On the Robustness of Co-prime Sampling

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***Abstract*—Coprime sampling has been shown to be an effective deterministic sub-Nyquist sampling scheme for estimating the power spectrum of wide sense stationary signals without any loss of information. In contrast to the existing results in coprime sampling which only assume an ideal setting, this paper considers both additive perturbation on the sampled signal, as well as sampling jitter, and analyzes their effect on the quality of the estimated correlation sequence. A variety of bounds on the error introduced by such non ideal sampling schemes are computed by considering a statistical model for the perturbations. They indicate that coprime sampling leads to stable estimation of the autocorrelation sequence, in presence of small perturbations. Additional results on identifiability in spatial spectrum estimation are derived using the Fisher Information Matrix, which indicate that with high probability, it is still possible to identify $O(M^2)$ sources with M sensors, with a perturbed coprime array.**
***Keywords* — Coprime Sampling, Spectrum Estimation, Co-Array, Jitter, Line Spectrum, Fisher Information Matrix.** [1]

I. Introduction

Co-prime sampling [1], [2] is a novel sampling technique that has been proposed to estimate the spectrum of wide-sense stationary (WSS) processes at sub-Nyquist rates. Unlike common compressive sensing approaches that use sparsity to enable sampling at sub-Nyquist rates, co-prime sampling only requires the random process to be WSS. For a line spectrum, co-prime sampling (with coprime numbers M,N) is shown to be capable of recovering $O(MN)$ sinusoidal frequencies [1]–[3]. However, existing results in co-prime sampling largely ignore any non-ideal settings such as additive noise and/or jitter (inaccuracy or uncertainty about the sampling instants) which are of great practical importance in implementing any sampling strategy .

The effect of jitter has been extensively studied in the context of uniform sampling [4]–[6]. Jitters usually occur due to inaccuracies of the system clock of A/D converters at high frequencies. A desirable sampling technique would be one that is tolerant to jitters as well as to additive noise so that small jitters or noise would lead only to relatively small reconstruction errors. This is also known as stability of sampling [7]. Jitter in spatial sampling leads to perturbation of sensor locations in an antenna array, which is well studied for uniform linear arrays (ULA). It is known that small perturbations in the location of the sensors, can cause relatively large errors in subspace based algorithms such as MUSIC [8], [9]. Several algorithms for ULAs try to jointly estimate the array perturbations and the source directions [10], [11]. Our recent work [12], addresses a similar situation for co-arrays. Nevertheless, the existing studies about sampling jitter and perturbations cannot be applied to the co-prime sampling, This is because, in co-prime sampling the goal is to reconstruct the autocorrelation sequence whereas in typical sampling problems the goal is to reconstruct the time domain signal.

This paper studies the effect of non ideal conditions on coprime sampling and how they affect the subsequent estimation of the autocorrelation sequence. One of the main results of ideal co-prime sampling was to show that it is possible to estimate autocorrelation sequence at Nyquist rate using sub-Nyquist samplers. In the context of spatial sampling, this means identification of $O(M^2)$ sources using M sensors. We want to investigate if coprime sampling is robust to small perturbations. In particular, we will explore if we can still reliably estimate autocorrelation sequence for temporal sampling, and identify $O(M^2)$ sources for spatial sampling, using a noisy version of co-prime samples. For spatial sampling, we derive a Cramér Rao lower bound and examine identifiability issues in the presence of array perturbations. In this paper, jitter in temporal sampling is modeled as a uniform random variable, whereas the array perturbation are assumed to be fixed unknown variables which are both natural assumptions in their corresponding problem settings.

This paper is organized as follows. Section III briefly reviews the concept of co-prime sampling and examines the effect of jitter and additive noise. Section IV establishes some results for the array perturbations in the context of spatial sampling, and also examines identifiability issues for this case. Section V validates the results of this paper through numerical simulations. Section VI concludes the paper.

II. Robustness to Additive Noise

Let $R_x(\tau)$ denote the autocorrelation function (ACF) of a wide-sense stationary (WSS) random process $x(t)$. The signal $x(n)$ is sampled with a pair of coprime samplers at the rate $1/MT$ and $1/NT$ ($M < N$) to obtain the samples $x_M[n] = x(MnT)$ and $x_N[m] = x(NmT)$, where $1/T$ corresponds to the Nyquist rate determined by the power spectrum density of $x(t)$. It can be shown [2] that $\{E[x_M[n]x_N[m], 0 \leq n \leq N-1, 0 \leq m \leq 2M-1\}$ generates correlation values $R_x(kT)$ for all lags $0 \leq k \leq MN-1$. Hence, it is possible to obtain samples of the autocorrelation function at the Nyquist rate $(1/T)$ by using sub-Nyquist samplers operating at M and N times slower than the Nyquist rate. In this section, we study the effect of perturbing the samples obtained from coprime samplers, on the subsequent computation of correlation lags $R_x(kT)$ at the Nyquist rate. Assuming additive perturbation on the samples, the signals obtained from the coprime samplers are:

$$x_1[n, l] = x\left(nMT + 2MNlT\right) + z\left(nMT + 2MNlT\right) \quad (1)$$

$$x_2[m, l] = x\left(mNT + 2MNlT\right) + z\left(mNT + 2MNlT\right) \quad (2)$$

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For the purpose of our analysis we make the following assumptions:

