PARAMETRIC ELLIPTICALLY-SHAPED ZERO-PHASE 2D IIR FILTER DESIGN

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ABSTRACT

This paper proposes an analytical design method for a class of 2D IIR filters, namely elliptically-shaped zero-phase filters. These also include orientation-selective filters with a narrow frequency response along a specified direction in the frequency plane. The approached filters are based on zero-phase low-pass 1D prototypes. We consider a maximally-flat filter with minimum errors in the pass-band. Its transfer function is transformed using a specific 1D to 2D frequency mapping. An efficient pre-warping is applied before using the bilinear transform and the filter matrices are obtained. The filters are parametric since their characteristics are specified by parameters occurring explicitly in the filter coefficients. Simulation results on a test image are provided.

***Index Terms*—**2D IIR filters, analytical design, frequency mapping, approximation

1. INTRODUCTION

The field of two-dimensional filters has largely developed along the last three decades and various design methods were proposed [1]. Generally the currently-used design methods for 2D recursive filters rely on 1D filter prototypes, using spectral transformations from s to z plane via bilinear or Euler approximations, with the aim to obtain a 2D filter with a desired frequency response [2]. A major issue regards the digital spectral transformations, approached for instance in [2]-[4]. A commonly-used design method relies on a prototype low-pass filter and transforms its transfer function to obtain a 2D filter with desired frequency response. In [4] the design of circular, elliptical-shaped and fan filters is approached. Anisotropic filters have also been studied extensively [5], [6] and are used in some interesting applications, like remote sensing for directional smoothing applied to weather images, and texture segmentation and pattern recognition. Applications of directional filters banks are approached in [7], [8]. Design methods for orientation-selective filters were also approached in [9]. Other design methods for elliptically-shaped and directional filters were proposed in [10], [11].

We approach here the design of a class of 2D filters, having an elliptical shape in the frequency plane. The design method is mainly analytical and uses approximations, but does not involve any numerical optimization algorithms. It is based on 1D low-pass, maximally-flat prototype filters and frequency mappings. Several design examples using the proposed method are given. Simulation results for filtering a real grayscale image are finally given, to prove the usefulness of these filters in image processing.

2. ZERO-PHASE LOW-PASS ANALOG PROTOTYPE FILTERS

A recursive analog filter of order N is described by the general transfer function in variable s:

$$H(s) = \frac{P(s)}{Q(s)} = \sum_{i=0}^{M} p_i \cdot s^i \Bigg/ \sum_{j=0}^{N} q_j \cdot s^j \tag{1}$$

A zero-phase prototype can be obtained from the general filter $H(s)$ if the magnitude $|H(j\omega)|$ is considered. Zero-phase filters are frequently used in image processing as they do not introduce phase distortions. To obtain an elliptically-shaped filter, we start from a maximally-flat 1D prototype.

Let us consider a Butterworth low-pass (LP) filter of order N having the following transfer function magnitude:

$$|H(j\omega)| = 1 \Big/ \sqrt{1 + (\omega/\omega_0)^{2N}} \tag{2}$$

where ω_0 is the filter cut-off frequency. We look for a rational expression $H_P(\omega)$ of the magnitude $|H(j\omega)|$, which has to be an approximation as accurate as possible on the range $[-\pi, \pi]$. The most convenient for our purpose is the Chebyshev-Padé expansion, as it yields an efficient uniform approximation along the entire specified interval. It can be found using a symbolic computation software like MAPLE. The frequency response of the derived zero-phase prototype filter will be real-valued and will generally have the form:

$$H_P(\omega) = |H(j\omega)| = \sum_{j=0}^{M} b_j \cdot \omega^{2k} \Bigg/ \sum_{k=0}^{N} a_k \cdot \omega^{2k} \tag{3}$$

where often $M = N$.

