PARAMETER ESTIMATION OF GAUSSIAN FUNCTIONS USING THE SCALED REASSIGNED SPECTROGRAM

Johan Brynolfsson and Maria Hansson-Sandsten

Dept. of Mathematical Statistics, Lund University, Sweden

ABSTRACT

In this paper we suggest an improved algorithm for estimation of parameters detailing Gaussian functions and expand it to handle linear combinations of Gaussian functions. Components in the signal are first detected in the spectrogram, which is calculated using a Gaussian window function. Scaled reassignment is then performed using a set of candidate scaling factors and the local Renyi entropy is used to measure the concentration of each component using every candidate scaling factor. Exploiting the fact that a Gaussian function may be perfectly reassigned into one single point given the correct scaling, one may identify the parameters detailing the Gaussian function. We evaluate the algorithm on both simulated and real data.

Index Terms— Reassigned spectrogram, Gaussian functions, Parameter estimation

1. INTRODUCTION

Time-frequency distributions have long been a useful tool in identification and classification of non-stationary signals. A common method for estimation of the time-frequency content of a signal is the spectrogram, where a time window is allowed to slide over the signal and the spectral content is estimated within each time-frame. The drawback of this method is the resolution trade-off, i.e. a short time-window renders good resolution in time, but bad resolution in frequency and vice versa for a long time-window. One proposed solution for this is the reassignment method, first proposed in [1] and then reintroduced and analyzed thoroughly in [2]. There each component, in a time-frequency or time-scale domain, is reassigned to its center of gravity. It has been shown to give perfect localization of the instantaneous frequency for impulses, sinusoids and chirp signals. Further discussion on the topic can be found in e.g. [3–5]. Often however, finding the exact time frequency distribution of the process is not the main goal, but rather a way to extract information about the signal. In [6] we present a version of the scaled reassignment, introduced in [7]. If scaling is done correctly, perfect localization of a Gaussian function is achieved. We also suggest how this may be used to estimate the shape parameter of a Gaussian function, by fixing the scaling parameters and finding the matched window function for the Gaussian signal. In this paper we improve this estimation in terms of computational complexity. Further, we suggest an extension, which makes the method viable if the signal consists of multiple, well separated, Gaussian components.

In this paper all integrals are assumed to range from $-\infty$ to ∞, unless stated otherwise. Real and imaginary parts will be denoted $\Re$ and $\Im$ respectively and $(\cdot)^*$ will denote conjugate transpose.

2. SIGNAL MODEL

We assume that the observed signal is comprised of a sum of Gaussian functions

$$x(t) = \sum_{k=1}^{K} \alpha_k e^{-\frac{(t-t_k)^2}{2\sigma_k^2}} e^{i\omega_k t} \tag{1}$$

where α_k, t_k, and ω_k denote amplitude, center time and center frequency respectively, and σ_k is the shape parameter, which we here consider the parameter of main interest. Further, we assume that the components are separated in time and/or frequency, meaning that no two components overlap each other enough in the time-frequency representation, to affect the reassignment of the neighboring component.

3. THE SCALED REASSIGNED SPECTROGRAM

The short-time Fourier transform (STFT) of the signal $x(t)$ is calculated using some appropriate window function $h(t)$ is defined as

$$F_x^h(t,\omega) = \int x(s)h^*(s-t)e^{i\omega s}ds \tag{2}$$

which renders the spectrogram as

$$S_x^h(t,\omega) = \left|F_x^h(t,\omega)\right|^2 \tag{3}$$

The reassignment of the spectrogram is defined, using the two-dimensional Dirac delta function, as

$$RS(t,\omega) = \iint S_x^h(s,\xi)\delta\left(t - \hat{t}_x(s,\xi), \omega - \hat{\omega}_x(s,\xi)\right) dsd\xi. \tag{4}$$

This work was supported by the Swedish Research Council.