1) **(A1):** $x(nT)$ is assumed to be a zero mean jointly Gaussian WSS process whose autocorrelation $R_x(kT)$ is assumed to be zero for $|k| \geq 2MN$. For instance, a moving-average (MA) process with order less than or equal to $2MN$ would satisfy this criterion. In particular, this implies $x(nT)$ and $x(nT + 2MNlT + k)$ are independent for $l, k \neq 0$ (since these variables are jointly Gaussian and uncorrelated). Also, for each n, the random variables $\{x(nT + k), |k| < 2MN\}$ are jointly Gaussian with correlation coefficients given by $R_x(kT), |k| <= 2MN$.
2) **(A2):** The perturbations $z(nT)$ are assumed to be zero mean i.i.d process, jointly Gaussian with (and independent of) $x(nT)$. The power of $z(nT)$ is σ_z^2.

We will suppress T in our notations. Assumptions (A1) and (A2) imply that $x_1[n,l]$ and $x_2[n,l]$ are jointly Gaussian for each n, l satisfying $E(x_1[n,l]x_2[m,l]) = \mathbf{R_x}(Nm - Mn) + \sigma_z^2\delta(Nm - Mn)$. In practice however, we estimate the autocorrelation sequence using L such observations as

$$\hat{R}_x(Nm - Mn) = \frac{1}{L}\sum_{l=1}^{L} x_1[n,l]x_2[m,l] \tag{3}$$

with $0 \leq m \leq N-1$, and $0 \leq n \leq 2M-1$. Our goal is to understand how *perturation and effect of finite samples* jointly influence the estimation of the correlation values using coprime samplers. The following theorem explicitly characterize such an effect:

Theorem 1. *Under assumptions (A1) and (A2) on $x(t)$, for each $k = Nm - Mn, 0 \leq n \leq 2M-1, 0 \leq m \leq N-1$, the perturbed autocorrelation $\hat{R}_x[k]$ estimated using L samples, differs from the actual autocorrelation $R_x(k)$ as*

$$P\left(|\hat{R}_x(k) - R_x(k)| > \epsilon\right) \tag{4}$$

$$\leq 2\epsilon^{L/2}\exp\left\{-\frac{\epsilon L}{R(0) + \sigma_z^2 + |R_x(k)|}\right\}, \epsilon \gg |\rho| \tag{5}$$

Proof. Since $R_x(k) = 0, |k| \geq 2MN$ samples, the random variables $z_{mn}[l] \triangleq x_1[n,l]x_2[m,l], l = 1, 2, \cdots L$ are jointly Gaussian and uncorrelated, and hence independent. Also, $E((x_1[n,l])^2) = E((x_2[m,l])^2) = R_x(0) + \sigma_z^2$, and $E(x_1[n,l]x_2[m,l]) = R_x(Nm - Mn) + \sigma_z^2\delta(Nm - Mn)$. Using Chernoff bound on $\hat{R}_x[Nm - Mn] = \frac{1}{L}\sum_{l=1}^{L} z_{mn}[l]$ (See Appendix VII), we can obtain the concentration inequality (4), in which we replaced a with $\epsilon/\left(R_x(0) + \sigma_z^2\right)$, and ρ with $(R_x(Mn-Nm)+\sigma_z^2\delta(Nm-Mn))/\left(R_x(0) + \sigma_z^2\right)$. $\square$

The result shows that for a given perturbation σ_z^2, the probability of large deviation of $\hat{R}_x(k)$ from its mean decays at least exponentially with L with a decay rate that is explicitly characterized by the perturbation strength. We will look into this result in more detail in [13].

III. Co-prime Sampling for Line Spectrum Estimation

In this section, we consider a specific class of WSS signals whose spectrum consists of lines, representing frequencies of sinusoids buried in noise. We briefly review the problem of estimating these frequencies using the concept of co-prime sampling [2]. Consider a signal $x(t)$ composed of K complex sinusoids, i.e., $x(t) = \sum_{k=1}^{K} A_k e^{j(2\pi f_k t + \phi_k)}$. Assume that the phases ϕ_k are uniformly distributed on $[0, 2\pi]$. The signal is sampled using two A/D converters operating at rates $\frac{1}{MT}$ and $\frac{1}{NT}$, in which $1/T = 2f_{\max}$ is the Nyquist rate, yielding