Let us consider the maximally-flat filter (for $\omega_0 = 0.8$) with the transfer function:

$$H_P(\omega) = \frac{1 - 0.216684\omega^2 + 0.0151434\omega^4 - 0.000343\omega^6}{1 - 0.199461\omega^2 + 0.002517\omega^4 + 0.002086\omega^6} \tag{4}$$

which can be factorized as below:

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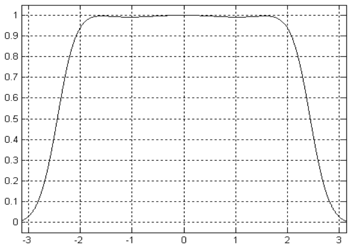

Fig. 1. Maximally-flat zero-phase prototype filter

$$H_P(\omega) = -0.1644 \cdot \frac{\left(\omega^2 - 10.294\right)\left(\omega^4 - 33.856 \cdot \omega^2 + 283.226\right)}{\left(\omega^2 + 12.23\right)\left(\omega^4 - 11.0234 \cdot \omega^2 + 39.1975\right)} \quad (5)$$

and since $s = j\omega$, $s^2 = -\omega^2$, the real function $H_P(\omega)$ can be expressed in the complex frequency variable s:

$$H_P(s) = -0.1644 \cdot \frac{\left(s^2 - 10.294\right)\left(s^4 + 33.856 \cdot s^2 + 283.226\right)}{\left(s^2 - 12.23\right)\left(s^4 + 11.0234 \cdot s^2 + 39.1975\right)} \quad (6)$$

The frequency response of this prototype filter is shown in Fig.1 and will be used to derive a 2D elliptically-shaped filter by applying a particular frequency transformation.

3. ELLIPTICALLY-SHAPED FILTER DESIGN

We propose an efficient design technique for a class of 2D low-pass filters having an elliptically-shaped horizontal section, based on the previously discussed 1D zero-phase filters, considered as prototypes. These filters will be specified by imposing the values of the semi-axes of the ellipse, and the orientation is given by the angle of the large axis with respect to ω_2-axis. Starting from the frequency response of a 1D zero-phase filter given by (3), we derive a 2D elliptically-shaped filter using the frequency mapping $\omega^2 \to E_\varphi(\omega_1, \omega_2)$, where:

$$\begin{aligned} E_\varphi(\omega_1,\omega_2) &= \omega_1^2\left(\frac{\cos^2\varphi}{E^2} + \frac{\sin^2\varphi}{F^2}\right) + \omega_2^2\left(\frac{\sin^2\varphi}{E^2} + \frac{\cos^2\varphi}{F^2}\right) \\ &+ \omega_1\omega_2\sin(2\varphi)\left(\frac{1}{F^2} - \frac{1}{E^2}\right) = a\cdot\omega_1^2 + b\cdot\omega_2^2 + c\cdot\omega_1\omega_2 \end{aligned} \quad (7)$$

The elliptically-shaped filter can be considered as derived from a circular filter through the linear transformation:

$$\begin{bmatrix}\omega_1\\ \omega_2\end{bmatrix} = \begin{bmatrix}E & 0\\ 0 & F\end{bmatrix}\cdot\begin{bmatrix}\cos\varphi & -\sin\varphi\\ \sin\varphi & \cos\varphi\end{bmatrix}\cdot\begin{bmatrix}\omega_1'\\ \omega_2'\end{bmatrix} \quad (8)$$

where usually we consider $E > F$; in (8), (ω_1, ω_2) are the current coordinates and (ω_1', ω_2') are the former (rotated) coordinates. Thus, the unit circle is stretched along the axes ω_1 and ω_2 with factors E and F, then counter-clockwise rotated with an angle φ, becoming an oriented ellipse.