978-0-9928626-3-3/15/$31.00 ©2015 IEEE

where F_x^{th} and $F_x^{dh/dt}$ are the STFTs using $t \cdot h(t)$ and $\frac{dh(t)}{dt}$ as window functions, respectively. In [2], it is shown that the reassignment functions in time and frequency, where each point is reallocated to its center point of gravity, can be expressed as

$$\hat{t}_x(t,\omega) = t + c_t \Re\left\{\frac{F_x^{th}(t,\omega)}{F_x^h(t,\omega)}\right\} \quad (5)$$

$$\hat{\omega}_x(t,\omega) = \omega - c_\omega \Im\left\{\frac{F_x^{dh/dt}(t,\omega)}{F_x^h(t,\omega)}\right\} \quad (6)$$

By setting $c_t = c_\omega = 1$ one gets the standard reassigned spectrogram, where chirps, sinusoids and impulses are perfectly reassigned. The objective in this paper however is to find scaling parameters such that a Gaussian function is perfectly concentrated after the reassignment. This can be achieved conditioned that a unit norm Gaussian window function with parameter λ is used when calculating the STFT

$$h(t) = \frac{1}{\pi^{1/4}\sqrt{\lambda}} e^{-\frac{t^2}{2\lambda^2}} \quad (7)$$

Additionally one needs the necessary STFTs from equation (5) and (6). Given that the observed is a single Gaussian function $x_k(t)$, on the form of equation (1) with parameters α_k, t_k, ω_k and σ_k, the STFTs are found as

$$F_{x_k}^h(t,\omega) = \quad (8)$$
$$\alpha_k \sqrt{\frac{2\lambda\sigma_k^2\sqrt{\pi}}{\lambda^2+\sigma_k^2}} e^{-\frac{1}{2}\left(\left(\frac{1}{\lambda^2+\sigma_k^2}\right)(t-t_k)^2 + \left(\frac{\lambda^2\sigma_k^2}{\lambda^2+\sigma_k^2}\right)(\omega-\omega_k)^2\right)}$$
$$\cdot\, e^{-i\frac{(\omega-\omega_k)(\sigma_k^2 t + \lambda^2 t_k)}{\lambda^2+\sigma_k^2}}$$

and

$$F_{x_k}^{th}(t,\omega) = -\frac{\lambda^2}{\lambda^2+\sigma_k^2}\left(t - t_k + i\sigma_k^2(\omega-\omega_k)\right) F_{x_k}^h(t,\omega) \quad (9)$$

and the last necessary STFT

$$F_{x_k}^{dh/dt}(t,\omega) = -\frac{1}{\lambda^2} F_{x_k}^{th}(t,\omega) \quad (10)$$

The spectrogram, for the single component, can then be calculated as

$$S_{x_k}^h(t,\omega) \quad (11)$$
$$= \frac{2\alpha_k^2\lambda\sigma_k^2\sqrt{\pi}}{\lambda^2+\sigma_k^2} e^{-\left(\left(\frac{1}{\lambda^2+\sigma_k^2}\right)(t-t_k)^2 + \left(\frac{\lambda^2\sigma_k^2}{\lambda^2+\sigma_k^2}\right)(\omega-\omega_k)^2\right)}$$

By defining the normalized reassignment vector as

$$r_{x_k}(t,\omega) = \frac{\hat{t}_{x_k}(t,\omega) - t}{\Delta t_h c_t} + \frac{\hat{\omega}_{x_k}(t,\omega) - \omega}{c_\omega \Delta\omega_h} \quad (12)$$

the reassignment of a Gaussian function using a Gaussian smoothing window can be expressed on a compact form as

$$r_{x_k}^h(t,\omega) = \frac{\sqrt{2}}{\lambda}\frac{F_{x_k}^{th}(t,\omega)}{F_{x_k}^h(t,\omega)} \quad (13)$$

Finally, applying the scaled reassignment method the spectrogram in equation (11)

$$RS_x^h(t,\omega) = \frac{2\alpha_k^2\sigma_k^2\lambda\sqrt{\pi}}{(\lambda^2+\sigma_k^2)\left|1-\frac{c_t\lambda^2}{\lambda^2+\sigma_k^2}\right|\left|1-\frac{c_\omega\sigma_k^2}{\lambda^2+\sigma_k^2}\right|}$$
$$e^{-\frac{(t-t_k)^2}{(\lambda^2+\sigma_k^2)\left(1-\frac{c_t\lambda^2}{\lambda^2+\sigma_k^2}\right)^2} + \frac{(\omega-\omega_k)^2}{(\lambda^2+\sigma_k^2)\left(1-\frac{c_\omega\sigma_k^2}{\lambda^2+\sigma_k^2}\right)^2}} \quad (14)$$