$$x_1[n] = \sum_{k=1}^{K} A_k e^{jw_k Mn + j\phi_k} + z_n \tag{6}$$

$$x_2[m] = \sum_{k=1}^{K} A_k e^{jw_k Mm + j\phi_k} + z_m, \tag{7}$$

where $w_k = 2\pi f_k T$. Now, let us construct the vectors $\mathbf{y}_1[l] = [x_1[2Nl] \;\; x_1[2Nl+1] \;\; \ldots \;\; x_1[2Nl+N-1]]^T$, $\mathbf{y}_2[l] = [x_2[2Ml+1] \;\; x_2[2Ml+2] \;\; \ldots \;\; x_2[2Ml+2M-1]]^T$, and $\mathbf{y}[l] = [\mathbf{y}_1[l]^T \;\; \mathbf{y}_2[l]^T]^T$. Following [2], the autocorrelation matrix of $\mathbf{y}$ can be derived as

$$\mathbf{R_y} = E\left(\mathbf{y}\mathbf{y}^H\right) = \sum_{k=1}^{K} A_k^2\mathbf{B}(w_k) + \sigma^2\mathbf{I} \tag{8}$$

where $\mathbf{B}(w_k) = \mathbf{a}(w_k)\mathbf{a}(w_k)^H$, $\mathbf{a}(w_k) = [\mathbf{a}_M(w_k)^T \;\; \mathbf{a}_N(w_k)^T]^T$ with $\mathbf{a}_M(w_k) = [1 \;\; e^{jw_kM} \;\; e^{2jw_kM} \;\; \ldots \;\; e^{jw_kM(N-1)}]$, $\mathbf{a}_N(w_k) = [e^{jw_kN} \;\; e^{2jw_kN} \;\; \ldots \;\; e^{jw_kN(2M-1)}]$. One can write the (8) in the vectorized form to get

$$\text{vec}(\mathbf{R_y}) = \sum_{k=1}^{K} \mathbf{b}(w_k)A_k^2 + \sigma^2\,\text{vec}(\mathbf{I}), \tag{9}$$

where $\mathbf{b}(w_k) = \mathbf{a}^*(w_k) \otimes \mathbf{a}(w_k)$, and $\otimes$ denotes the Kronecker product. The elements of $\mathbf{b}(w_k)$ have the form e^{jw_kp}, where p takes all the integer values between 0 and MN. This gives rise to possibility of detecting $O(MN)$ sinusoids by modifying the MUSIC algorithm [2].

A. Robustness to the Perturbation in Sampling Instants

Assume that due to imperfections of the A/D converters, the samples are picked with a random jitter. The perturbed samples can then be written as

$$\tilde{x}_1[n] = \sum_{k=1}^{K} A_k e^{jw_k(Mn + \delta_1[n]) + j\phi_k} + z_n \tag{10}$$

$$\tilde{x}_2[m] = \sum_{k=1}^{K} A_k e^{jw_k(Nm + \delta_2[m]) + j\phi_k} + z_m, \tag{11}$$

where $\delta_1[n], \delta_2[m]$ are i.i.d random variables distributed uniformly in $[-\frac{\rho}{2}, \frac{\rho}{2}]$. In this case, we can write $\tilde{\mathbf{y}}[k]$ as

$$\tilde{\mathbf{y}}[l] = \sum_{k=1}^{K} \tilde{\mathbf{a}}_k[l]e^{j2MNl + \phi_k}. \tag{12}$$

Here, $\tilde{\mathbf{a}}_k[l] = \mathbf{a}(w_k) \circ \mathbf{p}_k[l]$, where $\circ$ denotes the element-wise product, and $\mathbf{p}_k[l] = [e^{jw_k\delta_1[2Nl]} \;\; e^{jw_k\delta_1[2Nl+1]} \;\; \ldots \;\; e^{jw_k\delta_1[2Nl+N-1]} \;\; e^{jw_k\delta_2[2Ml]} \;\; e^{jw_k\delta_2[2Ml+1]} \;\; \ldots \;\; e^{jw_k\delta_2[2Ml+2M-1]}]^T$ is a vector consisting of the samples of $\delta_1[n]$ and $\delta_2[m]$ constructed the same way $\mathbf{y}$ is composed of the elements of $x_1[n]$ and $x_2[m]$. The perturbed autocorrelation matrix is given by the following theorem.

