STATIONARY GRAPH SIGNALS USING AN ISOMETRIC GRAPH TRANSLATION

Benjamin Girault

École Normale Supérieure de Lyon, Université de Lyon
IXXI – UMR 5668 (CNRS – ENS Lyon – UCB Lyon 1 – Inria)

ABSTRACT

We extend the concept of stationary temporal signals to stationary graph signals. Doing so, we introduce the concept of strict sense stationary and wide sense stationary graph signals as a statistical invariance through an isometric graph translation operator. Using these definitions, we propose a spectral characterisation of WSS graph signals allowing to study stationarity using only the spectral components of a graph signal.

Finally, we apply this characterisation on a synthetic graph in order to study a few important stochastic graph signals. Also, using geographic data, we study weather readings on a graph of weather stations and show evidence of stationarity in the temperature readings.

***Index Terms*—** Signal processing on graphs, stationary signals

1. INTRODUCTION

The evolution of computing capabilities or techniques have produced an avalanche of data, and more importantly structured data. Social networks connecting people, technological networks connecting computers or smart devices, or geographic data are a few examples among many.

The emerging field of signal processing aims at tackling the problem of studying these data using the powerful approach of signal processing. The question at hand is then how to extend successes obtained by signal processing on temporal data or images to the less ordered, more complex, yet very rich graph structures. Recent successes include, but are not limited to, filtering [1], short-time Fourier transform and wavelets [2], empirical mode decomposition [3], or incertitude principle [4].

This communication aims at being part of this journey to a powerful signal processing on graphs. We propose a definition of stationary graph signals, a class of signal that proved to be crucial in the context of temporal data [5]. A spectral characterisation of those signals completes the definition. Finally we apply those to a few illustrative applications.

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2. BACKGROUND

2.1. Stationary Time Signals

We denote $\mathbf{x}(t)$ a stochastic signal. Throughout this communication, variables written with a bold font are stochastic variables. Among the wide class of stochastic signals, one subclass is of particular interest: the stationary signals. The initial definition is that of *Strict-Sense Stationary* (SSS) signals verifying:

$$\forall t, \forall s, \quad \mathbf{x}(t) \stackrel{\mathrm{d}}{=} \mathbf{x}(s), \tag{1}$$

i.e. the probability density functions of $\mathbf{x}(t)$ and $\mathbf{x}(s)$ are equal.

This property is in practice impossible to verify since it involves the computation of all statistical moments. We usually are satisfied with a less strict definition named *Wide Sense Stationary* (WSS). A signal is WSS if its first two moments $\eta_{\mathbf{x}}(t) = \mathbb{E}[\mathbf{x}(t)]$ and $R_{\mathbf{x}}(t,s) = \mathbb{E}[\mathbf{x}(t)\mathbf{x}^*(s)]$ verify:

$$\eta_{\mathbf{x}}(t) = \eta_{\mathbf{x}} \tag{2}$$

$$R_{\mathbf{x}}(t,s) = \gamma_{\mathbf{x}}(t-s). \tag{3}$$

The first order property (2) is equivalent to the first moment being the Direct Current (DC) component of the signal. In the second order property (3), the function γ is called the *autocorrelation function* of the signal. In the context of WSS time series, R is a Toeplitz matrix with $R_{ij} = \gamma(t_i - t_j)$.

WSS signals admit a harmonic decomposition, *i.e.* they can be written as the Fourier-Stieljes integral:

$$\mathbf{x}(t) = \int_{-\infty}^{\infty} e^{-\imath\omega t} d\mathbf{X}(\omega), \tag{4}$$

with $d\mathbf{X}(\omega)$ a probability measure on the spectral decomposition of the signal named the *spectral increments*. Stationarity is then characterised by a *double orthogonality* [5]: The exponential basis is orthogonal, and the spectral increments are orthogonal:

$$\int_{-\infty}^{\infty} e^{\imath\omega t} e^{-\imath\omega' t} dt = 2\pi\delta(\omega - \omega') \tag{5}$$

$$\mathbb{E}[d\mathbf{X}(\omega) d\mathbf{X}(\omega')^*] = 2\pi\delta(\omega-\omega')\Gamma(\omega) d\omega d\omega', \tag{6}$$