Consequently, given a 1D prototype filter of the general form (3), we can obtain a corresponding 2D filter with an elliptical section, specified by the parameters E, F, φ which impose the shape and orientation using the mapping:

$$\omega^2 \to E_\varphi(\omega_1,\omega_2) = a\cdot\omega_1^2 + b\cdot\omega_2^2 + c\cdot\omega_1\omega_2 \quad (9)$$

Substituting the real variables ω_1 and ω_2 by the complex variables $s_1 = j\omega_1$ and $s_2 = j\omega_2$, the expression $E_\varphi(\omega_1,\omega_2)$ in the 2D Laplace domain becomes:

$$\begin{aligned} s^2 \to E_\varphi(s_1,s_2) &= a\cdot s_1^2 + b\cdot s_2^2 + c\cdot s_1 s_2 \\ &= \alpha\cdot s_1^2 + \beta\cdot s_2^2 + \gamma\cdot(s_1+s_2)^2 \end{aligned} \quad (10)$$

where

$$\begin{aligned} \alpha &= a - 0.5c = p + q\cdot\cos(2\varphi) + q\cdot\sin(2\varphi) \\ \beta &= b - 0.5c = p - q\cdot\cos(2\varphi) + q\cdot\sin(2\varphi) \\ \gamma &= 0.5c = -q\cdot\sin(2\varphi) \end{aligned} \quad (11)$$

Here we have used the notations:

$$p = 1/E^2 + 1/F^2, \; q = 1/E^2 - 1/F^2 \quad (12)$$

The next step is to find the discrete approximation $F_\varphi(z_1,z_2)$ of $E_\varphi(s_1,s_2)$ from (10). This can be achieved either using forward or backward Euler approximations, or otherwise the bilinear transform, which in principle gives better accuracy. The bilinear transform is a first-order approximation of the natural logarithm function, that is an exact mapping of the z-plane to the s-plane. For our purposes the sample interval is $T = 1$ so the bilinear transform has the form on the two axes:

$$s_1 = 2(z_1-1)/(z_1+1) \quad s_2 = 2(z_2-1)/(z_2+1) \quad (13)$$

Even if this method is straightforward, the designed 2D filter, corresponding to the transfer function in z_1, z_2 will inherently present large linearity distortions towards the frequency plane limits as compared to the ideal frequency response. This is due to the frequency warping effect of the bilinear transform, expressed by the frequency mapping:

$$\omega = (2/T)\cdot\mathrm{arctg}(\omega_a T/2) \quad (14)$$

where ω is a frequency of the discrete-time filter and ω_a is the corresponding frequency of the continuous-time filter. In order to correct this error, a pre-warping will be applied. Taking $T = 1$ in (14) we substitute the mappings:

$$\omega_1 \to 2\cdot\mathrm{arctg}(\omega_1/2) \quad \omega_2 \to 2\cdot\mathrm{arctg}(\omega_2/2) \quad (15)$$

Rational approximations are more suitable in handling these nonlinear mappings. Using again Chebyshev-Padé method, we get the accurate approximation on the range $[-\pi, \pi]$:

$$\mathrm{arctg}(\omega/2) \cong 0.4751\cdot\omega\big/\left(1 + 0.05\cdot\omega^2\right) \quad (16)$$

Thus the pre-warping correction consists in the substitution:

$$\omega \to 0.95\cdot\omega\big/\left(1 + 0.05\cdot\omega^2\right) \quad (17)$$

Even if (17) is useful in general, taking into account that the 2D filter function $H_C(s_1,s_2)$ has even parity in s_1 and s_2, i.e. the variables appear as s_1^2, s_2^2, a more efficient pre-warping can be made in this case. We obtain the following Chebyshev-Padé approximation on $[-\pi, \pi]$:

$$\left(\mathrm{arctg}(\omega/2)\right)^2 \cong \frac{(0.017251 + 0.218196\cdot\omega^2)}{(1 + 0.116048\cdot\omega^2)} \quad (18)$$

Consequently, from (15) and (18), the frequency pre-warping mapping can be written, on both frequency axes:

$$\omega_{1,2}^2 \to 7.521 \cdot \left(\omega_{1,2}^2 + 0.079\right) \Big/ \left(\omega_{1,2}^2 + 8.6171\right) \tag{19}$$

$$s_{1,2}^2 \to 7.521 \cdot \left(s_{1,2}^2 - 0.079\right) \Big/ \left(s_{1,2}^2 - 8.6171\right) \tag{20}$$

