# ANALYSIS AND SIMULATIONS OF MULTIFRACTAL RANDOM WALKS

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**ABSTRACT**

Multifractal time series, characterized by a scale invariance and large fluctuations at all scales, are found in many fields of natural and applied sciences. Here we consider a quite general type of multifractal time series, called multifractal random walk, as non stationary stochastic processes with intermittent stationary increments. We first quickly recall how such time series can be analyzed and characterized, using structure functions and arbitrary order Hilbert spectral analysis, and then we discuss the simulation approach. Here we review recent works on this topic. We provide an unification of the works published, and discuss how to choose parameters in stochastic simulations in order to simulate a multifractal series with desired properties. In the lognormal framework we provide a new $h-\mu$ plane expressing the scale invariant properties of these simulations.

***Index Terms*—** Scaling, Multifractal random walks, Intermittency, Stochastic modeling, Time series

## 1. INTRODUCTION: ANALYSIS OF MULTIFRACTAL TIME SERIES

We consider here the properties of a time series $X(t)$, assumed to have scaling statistics and intermittent fluctuations. In this section we quickly present the data analysis part, first the classical Fourier approach and structure functions, then Empirical Mode Decomposition associated with Hilbert spectral analysis. In the next section we discuss the modeling part.

### 1.1. Fourier analysis and structure functions

The scaling is usually revealed through a Fourier spectrum of the form $E_X(f) = Cf^{-\beta}$, where $C$ is a constant and $\beta > 0$ is the spectral exponent. This can also be done using wavelet analysis. This spectral analysis usually helps to detect the scaling range of the data, providing a smaller and larger scale (frequency scale or temporal scale) for the scaling regime. On this range of scales, the intermittency is classically expressed and characterized using the structure functions. We consider the moments of order $q > 0$ of fluctuations at scale $\ell$, $M_\ell(q) = \langle |X(t+\ell) - X(t)|^q \rangle$ of $X(t)$, called structure functions, where $\ell$ is the scale belonging to the scaling range. Since we are dealing with scaling processes that have stationary increments, we expect the following scaling behavior:

$$M_\ell(q) = A_q \ell^{\zeta(q)} \tag{1}$$

where $A_q$ is a parameter independent from the scale, and $\zeta(q)$ the scaling moment function, with $\beta = 1 + \zeta(2)$, and $1 < \beta < 3$, required by the convergency condition. The knowledge of the full $\zeta(q)$ function then provides much more information than the single parameter $\beta$. Some completely different stochastic processes may possess the same spectral exponent, showing that, when doing data analysis and model assessment, estimation of $\zeta(q)$ on a full range of values is much more useful than only estimating the single parameter $\beta$. Figure 1 shows the $\zeta(q)$ function for several classical linear stochastic processes (Brownian motion, fractional Brownian motion, Lévy stable motion); a multifractal time series with a nonlinear $\zeta(q)$ function is also shown.

### 1.2. Empirical Mode Decomposition and Hilbert Spectral Analysis

We also present another method, which has been used to characterize multifractal time series [1], based on an amplitude-time-frequency analysis: Empirical Mode Decomposition and Hilbert Spectral analysis.

Empirical Mode Decomposition (EMD) has been introduced in 1998 as a data-driven method especially adapted for nonlinear and non stationary time series [2, 3]. It is a way to decompose a series into a sum of modes, each mode being a time series with a dominant scale. Modes are extracted using an algorithm based on spline interpolation between local maxima and minima (see details in the original paper). After application of this algorithm, the orignal time series is written as:

$$X(t) = \sum_{i=1}^{n-1} C_i(t) + r_n(t) \tag{2}$$

where $C_i(t)$ is the mode number $i$ and $r_n(t)$ is the residual, a monotonic function.