$\Gamma(\omega)$ being the Fourier transform of $\gamma(t)$, and called the *power spectrum density*. Stochastic signal analysis of $\mathbf{x}$ often involves the study of $\gamma(\tau)$ and $\Gamma(\omega)$ making these two functions crucial to the field [6].

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Both definitions of stationarity naturally invoke the time shift operator. Let T_τ be the time shift operator acting on a signal x as $T_\tau\{x\}(t) = x(t-\tau)$. The SSS property reduces to a statistical invariance of a stochastic signal with respect to T_τ, while WSS is the invariance of the first two moments ($\eta_{\mathbf{x}} = \eta_{T_\tau\{\mathbf{x}\}}$ and $R_{\mathbf{x}} = R_{T_\tau\{\mathbf{x}\}}$).

In other words, given a time instant, the statistical properties of the signal and its translated version are statistically the same at that instant. Also, the samples of the signal at all time instants are statistically the same. For temporal signals these two properties are equivalent, but are worth noticing since only one of them will remain true for graph signals.

In addition, given $e_\omega(t) = e^{\imath\omega t}$ the Fourier mode of (angular) frequency ω, the expression of the time shift in the frequency domain is given by:

$$T_\tau\{e_\omega\}(t) = e^{-\imath\tau\omega} e_\omega(t). \tag{7}$$

The Fourier modes are eigenvectors of the time shift operator. Therefore, the time shift behaves as a phase shifting operator in the frequency domain. We refer the reader to [5, 6] for details on stationarity.

2.2. Signal Processing on Graphs

We now introduce the field of signal processing on graphs. Let $\mathcal{G} = (V, E)$ be a *graph*, with V the set of *vertices* and E the set of *edges* ij. We suppose the graphs to be *symmetric*, *i.e.* $ij \in E$ if and only if $ji \in E$. Let A be the *adjacency* matrix of $\mathcal{G}$, with a_{ij} the *weight* of the edge ij. Let D be the diagonal matrix of *degrees* $d_{ii} = \sum_j a_{ij} = d_i$. Since $\mathcal{G}$ is symmetric, A is a symmetric matrix. We define then $L = D - A$ the *Laplacian matrix* of $\mathcal{G}$. L is a semi-definite positive matrix. Let $L\chi_l = \lambda_l \chi_l$ be the eigendecomposition of L, with $0 = \lambda_0 \leq \lambda_1 \leq \dots$ [7, p. 4]. We will suppose the graph to be connected such that $\lambda_1 > 0$ [7, Prop. 1.3.7].

We call a *graph signal* a function assigning values to vertices: $X : V \to \mathbb{R}$ or $\mathbb{C}$. Most often, the *Fourier transform* is defined as the projection on the basis of eigenvectors of L [2]. Let F be the matrix of this Fourier transform. We have $L = F^*\Lambda F$, with $\Lambda = \operatorname{diag}(\lambda_0, \dots, \lambda_{N-1})$. Since L is semi-definite positive, $\{\chi_l\}_l$ is orthogonal and the matrix F is unitary. We have an equivalence of Parseval's identity: $\|X\|_2 = \|FX\|_2$, with $\|X\|_2^2 = \sum_i |X_i|^2$ the l_2-norm. We denote $\widehat{X} = FX$ the Fourier transform of a signal X.

Let H be a linear operator on graph signals. We use the notation H to denote both the operator and its matrix representation. Its action is that of $X \mapsto HX$, with X represented as a column vector. We denote $\widehat{H} = FHF^*$ the expression of H in the Fourier domain.