For more details of these calculations, see [6]. One may note that perfect localization will be achieved for every (positive valued) window parameters λ if the scaling parameters are set to

$$c_t = \frac{\lambda^2+\sigma_k^2}{\lambda^2} \quad (15)$$

$$c_\omega = \frac{\lambda^2+\sigma_k^2}{\sigma_k^2} \quad (16)$$

Due to the linearity of the Fourier transform, the reassignment vector for a composition of K components is found as

$$r_x^h(t,\omega) = \sum_{k=1}^{K} \frac{F_{x_k}^h(t,\omega)}{\sum_{j=1}^{K} F_{x_j}^h(t,\omega)} r_{x_k}(t,\omega) \quad (17)$$

where $F_{x_k}^h(t,\omega)$ and $r_{x_k}(t,\omega)$ are the STFT and the normalized reassignment vector of the k:th component, respectively. Since the components in this paper are assumed to be well separated, at most only one $F_{x_k}^h(t,\omega)$ will locally have values significantly different from zero. To give a good definition of well separated is difficult as it depends on every parameter detailing every neighboring component, as well as on the choice of time-window parameter λ in equation (7). For example, considering only two neighboring components, if the masses of each component affects the reassigning of the other component depends on λ, and the amplitude, center time, center frequency and shape parameter of both components in an intricate way.

4. PARAMETER ESTIMATION

In [6] we suggest an optimization scheme where the scale parameters are fixed, and the window parameter lambda is changed. The Renyi entropy [8], defined as

$$R_\alpha^{RS_x^h} = \frac{1}{\alpha-1}\log_2 \iint \left(RS_x^h(t,\omega)\right)^\alpha dt d\omega, \quad (18)$$

is then used to measure the concentration of the reassigned spectrogram. As per the discussions in [8, 9], we henceforth

Algorithm 1 Flow of Gaussian parameter estimation

1: Calculate the spectrogram using a Gaussian window with shape parameter λ. Equation (2), (3), and (7)
2: Set a noise threshold and find peaks in the spectrogram.
3: Compute the scaled reassigned spectrograms using candidate ς
4: Calculate the LRE for each identified component.
5: Identify minima in LRE of every component.

choose $\alpha = 3$. Given a single component setting, we show that the function of the Renyi entropy over the space of window parameters is unimodal and hence suggest line search to find the minimum. Here we suggest three changes to this parameter estimation method. First, we propose that we instead fix the window parameter and optimize over the scale parameters. It then follows that one only has to calculate the STFT once, and reassign the resulting spectrogram according to the scale parameters instead of having to calculate both STFT and reassignment for each window parameter.

Secondly we propose a change to the optimization method of using line search. The line search method is computationally efficient but is in this case sensitive to noise. When the Gaussian functions are observed in a noisy environment, the Renyi curve will no longer be unimodal and the line search method may then converge to a local minimum. To solve this problem, we suggest the more robust dictionary approach where a grid of candidate parameters is set up and the Renyi entropy of the scaled reassigned spectrogram is evaluated over the whole grid of candidate scaling parameters. The estimate is then taken as the grid point rendering the smallest value. Given one estimate, one may set up a new, more finely spaced, grid around this estimate and iterate the procedure.

Thirdly, as the observed signal possibly is multicomponent, the Renyi entropy is multi-modal and is affected by changes in all components in the entire spectrogram. To this end we propose that components are first identified in the spectrogram using peak detection. Further, we propose that the local Renyi entropy (LRE) is used, defined as

$$R_\alpha^{RS_x^{h,\varsigma}}(t_0,\omega_0) = \frac{1}{\alpha-1}\log_2 \int_{t_0-\Lambda_t}^{t_0+\Lambda_t}\int_{\omega_0-\Lambda_\omega}^{\omega_0+\Lambda_\omega} \left(RS_x^{h,\varsigma}(s,\xi)\right)^\alpha ds d\xi \quad (19)$$