Theorem 2. *In the presence of random jitter in the sampling, the perturbed autocorrelation matrix is given by*

$$\tilde{\mathbf{R}}_{\mathbf{y}} = \sum_{k=1}^{K} A_k^2 \mathbf{B}(w_k) \circ \mathbf{E}_k + \sigma^2 \mathbf{I} \tag{13}$$

where $\mathbf{E}_k$ *is a matrix with ones on the diagonal and* $\mathrm{sinc}^2(w_k\rho/2)$ *elsewhere.*

Proof. The perturbed autocorrelation matrix is obtained by

$$\tilde{\mathbf{R}}_{\mathbf{y}} = E\left(\tilde{\mathbf{y}}[l]\tilde{\mathbf{y}}^H[l]\right) = E_\delta\left(\sum_{k=1}^{K} A_k^2 \tilde{\mathbf{B}}_k[l] + \sigma^2\mathbf{I}\right),$$

where $\tilde{\mathbf{B}}_k[l] = \tilde{\mathbf{a}}_k[l]\tilde{\mathbf{a}}_k^H[l]$ and $\tilde{\mathbf{B}}_k[l]$ can be written as $\tilde{\mathbf{B}}_k[l] = \mathbf{B}(w_k)\circ\mathbf{P}_k[l]$ with $\mathbf{P}_k[l] = \mathbf{p}_k[l]\mathbf{p}_k^H[l]$. The diagonal elements of the matrix $\mathbf{P}_k[l]$ are all 1 and the off diagonal elements are of the form $e^{jw_k\beta_{rs}[l]}, 1 \le r, s \le N+2M-1, r \ne s$, in which $\beta_{rs}[l]$ is the difference of two independent random variables with uniform distribution in $[-\frac{\rho}{2}\ \frac{\rho}{2}]$. As a result, the pdf of $\beta_{rs}[l]$ will be a triangular function spanning from $-\rho$ to ρ:

$$f_{\beta_{rs}[l]}(\beta) = \begin{cases} \frac{1}{\rho^2}(\rho+\beta) & \beta < 0 \\ \frac{1}{\rho^2}(\rho-\beta) & \beta \ge 0 \end{cases}$$

Hence, with integration we obtain $E_\delta(e^{jw_k\beta_{rs}[l]}) = \left(\frac{\sin(w_k\rho/2)}{w_k\rho/2}\right)^2 \triangleq \mathrm{sinc}^2(w_k\rho/2)$. This leads to (13) in which $\mathbf{E}_k$ is a matrix with ones on the diagonal and $\mathrm{sinc}^2(w_k\rho/2)$ elsewhere. □

Corollary 1. *Let* $\rho w_k \ll 1$ *for all* k*. The deviation of perturbed autocorrelation matrix* $\tilde{\mathbf{R}}_{\mathbf{y}}$ *from the ideal autocorrelation matrix* $\mathbf{R}_{\mathbf{y}}$ *is given by*

$$\|\mathbf{R}_{\mathbf{y}} - \tilde{\mathbf{R}}_{\mathbf{y}}\|_F \le (N+2M-1)\frac{\rho^2}{12}\sqrt{K\sum_{k=1}^{K} A_k^4 w_k^4} \tag{14}$$

Proof. Using (13), we obtain $\mathbf{R}_{\mathbf{y}} - \tilde{\mathbf{R}}_{\mathbf{y}} = \sum_{k=1}^{K} A_k^2\mathbf{B}(w_k)\circ(\mathbf{1}-\mathbf{E}_k)$ in which $\mathbf{1} \in \mathbb{C}^{(N+2M-1)\times(N+2M-1)}$ is an all-ones matrix. Each off-diagonal element of $\mathbf{R}_{\mathbf{y}} - \tilde{\mathbf{R}}_{\mathbf{y}}$ can be upper-bounded by

$$\left|\left(\mathbf{R}_{\mathbf{y}} - \tilde{\mathbf{R}}_{\mathbf{y}}\right)_{r,s}\right|^2 \le K\sum_{k=1}^{K} A_k^4\left(1-\mathrm{sinc}^2(\frac{\rho w_k}{2})\right)^2, \tag{15}$$

and the diagonal entries of $\mathbf{R}_{\mathbf{y}} - \tilde{\mathbf{R}}_{\mathbf{y}}$ are obviously equal to zero. Hence, we immediately get

$$\|\mathbf{R}_{\mathbf{y}} - \tilde{\mathbf{R}}_{\mathbf{y}}\|_F \le (N+2M-1)\sqrt{K\sum_{k=1}^{K} A_k^4\left(1-\mathrm{sinc}^2(\frac{\rho w_k}{2})\right)^2}$$

Using the assumption $\rho w_k \ll 1$ for all k, we can approximate $1-\mathrm{sinc}^2(\frac{\rho w_k}{2})$ as $\frac{\rho^2 w_k^2}{12}$ to obtain (14). □