Let us now apply the bilinear transform on both axes, i.e. we substitute the relations (13) into the right side of mapping (20); we get a simple relation, valid on both axes ω_1 and ω_2:

$$s_1^2 \to 6.3868 \cdot \frac{\left(z_1^2 - 2.080651 \cdot z_1 + 1\right)}{\left(z_1^2 + 5.465361 \cdot z_1 + 1\right)} = 6.3868 \cdot \frac{P_1(z_1)}{Q_1(z_1)} \tag{21}$$

$$s_2^2 \to 6.3868 \cdot \frac{\left(z_2^2 - 2.080651 \cdot z_2 + 1\right)}{\left(z_2^2 + 5.465361 \cdot z_2 + 1\right)} = 6.3868 \cdot \frac{P_2(z_2)}{Q_2(z_2)} \tag{22}$$

The analog-discrete mappings (21) and (22) are therefore corrected forms of bilinear transform including pre-warping in the region of interest, namely $\omega_{1,2} \in [-\pi, \pi]$. It is noteworthy that pre-warping has not increased the order. The mappings (21), (22) in terms of frequency ω have the following forms:

$$\omega_{1,2}^2 \to 6.3868 \cdot \left(1.0403 - \cos\omega_{1,2}\right) \Big/ \left(2.7326 + \cos\omega_{1,2}\right) \tag{23}$$

This mapping is plotted in Fig.2 as curve 1 and is almost linear at least on the range $\omega \in [0, 0.7\pi]$, as compared to the mapping (15) – curve 2. Therefore the proposed correction compensates the distortions introduced by the bilinear transform. As shown further in the design examples, using the simple frequency mappings (21), (22) we obtain 2D filters with remarkable elliptical symmetry, even for frequencies close to $\pm\pi$, near the frequency plane limits .

Since we have determined the mapping $s \to z = e^s$ given by (21), (22) we now obtain a corresponding mapping for the sum $s_1 + s_2 \to e^{s_1 + s_2} = e^{s_1} \cdot e^{s_2} = z_1 \cdot z_2$. Using (21) the following mapping results, with $k = 6.3868$:

$$\begin{aligned}(s_1 + s_2)^2 &\to k \cdot \frac{\left(z_1^2 z_2^2 - 2.080651 \cdot z_1 z_2 + 1\right)}{\left(z_1^2 z_2^2 + 5.465361 \cdot z_1 z_2 + 1\right)} \\ &= k \cdot P_{12}(z_1, z_2) \big/ Q_{12}(z_1, z_2)\end{aligned} \tag{24}$$

The next step is finding the discrete expression for the mapping $F_\varphi : \mathbb{R} \to \mathbb{C}^2$, $\omega^2 \to F_\varphi(z_1, z_2)$. This is easy because we only have to substitute the expressions (21), (22), (24) of the terms s_1^2, s_2^2 and $(s_1 + s_2)^2$ into (10). The following mapping results:

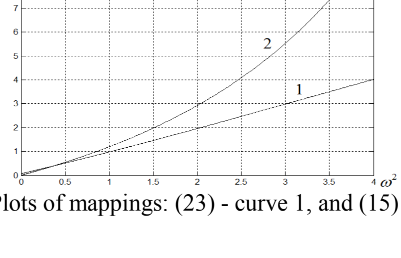

Fig. 2. Plots of mappings: (23) - curve 1, and (15) - curve 2

$$\begin{aligned} s^2 &\to E_\varphi(s_1, s_2) = \alpha \cdot s_1^2 + \beta \cdot s_2^2 + \gamma \cdot (s_1 + s_2)^2 \to F_\varphi(z_1, z_2) \\ &= k \cdot \left(\alpha \cdot \frac{P_1(z_1)}{Q_1(z_1)} + \beta \cdot \frac{P_2(z_2)}{Q_2(z_2)} + \gamma \cdot \frac{P_{12}(z_1, z_2)}{Q_{12}(z_1, z_2)} \right) = k \cdot \frac{P(z_1, z_2)}{Q(z_1, z_2)} \end{aligned} \tag{25}$$

where the constant has the value $k = 6.3868$.