3. STATIONARITY FOR GRAPH SIGNALS

Stochastic analysis for temporal signals have produced very interesting results such as optimal noise filtering (Wiener filters), predictive systems (ARMA models) or non-stationary signal analysis (wavelet-based statistical inference, *e.g.* denoising), and we naturally wish to leverage the full potential of the statistics of graph signals for their analysis. In particular, Wiener filters are of interest for graph filters [8]. Naturally, we need definitions, concepts and associated characterisations for that purpose. As seen in the previous section, stationnarity is one of those important concepts and we propose to define its counterpart for graph signals.

For temporal signals, stationarity can be interpreted as the statistical invariance by time shifting. There exists in the literature two operators defining equivalents of the time shift for graph signals. The generalized translations in [9] define operators acting as generalized convolution by a delta centred on vertex i whose expression in the Fourier domain is $\widehat{T_i X}(l) = \sqrt{N}\widehat{\delta}_i(l)\widehat{X}(l)$. These operators, when applied to very specific input signals, localise their input around vertex i. The authors of [1] proposed the graph shift as the matrix multiplication $X \mapsto AX$. The graph shift is then an operator diffusing a sample from one vertex to its neighbours according to the edge weights.

Unfortunately, neither operator is isometric with respect to the l_2-norm, *i.e.* $\|T_i X\|_2 \neq \|X\|_2$ and $\|AX\|_2 \neq \|X\|_2$. As opposed to the time shift, these two operators lack the mathematical comfort of isometry to define statistical invariance. In [10], we proposed an alternative shift operator for graph signals which is isometric by design. After recalling its definition, we use it to propose a definition of stationary graph signals.

3.1. Graph Translation

In [10] we proposed the graph translation as a new operator defined by analogy to the time shift for temporal signals. To this end, the graph translation is defined as an isometric operator shifting the phase of each of the spectral components of the input, similarly to (7). More precisely, we define graph frequencies as:

$$\omega_l = \pi\sqrt{\lambda_l/\rho_{\mathcal{G}}}, \tag{8}$$

where $\rho_{\mathcal{G}}$ is an upper bound of λ_l [11, Cor. 3.3]:

$$\rho_{\mathcal{G}} = \max_{i \in V} \sqrt{2d_i(d_i + \bar{d}_i)} \quad \text{with} \quad \bar{d}_i = \frac{\sum_{ij \in E} w_{ij} d_j}{d_i}.$$

The graph frequencies lie in $[0, \pi]$, and a small ω_l corresponds to a Fourier mode χ_l of lower frequency.

The graph translation is then defined as a phase shifting operator on the spectral components, and verifies $T_{\mathcal{G}}\chi_l = e^{-\imath\omega_l}\chi_l$:

Definition 1 (Graph Translation)**.** *The* graph translation *operator is defined using its matrix representation as:*

$$T_{\mathcal{G}} = \exp\left(-\imath\pi\sqrt{L/\rho_{\mathcal{G}}}\right).$$

Details on the construction, justification of the formula, and comparison to the generalized translations and graph shift can be found in [10][1].

3.2. Definition of stationary signals

We now propose to define stationarity for graph signals as the statistical invariance to the graph translation:

Definition 2 (Strict-Sense Stationary)**.** *A stochastic signal* $\mathbf{X}$ *on the graph* $\mathcal{G}$ *is* Strict-Sense Stationary *(SSS) if and only if:*

$$\mathbf{X} \stackrel{d}{=} T_{\mathcal{G}}\mathbf{X}. \tag{9}$$

Def. 2 is tractable thanks to the isometric nature of the graph translation. However, as for temporal signals, SSS is difficult to verify in practice, and we introduce a weaker, more practical definition of stationarity:

Definition 3 (Wide-Sense Stationary)**.** *A stochastic signal* $\mathbf{X}$ *on the graph* $\mathcal{G}$ *is* Wide-Sense Stationary *(WSS) if and only if:*

$$\mathbb{E}[\mathbf{X}] = \mathbb{E}[T_{\mathcal{G}}\mathbf{X}] \tag{10}$$

$$\mathbb{E}[\mathbf{X}\mathbf{X}^*] = \mathbb{E}[(T_{\mathcal{G}}\mathbf{X})(T_{\mathcal{G}}\mathbf{X})^*]. \tag{11}$$

The invariance property is interpreted as follows: Given a vertex $i \in V$, the random variables $\mathbf{X}_i$ and $(T_{\mathcal{G}}\mathbf{X})_i$ are statistically equal (SSS) or have identical moments of the first and second orders (WSS). However, in contrast to temporal signals, given two different vertices i and j, there is in general no equality in law between $\mathbf{X}_i$ and $\mathbf{X}_j$.

The definition of WSS signals is introduced in the vertex domain. As for temporal signals, we propose a dual characterisation of WSS signals in the spectral domain. We begin by the first moment. Let $\eta = \mathbb{E}[X]$ be the mean of the graph signal. Eq. 10 is then equivalent to $\eta = T_{\mathcal{G}}\eta$. In other words, η is an eigenvector of $T_{\mathcal{G}}$ associated to the eigenvalue $1 = e^0 = e^{-\omega_0}$. Since $\lambda_1 > 0$, η is collinear to χ_0: The vector of mean is collinear to the Fourier mode of frequency $\omega_0 = 0$. As for temporal signals, the mean of the signal is the equivalent of its DC component.

Next, we characterise the second moment. Let $R = \mathbb{E}[\mathbf{X}\mathbf{X}^*]$ be the autocorrelation matrix of the signal. Eq. 11 gives then:

$$R = \mathbb{E}[(T_{\mathcal{G}}\mathbf{X})(T_{\mathcal{G}}\mathbf{X})^*] = T_{\mathcal{G}}\mathbb{E}[\mathbf{X}\mathbf{X}^*]T_{\mathcal{G}}^* = T_{\mathcal{G}}RT_{\mathcal{G}}^*.$$

Let $S = \mathbb{E}[(F\mathbf{X})(F\mathbf{X})^*]$ be the autocorrelation matrix of the Fourier transform of the signal. By linearity of the Fourier transform, we obtain $S = FRF^*$ and $R = F^*SF$. The equality above becomes:

$$S = \widehat{T_{\mathcal{G}}}S\widehat{T_{\mathcal{G}}}^*. \tag{12}$$

Assuming that all eigenvalues of L are distinct, then (12) is verified if and only if S is diagonal. In other cases, S diagonal is a sufficient, but not necessary, condition. We can now formally write the spectral characterisation of WSS signals:

Proposition 1 (Second Moment Characterisation)**.** *The second moment of a graph signal is invariant through graph translation if its spectral components are uncorrelated (sufficient condition). This condition is also necessary if all graph frequencies are distinct.*

The eigenvalue uniqueness condition is actually verified on many real world graphs. *i.e.* when weights depend on measurements. This comes from the fact that a small random perturbation of weights will slightly change the eigenvalues making them easily unequal. On the contrary, synthetic graphs showing high regularity, such as cycles, or regular grids with unit weights have many multiple eigenvalues [7, Sec. 1.4.6].

Prop. 1 is similar to the spectral characterisation of WSS time series (*resp.* WSS signals) where the spectral components (*resp.* the spectral increments) are uncorrelated. We have then a doubly orthogonal decomposition.

3.3. Discussion

We remark first that the operator $T_{\mathcal{G}}$ is in general a complex operator, meaning that if $\mathbf{X}$ is real-valued, then $T_{\mathcal{G}}\mathbf{X}$ is usually complex. For WSS signals, this is not an issue since Prop. 1 shows that only the correlation between spectral components matters: A signal can be both WSS and real, and both sides of (11) are real for WSS signals.