IV. Identifiability in Presence of Spatial Perturbation: A Cramer Rao Bound Based Study

We now turn to studying the effect of perturbation in spatial coprime sampling, in the context of direction-of-arrival (DOA) estimation. We consider a grid-based model for the DOA estimation problem and examine the effect of perturbation in the location of the sensors to the estimated DOAs. Consider an array of M sensors receiving signals from K uncorrelated narrowband sources. In the grid-based model, the range of all possible source directions is quantized into N_θ grid points. Denoting δ_m to be perturbation corresponding to the $m$the sensor, the received signal model can then be written as

$$\mathbf{y}[l] = \mathbf{A}_{\mathrm{grid}}\mathbf{x}[l] + \mathbf{w}[l] \tag{16}$$

in which $\mathbf{y}[l], \mathbf{w}[l] \in \mathbb{C}^{M\times 1}$, $\mathbf{x}[l] \in \mathbb{C}^{N_\theta\times 1}$, $\mathbf{A}_{\mathrm{grid}} = [\mathbf{a}(\theta_1)\ \mathbf{a}(\theta_2)\ \dots\ \mathbf{a}(\theta_{N_\theta})]$, θ_i's are the grid points, and $\mathbf{a}(\theta) \in \mathbb{C}^{M\times 1}$ is the steering vector for the direction θ, whose mth element is given by $e^{j\pi(d_m+\delta_m)\sin(\theta)}$ with d_m denoting the location of the mth sensor of the array. The correlation matrix of the received signals can be written as

$$\mathbf{R}_{\mathbf{y}} = \mathbf{A}_{\mathrm{grid}}\mathbf{R}_{\mathbf{x}}\mathbf{A}_{\mathrm{grid}}^H + \sigma_w^2\mathbf{I}. \tag{17}$$

We can also write the vectorized form of (18) to obtain

$$\mathrm{vec}(\mathbf{R}_{\mathbf{y}}) = \mathbf{A}_{\mathrm{ca}}\boldsymbol{\gamma} + \sigma_w^2\,\mathrm{vec}(\mathbf{I}). \tag{18}$$

where $\mathbf{A}_{\mathrm{ca}} = \mathbf{A}_{\mathrm{grid}}^* \odot \mathbf{A}_{\mathrm{grid}}$ is the co-array manifold with $\odot$ denoting column-wise Khatri-Rao matrix product, and $\boldsymbol{\gamma}$ is the diagonal of $\mathbf{R}_{\mathbf{x}}$. For certain structure of arrays such as nested and coprime arrays, it is shown that we can resolve up to $O(M^2)$ sources using only M sensors. In the following sections, we examine the effect of array perturbations (uncertainty about the sensor locations) on the DOA estimation, by studying the Cramér Rao bound for the perturbed model.

A. Cramér Rao Bound

Let us denote $\mathbf{W} = {\mathbf{R}_{\mathbf{y}}}^{-T} \otimes {\mathbf{R}_{\mathbf{y}}}^{-1}$, $\mathbf{H}_\delta = [\mathrm{vec}(\mathbf{R}_{\delta_2})\ \mathrm{vec}(\mathbf{R}_{\delta_3})\ \dots\ \mathrm{vec}(\mathbf{R}_{\delta_M})]$ with $\mathbf{R}_{\delta_i} \triangleq \frac{\partial\mathbf{R}_{\mathbf{y}}}{\partial\delta_i} = \mathbf{A}\mathbf{R}_{\mathbf{x}}\mathbf{D}_{\boldsymbol{\delta}_i}^H + \mathbf{D}_{\boldsymbol{\delta}_i}\mathbf{R}_{\mathbf{x}}\mathbf{A}^H$, and $\mathbf{D}_{\boldsymbol{\delta}_i} = \frac{\partial\mathbf{A}}{\partial\delta_i}$. Also, let $\Pi_{\mathbf{X}}^{\perp} = \mathbf{I} - \mathbf{X}\mathbf{X}^\dagger$ denote the projection onto null-space of a matrix $\mathbf{X}$, and $(.)^\dagger$ represent the Moore-Penrose pseudo-inverse. Then, the following theorem provides a closed form for the Cramér Rao bound for the perturbed model (16).

Theorem 3. *Defining* $\boldsymbol{\psi} = [\boldsymbol{\gamma}^T\ \boldsymbol{\delta}^T]^T$ *as the parameters to be estimated, the Cramer Rao lower bound is given by*

$$\frac{1}{L}\left(CRB_{\gamma\gamma}\right)^{-1} = \mathbf{A}_{\mathrm{ca}}^H\mathbf{W}^{1/2}\Pi_{\mathbf{W}^{1/2}\mathbf{H}_\delta}^{\perp}\mathbf{W}^{1/2}\mathbf{A}_{\mathrm{ca}} \tag{19}$$

Proof. For Gaussian distributed random variables with covariance matrix $\mathbf{R}_{\mathbf{y}}$, the Fisher Information Matrix (FIM) can be derived as

$$\mathbf{J}_{ij} = \mathrm{vec}(\frac{\partial\mathbf{R}_{\mathbf{y}}}{\partial\psi_i})^H({\mathbf{R}_{\mathbf{y}}}^{-T}\otimes{\mathbf{R}_{\mathbf{y}}}^{-1})\,\mathrm{vec}(\frac{\partial\mathbf{R}_{\mathbf{y}}}{\partial\psi_j}) \tag{20}$$