The numerator and denominator polynomials result as:

$$\begin{aligned} P(z_1, z_2) = &\ \alpha \cdot P_1(z_1) \cdot Q_2(z_2) \cdot Q_{12}(z_1, z_2) \\ &+ \beta \cdot Q_1(z_1) \cdot P_2(z_2) \cdot Q_{12}(z_1, z_2) \\ &+ \gamma \cdot Q_1(z_1) \cdot Q_2(z_2) \cdot P_{12}(z_1, z_2) \end{aligned} \tag{26}$$

$$Q(z_1, z_2) = Q_1(z_1) \cdot Q_2(z_2) \cdot Q_{12}(z_1, z_2) \tag{27}$$

The numerator $P(z_1, z_2)$ contains the information about the filter specifications (ellipse axes *E, F* orientation angle φ) in the coefficients α, β, γ. The denominator $Q(z_1, z_2)$ is a polynomial with constant coefficients.

Next we express the polynomials $P(z_1, z_2)$ and $Q(z_1, z_2)$ in matrix form, more useful from the implementation point of view. Grouping together the constant terms and the terms containing $\cos(2\varphi)$ and $\sin(2\varphi)$, we can finally write:

$$\mathbf{P} = p \cdot \mathbf{M_0} + (\cos 2\varphi) \cdot q \cdot \mathbf{M_1} + (\sin 2\varphi) \cdot q \cdot \mathbf{M_2} \tag{28}$$

Finally the matrix form of the mapping is:

$$s^2 \to F(z_1, z_2) = \frac{P(z_1, z_2)}{Q(z_1, z_2)} = \frac{\mathbf{z_1} \times \mathbf{P} \times \mathbf{z_2}^T}{\mathbf{z_1} \times \mathbf{Q} \times \mathbf{z_2}^T} \tag{29}$$

where the vectors $\mathbf{z_1}$ and $\mathbf{z_2}$ are:

$$\begin{aligned} \mathbf{z_1} &= \begin{bmatrix} 1 & z_1 & z_1^2 & z_1^3 & z_1^4 \end{bmatrix} \\ \mathbf{z_2} &= \begin{bmatrix} 1 & z_2 & z_2^2 & z_2^3 & z_2^4 \end{bmatrix} \end{aligned} \tag{30}$$

The rational expression $F(z_1, z_2)$ is written in matrix form; $\mathbf{P}$ and $\mathbf{Q}$ are square matrices of size 5×5. Matrix $\mathbf{P}$ depends on the ellipse parameters given in the specifications (*E*, *F*, φ), while $\mathbf{Q}$ is a matrix with constant elements:

$$\mathbf{Q} = \begin{bmatrix} 0.01 & 0.05465 & 0.01 & 0 & 0 \\ 0.05465 & 0.35336 & 0.35336 & 0.05465 & 0 \\ 0.01 & 0.35336 & 1.65250 & 0.35336 & 0.01 \\ 0 & 0.05465 & 0.35336 & 0.35336 & 0.05465 \\ 0 & 0 & 0.01 & 0.05465 & 0.01 \end{bmatrix} \tag{31}$$

The component matrices $\mathbf{M_0}$, $\mathbf{M_1}$, $\mathbf{M_2}$ are given below.

$$\mathbf{M_0} = \begin{bmatrix} 0.02 & 0.03385 & 0.02 & 0 & 0 \\ 0.0338 & -0.11812 & 0.21884 & 0.10931 & 0 \\ 0.02 & 0.21884 & -1.203 & 0.21884 & 0.02 \\ 0 & 0.10931 & 0.21884 & -0.11812 & 0.0338 \\ 0 & 0 & 0.02 & 0.03385 & 0.02 \end{bmatrix} \tag{32}$$

$$\mathbf{M_1} = \begin{bmatrix} 0 & 0.0754 & 0 & 0 & 0 \\ -0.0754 & 0 & 0.3369 & 0 & 0 \\ 0 & -0.3369 & 0 & -0.3369 & 0 \\ 0 & 0 & 0.3369 & 0 & -0.0754 \\ 0 & 0 & 0 & 0.0754 & 0 \end{bmatrix} \tag{33}$$

$$\mathbf{M_2} = \begin{bmatrix} 0.01 & -0.0208 & 0.01 & 0 & 0 \\ -0.0208 & -0.3960 & 0.2779 & 0.1301 & 0 \\ 0.01 & 0.2779 & -0.6014 & 0.2779 & 0.01 \\ 0 & 0.1301 & 0.2779 & -0.3960 & -0.0208 \\ 0 & 0 & 0.01 & -0.0208 & 0.01 \end{bmatrix} \quad (34)$$