where $R_\alpha^{RS_x^{h,\varsigma}}(t_0,\omega_0)$ denotes the LRE around the point (t_0,ω_0) of the reassigned spectrogram, calculated using candidate parameter ς. Further Λ_t and Λ_ω are parameters determining the area where the LRE is calculated. We note that the algorithm is insensitive to the choice of Λ_t and Λ_ω as long as the area contains the center time t_k and center frequency ω_k for the component of interest and does not contain another component. Hence they may be set using some crude heuristic, e.g. using *a priori* knowledge of the resolution of components or utilizing the minimum distance between all found components. By denoting the candidate parameters as ς, the candidate scaling parameters are set up as

$$c_t(\varsigma) = \frac{\lambda^2+\varsigma^2}{\lambda^2} \quad (20)$$

$$c_\omega(\varsigma) = \frac{\lambda^2+\varsigma^2}{\varsigma^2} \quad (21)$$

which means that the Gaussian component with scale parameter σ will be perfectly reassigned to one single point when $\varsigma = \sigma$. It then follows that the LRE will have its minimum in this point. For each identified component in the signal, the shape parameter is estimated as

$$\hat{\sigma_k} = \arg\min_\varsigma R_\alpha^{RS_x^{h,\varsigma}}(t_k,\omega_k) \quad (22)$$

In other words, reassign the entire spectrogram using the candidate scaling parameters. Then calculate the LRE around

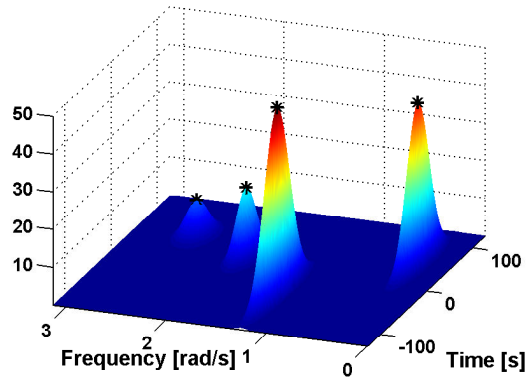

Fig. 1: The spectrogram showing the four components. The detected component peaks are marked by stars.

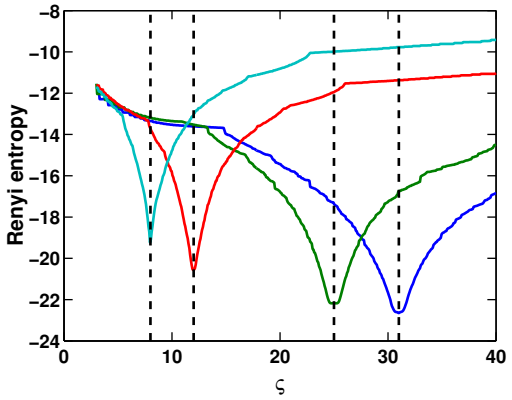

Fig. 2: The local Renyi entropy for the four components as function of candidate parameters ς. The true parameter values for each of the four components are plotted as dashed horizontal lines.

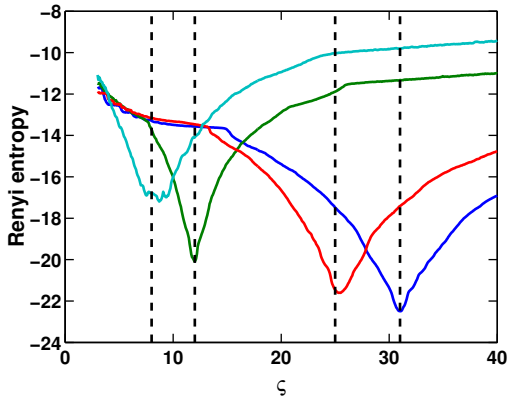

Fig. 3: The local Renyi entropy for the four components, observed with white noise, as function of candidate parameters ς. The true parameter values for each of the four components are plotted as dashed horizontal lines.

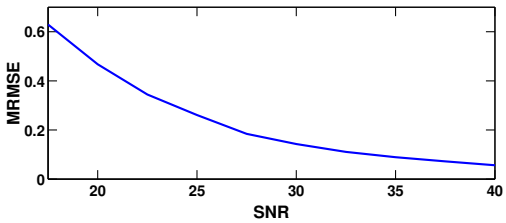

Fig. 4: The mean RMSE of the four components as function of SNR.