The Fisher information Matrix (FIM) is

$$\mathbf{J} = \begin{pmatrix} \mathbf{J}_{\gamma\gamma} & \mathbf{J}_{\gamma\boldsymbol{\delta}} \\ \mathbf{J}_{\gamma\boldsymbol{\delta}}^H & \mathbf{J}_{\boldsymbol{\delta}\boldsymbol{\delta}} \end{pmatrix} \tag{21}$$

Similar to [14], we get

$$\mathbf{J}_{\gamma\gamma} = \mathbf{A}_{\mathrm{ca}}^H\mathbf{W}\mathbf{A}^{\mathrm{ca}}, \quad \mathbf{J}_{\gamma\boldsymbol{\delta}} = \mathbf{A}_{\mathrm{ca}}^H\mathbf{W}\mathbf{H}_\delta, \quad \mathbf{J}_{\boldsymbol{\delta}\boldsymbol{\delta}} = \mathbf{H}_\delta^H\mathbf{W}\mathbf{H}_\delta$$

Computing the Schur-complement of the FIM, the block of CRB matrix corresponding to the source powers can be derived as

$$\frac{1}{L}\left(\mathrm{CRB}_{\boldsymbol{\gamma\gamma}}\right)^{-1} = \mathbf{J}_{\boldsymbol{\gamma\gamma}} - \mathbf{J}_{\boldsymbol{\gamma\delta}}\mathbf{J}_{\boldsymbol{\delta\delta}}^{-1}\mathbf{J}_{\boldsymbol{\gamma\delta}}^{H} \tag{22}$$

$$= \mathbf{A}_{\mathrm{ca}}^{H}\mathbf{W}\mathbf{A}_{\mathrm{ca}} - \mathbf{A}_{\mathrm{ca}}^{H}\mathbf{W}\mathbf{H}_{\delta}\left(\mathbf{H}_{\delta}^{H}\mathbf{W}\mathbf{H}_{\delta}\right)^{-1}\mathbf{H}_{\delta}^{H}\mathbf{W}\mathbf{A}_{\mathrm{ca}} \tag{23}$$

$$= \mathbf{A}_{\mathrm{ca}}^{H}\mathbf{W}^{1/2}\Pi^{\perp}_{\mathbf{W}^{1/2}\mathbf{H}_{\delta}}\mathbf{W}^{1/2}\mathbf{A}_{\mathrm{ca}} \tag{24}$$

□

Corollary 2. *If $N_\theta > \mathrm{rank}(\mathbf{A}_{\mathrm{ca}})$, $(CRB_{\boldsymbol{\gamma\gamma}})^{-1}$ is singular.*

Proof. A sufficient condition for the matrix $(\mathrm{CRB}_{\boldsymbol{\gamma\gamma}})^{-1}$ to be singular is that $\mathbf{A}_{\mathrm{ca}}$ be rank deficient. This is true if $N_\theta > \mathrm{rank}(\mathbf{A}_{\mathrm{ca}})$. □

The singularity of the CRB matrix implies the **non identifiability** of parameters [15]. Denoting M_{ca} as the number of distinct elements in the co-array of the given sensor array, it is easy to see that $\mathbf{A}_{\mathrm{ca}}$ is full rank if $N_\theta \leq M_{ca}$. However, in presence of perturbation, this may not still hold. Since the rank of $\mathbf{A}_{\mathrm{ca}}$ plays crucial role in deciding the identifiability of γ, we next investigate how perturbation affects the rank of $\mathbf{A}_{\mathrm{ca}}$ and if it is still possible to argue that it is full rank with high probability.

B. The probability of rank deficiency in the presence perturbations

In this section, we calculate the probability of the event under which the perturbed coarray $\mathrm{rank}(\mathbf{A}_{\mathrm{ca}})$ has rank at least M_{ca}. Assuming that all the *non-zero* singular values of $\bar{\mathbf{A}}_{\mathrm{ca}}$ are simple, we can linearly approximate the kth singular value of $\mathbf{A}_{\mathrm{ca}}$ as

$$\hat{\sigma}_m = \bar{\sigma}_k + \sum_{i=1}^{M}\bar{\mathbf{u}}_k^H\mathbf{D}_{\boldsymbol{\delta}_i}\bar{\mathbf{v}}_k\delta_i \tag{25}$$

in which $\bar{\mathbf{u}}_k$ and $\bar{\mathbf{v}}_k$ denote the kth left and right singular vector of the unperturbed co-array $\bar{\mathbf{A}}_{\mathrm{ca}}$, respectively, and $\mathbf{D}_{\boldsymbol{\delta}_i} = \frac{\partial \mathbf{A}_{\mathrm{ca}}}{\partial \delta_i}\big|_{\boldsymbol{\delta}=0}$. Using this approximation, the probability of the event that $\bar{\mathbf{A}}_{\mathrm{ca}}$ does not lose rank under the perturbations can be lower bounded using a union bound as follows:

$$P\left(\mathrm{rank}(\mathbf{A}_{\mathrm{ca}}) \geq M_{\mathrm{ca}}\right) \geq P\left(\bigcap_{m=1}^{M_{\mathrm{ca}}}\hat{\sigma}_m > 0\right)$$