The matrices $\mathbf{Q}, \mathbf{M_0}, \mathbf{M_1}, \mathbf{M_2}$ are symmetric with respect to their central element, since the elliptical filter is symmetric to the origin of the frequency plane. Each of the 4-th order fractions in the factorized function $H_P(s)$ in (3), in particular (5) can be generally written:

$$H_{Pi}(s) = \left(s^4 + b_1 \cdot s^2 + b_0\right) / \left(s^4 + a_1 \cdot s^2 + a_0\right) \quad (35)$$

Since finally we must reach a transfer function of a discrete filter in the complex variables z_1, z_2, we have to determine the discrete counterpart of the general factor $H_{Pi}(s)$. We only have to make the substitution of (25) in $H_{Pi}(s)$ given in (35) and we obtain the desired 2D transfer function:

$$\begin{aligned} H_2(z_1,z_2) &= B_2(z_1,z_2) / A_2(z_1,z_2) \\ &= \frac{P^2(z_1,z_2) + b_1 \cdot P(z_1,z_2) \cdot Q(z_1,z_2) + b_0 \cdot Q^2(z_1,z_2)}{P^2(z_1,z_2) + a_1 \cdot P(z_1,z_2) \cdot Q(z_1,z_2) + a_0 \cdot Q^2(z_1,z_2)} \end{aligned} \quad (36)$$

where $b_1 = 33.856$, $b_0 = 283.2259$, $a_1 = 11.0234$, $a_0 = 39.1975$ are the coefficients of the second factor of prototype (6). In our case $H_2(z_1,z_2)$ results of order 4. Similarly, a function $H_1(z_1,z_2)$ is found for the first 2-nd order fraction in (6). Since the 2D function $H_2(z_1,z_2)$ in (36) can also be described in terms of 5×5 matrices $\mathbf{B_2}, \mathbf{A_2}$ corresponding to $B_2(z_1,z_2)$ and $A_2(z_1,z_2)$, we can write equivalently:

$$\begin{aligned} \mathbf{B_2} &= \mathbf{P} * \mathbf{P} + b_1 \cdot \mathbf{P} * \mathbf{Q} + b_0 \cdot \mathbf{Q} * \mathbf{Q} \\ \mathbf{A_2} &= \mathbf{P} * \mathbf{P} + a_1 \cdot \mathbf{P} * \mathbf{Q} + a_0 \cdot \mathbf{Q} * \mathbf{Q} \end{aligned} \quad (37)$$

where $*$ denotes two-dimensional convolution.

Fig.3 (a), (b) shows the frequency response and contour plot for an elliptically-shaped filter with given parameters. Its shape in the frequency plane results perfectly elliptical, due to the applied pre-warping. There are no visible shape distortions even near the frequency plane limits.

In Fig.3 (c), (d) an elliptically-shaped filter with a large ratio E/F is shown; with $E = 1.2$, $F = 0.1$ we get a very selective directional filter which forms an angle $\varphi = \pi/4$ with ω_1 axis and has a very narrow frequency response. These filters may be used in detecting straight lines with a given orientation in an image [8], as shows the simulation example below. Two other filters are shown in Fig.3 (e), (f).

The stability of the designed filters was not approached here, but will be studied in detail in further work. If the prototype filter is stable and the frequency transformations preserve stability, the derived 2D filters should also be stable. Various stability criteria exist [12] and stabilization methods which can be applied for some unstable filters [13].

