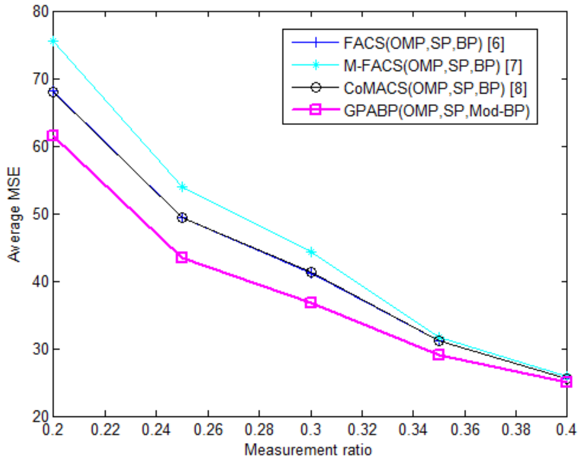

Fig. 4. Compressible ECG signals - Noisy measurements (SMNR = 15 dB): Average MSE versus measurement ratio

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