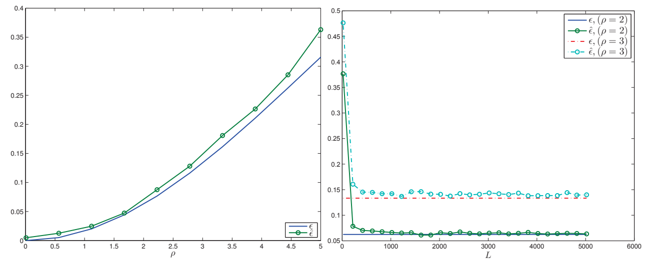
$$\geq 1 - P\left(\bigcup_{m=1}^{M_{\mathrm{ca}}}\hat{\sigma}_m \leq 0\right) \geq 1 - \sum_{m=1}^{M_{\mathrm{ca}}}P\left(\hat{\sigma}_m \leq 0\right) \tag{26}$$

Moreover, since the perturbation are assumed to be bounded on $[-\frac{\rho}{2}, \frac{\rho}{2}]$, using a Hoeffding bound we get

$$P\left(\hat{\sigma}_m \leq 0\right) = P\left(\sum_{i=1}^{M_{\mathrm{ca}}}\bar{\mathbf{u}}_m^H\mathbf{D}_{\boldsymbol{\delta}_i}\bar{\mathbf{v}}_m\delta_i \leq -\bar{\sigma}_m\right)$$

$$\leq e^{-\frac{2\bar{\sigma}_m^2}{\rho^2\sum_{i=1}^{M}\left(\bar{\mathbf{u}}_m^H\mathbf{D}_{\boldsymbol{\delta}_i}\bar{\mathbf{v}}_m\right)^2}} \tag{27}$$

Plugging (27) into (26) gives the probability under which the perturbed co-array manifold loses rank under perturbations. We observe that as long as the perturbations are small, the perturbed co-array matrix $\mathbf{A}_{\mathrm{ca}}$ will have rank at least M_{ca} with high probability.



(a) Deviation vs ρ (b) Deviation vs L, for fixed $\rho = 2$, $\rho = 3$.

Fig. 1. ϵ designates the deviation $\|\mathbf{R_y} - \tilde{\mathbf{R}}_{\mathbf{y}}\|_F / \|\mathbf{R_y}\|_F$ for the autocorrelation. $\hat{\epsilon}$ is the same quantity for the sample autocorrelation matrices.

V. Simulations

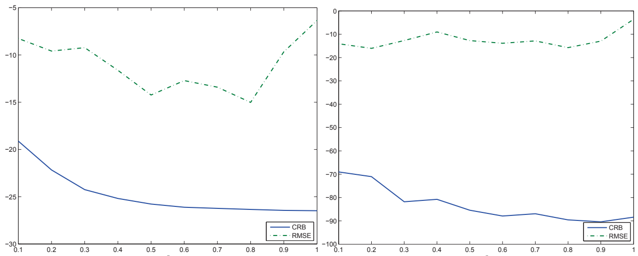
We perform simulations to confirm some of the theoretical claims developed in this paper. Figure 1a shows the deviation of the autocorrelation matrix ($\|\mathbf{R_y} - \tilde{\mathbf{R}}_{\mathbf{y}}\|_F / \|\mathbf{R_y}\|_F$) due to perturbations in sampling instants calculated using equation (13). The same quantity is plotted for the *sample* autocorrelation matrix for $L = 500$ samples, $M = 3, N = 7$, and 10 sinusoids with frequencies uniformly distributed between 10Hz and 200Hz, $T = 5 \times 10^{-4}$. We observed that the empirical values match with what we obtained in equation (13). Each plot in Figure 1b illustrate this deviation versus the number of samples L, for fixed values of ρ. As we increase L, the empirical values get close to the expected covariance matrices.

Figure 2a shows the *averaged* Cramér Rao bound (CRB) for a co-prime array with respect to ρ, which is the range of perturbations. Also, $M = 3, N = 7, N_\theta = 50$, and $L = 1000$. By *averaged* CRB we mean the inverse of averaged Fisher Information Matrix over several realizations of uniformly distributed random perturbations. The number of sources fixed to be $K = 10$. The sources are uniformly distributed on the grid with unit powers, i.e., the $\lfloor \frac{kN_\theta}{K} \rfloor$ element of γ is one, and the rest are zero ($k = 1 \ldots K$). The Root Mean Square Error (RMSE) of the estimated source powers using our algorithm [12] is also compared with the CRB. Figure 2b shows the same plot for a ULA array of 12 sensors with the same setting except for $N_\theta = 20$, and $K = 3$.