As regards the relationship to prior work in the field, the analytical design of elliptically-shaped filters, either FIR or IIR, has not been systematically approached previously. There are several types of oriented (anisotropic) filters, but to author's knowledge no filters were analytically designed with elliptically-shaped frequency response starting from a specified prototype, e.g. maximally-flat. For instance [6] approaches a separable Gaussian filter with elliptical section but it is a particular case. An earlier version of the proposed filters was developed in [10].

4. SIMULATION RESULTS

Fig.4 (a) shows a real grayscale image representing a straw texture. The straws have random directions and choosing different filter orientations we select the ones with roughly similar orientations and filter out the rest. A very selective elliptical filter was used, with different orientation angles ($\varphi = \pi/6$, $\varphi = \pi/3$), obtaining the filtered images (b), (c).

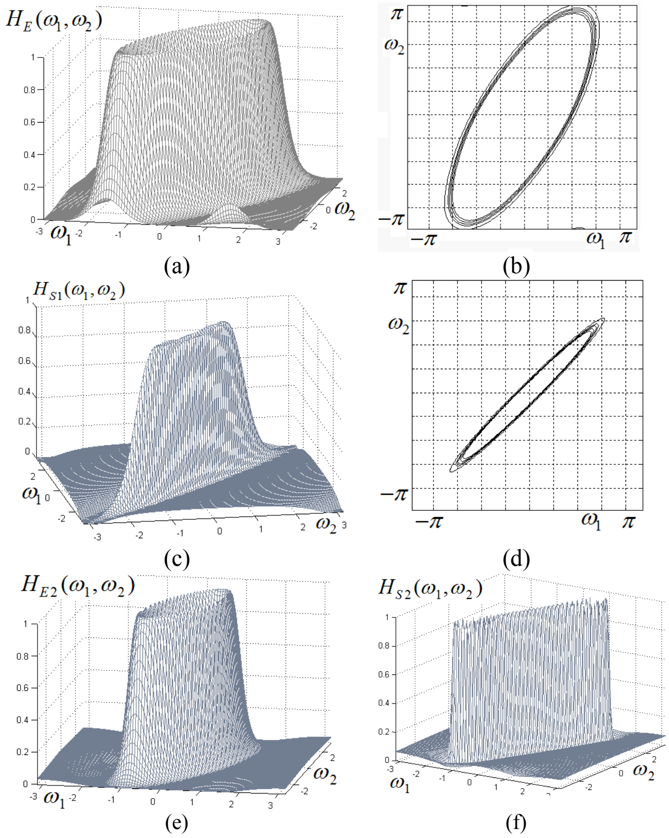

Fig. 3. Frequency responses and contour plots of elliptical filters: (a), (b) $E = 1.7$, $F = 0.4$, $\varphi = \pi/6$; (c), (d) $E = 1.2$, $F = 0.1$, $\varphi = \pi/4$; (e) $E = 0.9$, $F = 0.7$, $\varphi = \pi/9$; (f) $E = 1.7$, $F = 0.1$, $\varphi = \pi/7$;

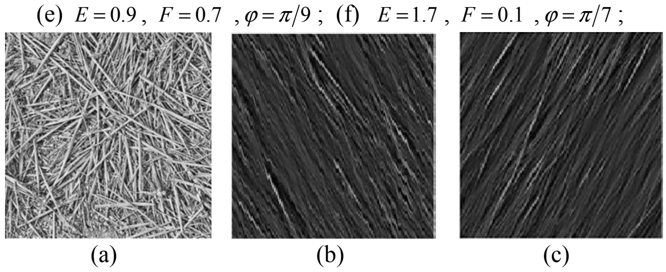

Fig. 4. Grayscale texture image and directional filtering results

5. CONCLUSION

The proposed analytical design method is simple and efficient. Once found the suitable zero-phase 1D recursive prototype filter, a specific frequency transformation is applied and the matrices of the 2D elliptical filter are determined in a straightforward way. Using a pre-warping correction, the filters result with negligible shape distortions. The designed 2D filters are parametric, in the sense that their matrices depend explicitly on the specified parameters. For different parameter values, the particular filter matrices result directly. An advantage of the method is versatility; the design need not be remade each time again from the start for various specifications.

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