We consider now the SSS property in the particular case of real-valued signals. Let $\mathbf{X}$ be such a signal. Def. 2 and the Fourier transform give then:

$$\forall i, \forall x, \mathbb{P}[\mathbf{X}_i = x] = \mathbb{P}[(T_{\mathcal{G}}\mathbf{X})_i = x],$$

and by linear combination of the equalities above:

$$\begin{aligned} & \forall l, \forall \widehat{x}, \mathbb{P}[\widehat{\mathbf{X}}_l = \widehat{x}] = \mathbb{P}[(\widehat{T_{\mathcal{G}}}\widehat{\mathbf{X}})_l = \widehat{x}] \\ \Leftrightarrow \quad & \forall l, \forall \widehat{x}, \mathbb{P}[\widehat{\mathbf{X}}_l = \widehat{x}] = \mathbb{P}[e^{-\imath\omega_l}\widehat{\mathbf{X}}_l = \widehat{x}]. \end{aligned}$$

When $l \neq 0$, $e^{-\imath\omega_l}\widehat{\mathbf{X}}_l$ is complex and $\widehat{\mathbf{X}}_l$ is real, such that the probabilities above are non zero only for $\widehat{x} = 0 = \widehat{\mathbf{X}}$. Therefore:

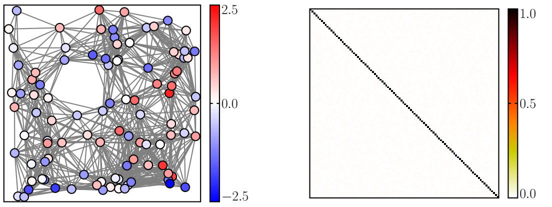
$$\forall l \neq 0, \mathbb{P}[\widehat{\mathbf{X}}_l = 0] = 1.$$

A real SSS signal is reduced to a DC component of random amplitude.

On the other hand, complex graph signals do not suffer from this constraint. This contrasts with temporal SSS signals that can be real without reducing to a DC component.

Another remark concerns the autocorrelation matrix R. We know that for temporal WSS signals, this matrix is Toeplitz. This is not the case for graph signals. This is to be expected since having such a structure would mean having an autocorrelation function γ which is a function of the difference between vertex indices. In general such a difference does not make sense. For example, shifting all indices by one would leave the matrix R unchanged. However, the graph translation does not perform such a shift such that it is unlikely that the signal is invariant to both the graph translation and this vertex index shifting operation.

[1] and animations on http://perso.ens-lyon.fr/benjamin.girault/.



(a) A realisation. (b) Empirical S.

Fig. 1: White noise on a graph with $\sigma^2 = 1$ and zero mean. (a) shows a realisation of this white noise, and (b) the empirical spectral correlation matrix computed using 50k realisations.

The characterisation of the first order moment in the previous section is actually independent of the underlying graph structure. Indeed, χ_0 being always a constant vector, the mean of a WSS signal is constant across vertices. However, several definitions of Fourier transform for graph signals exists, F being one of them. There exists an alternative definition based on the *normalized Laplacian* $\mathcal{L} = D^{-1/2}LD^{-1/2}$ [2]. Both matrices share similar properties. In particular, they are both semi-definite positive. We denote ψ_l the eigenvector associated to the eigenvalue μ_l of $\mathcal{L}$. Prop. 1 is still valid for the Fourier transform $\mathcal{F}$. However, and as opposed to χ_0, ψ_0 is not constant and equals $(\sqrt{d_1}, \dots, \sqrt{d_N})^T$. We denote $\mathcal{T}_\mathcal{G}$ the associated graph translation. We can then use this graph translation to obtain an alternative definition of stationarity. A connected graph verifies $\mu_1 > 0$ such that a WSS signal has a mean vector collinear to ψ_0. Finally, results on the second moment can be adapted to $\mathcal{T}_\mathcal{G}$ directly.