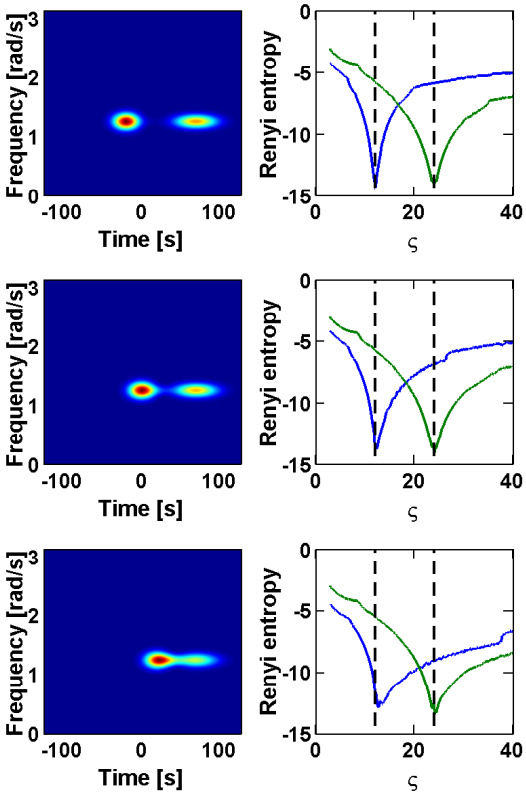

Fig. 5: The figure shows an example of how the Renyi curves are shifted due to influence from neighboring components. Note that as the two components approaches each other the troughs in the local Renyi curves deviates more from the true values.

each component and identify the trough for each Renyi curve and every candidate parameter. The parameter rendering the lowest LRE is taken as the scale parameter estimate. The algorithm is shown in Algorithm 1.

5. SIMULATIONS

We will show the performance of the algorithm by presenting some simulated examples and end with a real-data case. In all estimations below, the candidate parameters ς were set up as an equidistant grid of points in the range [5,40] with 500 points.

We begin by showing the proposed method on a four component signal with parameters shown in Table 1, when the signal is observed noise free. The spectrogram of the signal, shown in Figure 1, is calculated using a time-window on the form of equation (7) with $\lambda = 20$. The peaks identified in the spectrogram are marked by stars. The LRE for each of the four components are plotted in Figure 2 as functions of candidate shape parameters ς, together with the true values marked as dashed lines. One can clearly see the troughs lining up with the true parameter values. The estimated parameter values are $\hat{\sigma} = (30.95\,, 12.04, 7.79, 24.80)$, where the small errors are due to the resolution of both the spectrogram and the candidate parameters. We then apply the algorithm to a realization of the same four component signal, but this time observed with additive white noise. By defining the signal-to-noise ratio (SNR) for the k:th component as the squared amplitude value of the component divided by the noise variance, we set the SNR=20dB. In Figure 3, the local Renyi curves are shown together with the true shape parameters plotted as dashed vertical lines. The estimated shape parameters are $\hat{\sigma} = (30.81, 12.34, 7.82, 24.01)$. This experiment was repeated 250 times for different settings of the SNR and the mean root-mean-squared error (RMSE), i.e. the mean of the RMSE for the four components, are shown in Figure 4 as function of the SNR.

We go on by showing an example of how a neighboring component may influence the parameter estimates. In Figure 5, we see two components where the left component is successively moved closer to the right component. The left

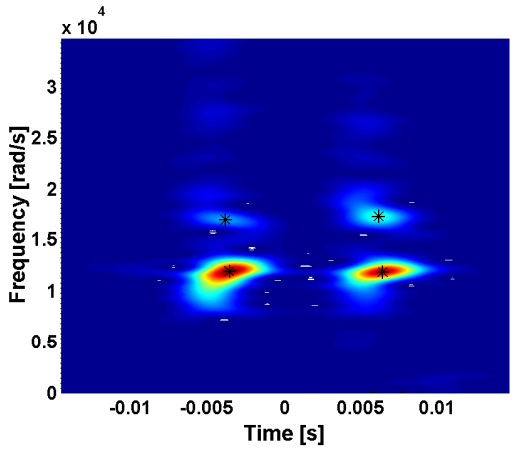

Fig. 6: The spectrogram of a recorded dolphins echo-location click. The Identified peaks are plotted with black stars.

comp.	1	2	3	4
t_k	-50	20	50	50
$\omega_k/2\pi$	0.2	0.3	0.4	0.1
σ_k	31	12	8	25
α_k	1	1	1	1

Table 1: Parameter values of components in first example.

row in Figure 5 shows the spectrogram in each case and the right column shows the calculated Renyi curves for the two components. One may note that the closer the two components are, the more biased the troughs are as compared to the true values. Still, the bias is quite small, and still renders renders reasonable estimates.