Figure 3a shows the probability of $\mathbf{A}_{\mathrm{ca}}$ having rank at least K (using equation (26)) for various K values, different array structures and different ρ's. Also, 3b demonstrates the probability of $\mathbf{A}_{\mathrm{ca}}$ having rank at least M_{ca} with respect to different ρ's. As it can be seen, for small perturbations, the perturbed co-array is guaranteed to preserve the rank of M_{ca}. Note that this plot is a lower bound for the actual probability.

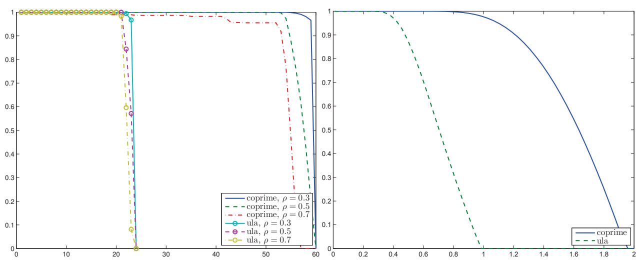
VI. Conclusion

The effects of additive perturbation and jitter in coprime sampling are studied. It is shown that such non idealness in sampling leads to errors in the estimated correlation which can be bounded under certain mild assumptions on the spectrum of the underlying WSS process. The robustness of coprime sampling is thereby established for a generic class of WSS signals, as well as for line spectrum processes, under small values of the perturbation. The issue of identifiability in spatial spectrum sensing is also addressed and it is shown that a



(a) Coprime array (b) ULA

Fig. 2. The CRB, and the RMSE of the recovered source powers. The source powers are recovered using our algorithm [12]



(a) $P\left(\mathrm{rank}(\mathbf{A}_{\mathrm{ca}}) \geq K\right)$. (b) $P\left(\mathrm{rank}(\mathbf{A}_{\mathrm{ca}}) \geq M_{\mathrm{ca}}\right)$.

Fig. 3. The lowerbound for the probability of $\mathbf{A}_{\mathrm{ca}}$ having rank at least K (Fig. 3a), and M_{ca} (Fig. 3b), for different values of ρ.

perturbed coprime sensor array (with unknown perturbations) can still identify $O(M^2)$ sources with high probability. Future research in this direction will aim at tightening some of the bounds derived in this paper and proposing robust algorithms for perturbed coprime sampling.

VII. Appendix

Following [16], the moment generating function of product of two zero mean correlated Gaussian variables with unit variance is given by

$$M_y(t) = \frac{1}{\sqrt{(1-\rho^+ t)(1+\rho^- t)}} \tag{28}$$

in which $Y = X_1 X_2$, and X_1, X_2 are the Gaussian variables, $\rho^+ = 1+\rho$, and $\rho^- = 1-\rho$ where $\rho = E(X_1 X_2)$. Assuming that we have L independent such Y variables, we can derive a Chernoff bound

$$P(|\frac{1}{L}\sum_{l=1}^{L} Y_l - \rho| > a) \leq \zeta_+^L + \zeta_-^L \tag{29}$$

where $\zeta_+ = \inf_{t>0}\left\{e^{-(a+\rho)t} M_y(t)\right\}$, $\zeta_- = \inf_{t<0}\left\{e^{-(\rho-a)t} M_y(t)\right\}$. For brevity, we only consider the case where $a > |\rho|$, which is of more interest since we want to derive a tail bound. We will consider other cases in our future work [13]. Taking the derivatives with respect to t, equating with zero, and performing the required operations we obtain

$$\zeta_i = \frac{\sqrt{2}|c_i|}{\sqrt{-\rho' + \sqrt{d_i^2 + 4\rho' c_i a}}} e^{-\frac{\sqrt{d_i^2+4\rho' c_i a} - d_i}{2\rho'}} \tag{30}$$

in which i can be either $+$ or $-$, and $c_+ = \rho + a, c_- = a - \rho, d_+ = 1 + \rho^2 + 2a\rho, d_- = 1 + \rho^2 - 2a\rho, \rho' = 1 - \rho^2$. Considering the asymptotic behavior of ζ_i for $a \gg \rho$, we get

$$\begin{aligned} P(|\frac{1}{L}\sum_{l=1}^{L} Y_l - \rho| > a) &\leq a^{\frac{L}{2}} \left(e^{-\frac{aL}{1+\rho}} + e^{-\frac{aL}{1-\rho}}\right) \\ &\leq 2a^{\frac{L}{2}} e^{-\frac{aL}{1+|\rho|}} \end{aligned} \tag{31}$$

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