The use of F or $\mathcal{F}$ carries then a different notion of DC component, adapted or not to the local structural properties of the vertices. This accounts for the heterogeneity between vertices weighing differently in the graph structure. Using the Laplacian matrix or the normalized Laplacian matrix carries then different meanings, and using one or the other depends on the application at hand. The rest of this communication keeps F as the Fourier transform.

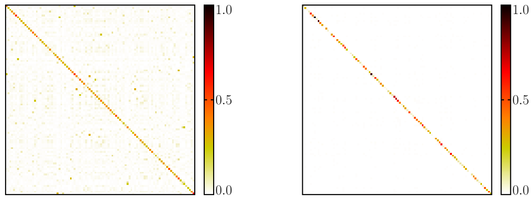
Finally, we remark that we can use a different invariant operator H to define stationarity and still obtain the same spectral characterisation. Indeed, if H is convolutive and isometric, Eq. 12 is still valid with the necessary condition being distinct eigenvalues of H. However, the first moment is not null if and only if 1 is an eigenvalue of H.

4. APPLICATIONS

In this section, we wish to apply the notion of stationary graph signals we introduced to concrete graph signals. First, using synthetic signals, and then to study a dataset and the stationarity of a real world signal.

4.1. Synthetic data

We study now several simple stochastic signals. To that end, we use the graph of Fig. 1(a) based on [9, Example 1], with parameters $\sigma_1^2 = 1/150$ and $\sigma_2 = 0.3$. This graph has 100 vertices randomly sampled in the unit plane connected by edges weighted by a Gaussian kernel of the Euclidean distance in the plane: $a_{ij} = \exp(-\operatorname{d}(p_i, p_j)^2/(2\sigma_1^2))$.



(a) R. (b) S.

Fig. 2: Empirical second moments of a WSS signal with zero mean and non uniform spectral power with 50k realizations.

We propose to define *white noise* on graphs as a stochastic signal having a flat power spectrum, *i.e.* with $S = \sigma^2 I_N$, with mean collinear to χ_0. A realization of white noise with zero mean is shown on Fig. 1(a). Using Prop. 1, a white noise is a WSS signal. Since the Fourier matrix F is unitary, we have also $R = \sigma^2 I_N$. Therefore, the samples on vertices are uncorrelated, and of equal variance.

We consider now a signal $\mathbf{X}$ with samples on the vertices independent and identically distributed (i.i.d.). Then, $R = \sigma^2 I_N = S$. $\mathbf{X}$ is therefore a white noise. The whole class of white noise is actually independent of the underlying graph structure. Indeed, the samples being uncorrelated, the edges play no role in explaining any correlation between them.

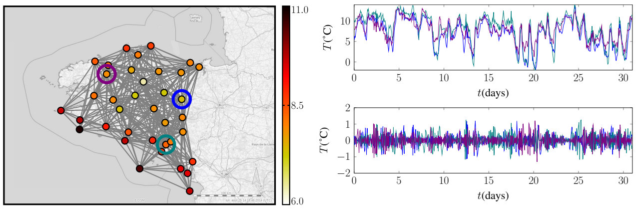
The third signal we consider is a WSS signal with non-constant power spectrum, *i.e.* with S diagonal but of non constant diagonal. Without loss of generality, we suppose the mean of the graph signal to be zero. Fig. 2 shows the empirical correlation matrices R and S obtained using 50k realisations. In general, such a signal has correlated samples, but in contrast to temporal signals, the second moment is in general not constant across vertices, *i.e.* the diagonal of R is not constant. Therefore, the mere fact that the signal is WSS is not a guarantee that the samples have the same variance.

This is an illustration of the property that WSS depends on the underlying graph structure. Also, as soon as the signal shows some correlations between vertices, the edges of the graph contribute to explaining those correlations.