Finally we show a real world example. Figure 6 depicts the spectrogram of echo-location clicks made by a Risso's dolphin. The clicks are oscillating impulses, that may be well modeled as Gaussian functions. The algorithm is applied to the data which identifies four components and estimates the parameters to $(11.6,\ 11.08)$ for the two lower frequency terms and $(16.1.\ 15.1)$ for the two higher frequency components. In Figure 7 the LRE are shown for the identified components.

6. CONCLUSIONS AND FUTURE WORK

In this paper we suggest an improved algorithm for estimating the parameters of oscillating Gaussian functions. This is done by calculating the spectrogram using a Gaussian window function. Scaled reassignment is then performed using a large set of candidate parameters, ς. Finally, the LRE is computed for each component and the minima are identified.

The algorithm presented here will be more thoroughly analyzed concerning how neighboring components influence each other and the parameter estimation.

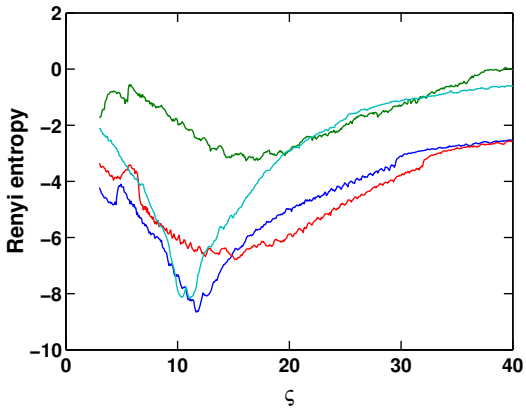

Fig. 7: The figure the four LRE curves for the components identified in the dolphin echo-location clicks.

REFERENCES

[1] K. Kodera, R. Gendrin, and D. De.Villedary, "Analysis of Time-Varying Signals with Small BT," *IEEE Transactions on Acoustics Speech and Signal Processing*, vol. ASSP-26, pp. 64–76, February 1978.

[2] F. Auger and P. Flandrin, "Improving the Readability of Time-Frequency and Time-Scale Representations by the Reassignment Method," *IEEE Transactions on Signal Processing*, vol. 43, pp. 1068–1089, 1995.

[3] B. Boashash, *Time Frequency Signal Analysis and Processing*. Elsevier, 2003.

[4] F. Hlawatsch and F. Auger, eds., *Time-Frequency Analysis*. ISTE Ltd, 2008.

[5] M. Hansson-Sandsten, "Evaluation of Non-Linear Combinations of Rescaled Reassigned Spectrograms," in *22nd European Signal Processing Conference*, (Lisbon, Portugal), 1-5 Sept. 2014.

[6] M. Hansson-Sandsten and J. Brynolfsson, "The Scaled Reassigned Spectrogram with Perfect Localization for Estimation of Gaussian Functions," *IEEE Signal Processing Letters*, vol. 22, pp. 100–104, January 2015.

[7] F. Auger, E. Chassande-Mottin, and P. Flandrin, "Making Reassignment Adjustable: The Levenberg-Marquardt Approach," in *37th IEEE Int. Conf. on Acoustics, Speech and Signal Processing*, (Kyoto, Japan), 25-30 Mar. 2012.

[8] R. Baraniuk, P. Flandrin, A. Janssen, and O. Michel, "Measuring Time-Frequency Information Content Using the Rényi Entropies," *IEEE Transactions on Information Theory*, vol. 47, pp. 1391–1409, May 2001.

[9] J. Xiao and P. Flandrin, "Multitaper Time-Frequency Reassignment for Nonstationary Spectrum Estimation and Chirp Enhancement," *IEEE Transactions on Signal Processing*, vol. 55, pp. 2851–2860, June 2007.