4.2. Real dataset

The French national meteorological service published in open access a dataset of hourly weather observations in Brittany for the month of January 2014[2]. From this dataset, we wish to study the temperature readings on the graph of ground weather stations. We define the graph of ground stations to be an image of the spatial distances between stations. Fig. 3(a) shows it. Each vertex (*i.e.* ground station) is connected to other vertices by edges weighted by a Gaussian kernel ($\sigma_1^2 = 5.10^8$, $\sigma_2 = 10^5$) of the spatial distance between them.

[2]https://www.data.gouv.fr/: hourly data of the Molène zone.



(a) Mean temperature. (b) Temperature variations.

Fig. 3: Spatial and temporal variations of temperatures in Brittany in January 2014. (a) mean temperature of each station. (b) temporal variations of three stations circled in (a) on top, and first IMF of these time series below.

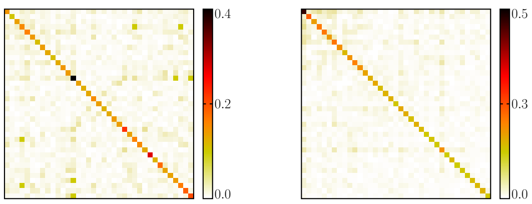
Note that these arbitrary parameters can be slightly changed without altering the results.

We wish to assess the stationarity of the temperature readings. Let $X_i(t)$ be the temperature at vertex i at time t. Fig. 3(a) shows the mean temperature of each station. This mean is obviously not constant, hence not collinear to χ_0. Therefore, we cannot model the readings $X_i(t)$ using a stationary graph signal of which the readings are realisations.

Fig. 3(b) shows the temperature variation over time of three ground stations. We observe that these time series have a significant trend together with seasonal variations. Unfortunately, for empirical estimators to be valid (ergodicity hypothesis), we need to remove these trend and variations to obtain stationary time series. To this end, we consider the first Intrinsic Mode Function (IMF) of the Empirical Mode Decomposition of each of the time series X_i [12]. The first IMF goal is precisely to remove the trend of a time series. We denote the first IMFs Y_i. Since the time series Y_i are centred, the first empirical moment of $\mathbf{Y}$ is zero.

Fig. 4 shows the empirical correlation matrices of $\mathbf{Y}$ and of its spectral decomposition $\widehat{\mathbf{Y}}$. The matrix $R_{\mathbf{Y}}$ shows some significant off-diagonal elements, suggesting spatial correlations between vertices. On the other hand, $S_{\mathbf{Y}}$ has fewer significant off-diagonal elements. The spatial correlations are therefore interpreted by uncorrelated spectral components: The graph structure interprets $\mathbf{Y}$ as a stationary graph signal.

More precisely, the temperature readings are split into two parts summing up: the trend and seasonal variations removed by the EMD on one hand, and the spatial correlations explained by a stationary graph signal on the other hand. More details on the stationarity assessment of this dataset can be found in [13].



(a) $|R_{\mathbf{Y}}|$. (b) $|S_{\mathbf{Y}}|$.

Fig. 4: Correlation matrix (a) and spectral correlation matrix (b) of the IMF1 $\mathbf{Y}$ of Fig. 3(b).

We stress here that this assessment of stationarity is performed with respect to the graph structure we defined. This graph can be defined differently leading to another definition of Fourier transform, and eventually of stationarity.

5. CONCLUSION

This communication introduces two definitions of stationary graph signals. In particular, our definition of WSS graph signals is simple yet rich enough to allow a tractable spectral characterisation. These definitions and their consequences share many properties with temporal signals, however, the more complex structural properties of graphs introduce fundamental differences with time, accounting for the heterogeneity of the structure.

Facing the still infancy of graph signal processing, our contribution brings in an important ingredient that will serve as the starting point of further development towards the constitution of an entire toolbox for analysing real-world measurements modelled as stochastic signals on graphs. A few examples in that toolbox include the study of more classes of stochastic signals, being stationary or not, optimal filtering, or the development of a stationarity test.

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