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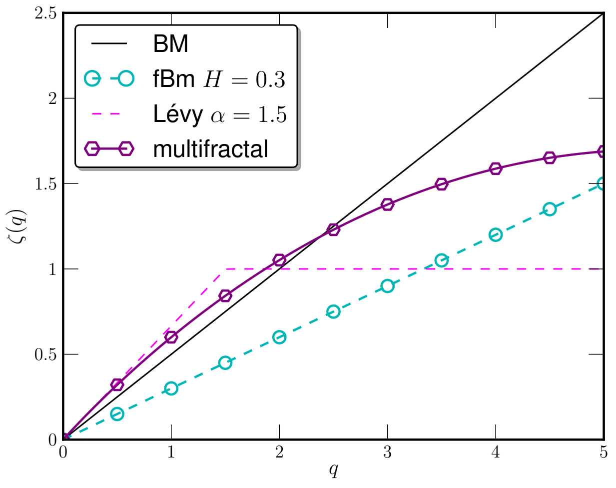


**Fig. 1.** Typical moment functions $\zeta(q)$ for several cases, Brownian motion, fractional Brownian motion for $H = 0.3$, Lévy stable motion with $\alpha = 1.5$ and a multifractal time series with a nonlinear moment function.

With extracted modes, one can apply the associated Hilbert spectral analysis to each component $C_i$, in order to extract an energy time-frequency information from the data [1]. The Hilbert transform is applied to the mode series $C(t)$ as

$$\tilde{C}(t) = \frac{1}{\pi} \int \frac{C(t')}{t - t'} dt' \quad (3)$$

This enables to construct the analytical signal [4] $\mathbf{C}_i(t) = C_i(t) + j\tilde{C}_i(t) = \mathcal{A}_i(t)e^{j\theta_i(t)}$. This way a local amplitude and phase can be extracted from the mode $C_i(t)$, at each time $t$: $\mathcal{A}_i(t) = [C_i(t)^2 + \tilde{C}_i^2(t)]^{1/2}$ and $\theta_i(t) = \arctan\left(\tilde{C}_i(t)/C_i(t)\right)$. The instantaneous frequency is defined as the derivative of the phase function $\omega_i(t) = \frac{1}{2\pi} d\theta_i(t)/dt$. This procedure is the classical Hilbert spectral analysis [4, 5]. The combination of EMD and HSA is also called Hilbert-Huang Transform (HHT). In general, this Hilbert-based transform can be seen as a generalization of the Fourier transform, since it allows frequency modulation and amplitude modulation simultaneously. The EMD-HSA methodology is a time-energy-frequency approach: it provides at each time $t$, $n$ values of instantaneous amplitude $\mathcal{A}_i(t)$ and instantaneous frequency $\omega_i(t)$, where $n$ is the number of modes.

In recent works (Huang et al., 2008, 2011), we have developed a generalization of the EMD-HSA methodology, to help characterizing the intermittent properties of multifractal random walk time series in the amplitude-frequency space. At each time step, and for each $n$ modes, a local amplitude $\mathcal{A}$ and a local frequency $\omega$ are extracted. This can be used to generate a joint probability density function (pdf) $p(\omega, \mathcal{A})$. With the joint pdf $p(\omega, \mathcal{A})$, the Hilbert marginal spectrum is estimated as $h(\omega) = \int_0^{+\infty} p(\omega, \mathcal{A})\mathcal{A}^2 d\mathcal{A}$. This definition is a second-order statistical moment. In order to study the intermittency of the time series fluctuations, it is thus natural to generalize this approach to arbitrary-order moment $q \geq 0$ [1, 6, 7]:

$$\mathcal{L}_q(\omega) = \int_0^{+\infty} p(\omega, \mathcal{A})\mathcal{A}^q d\mathcal{A} \quad (4)$$

In the case of scale invariance, the following power-law behavior is expected:

$$\mathcal{L}_q(\omega) \sim \omega^{-\xi(q)} \quad (5)$$

in which $\xi(q)$ is the Hilbert-based scaling exponent function. Due to the integration operator, $\xi(q) - 1$ can be associated with $\zeta(q)$ from structure-function analysis.

This provides another way to extract the scaling exponent function $\zeta(q)$, which is useful when there are energetic structures at a given scale (i.e. large scale forcing) since in such situation the structure function approach fails [1, 6, 7].

## 2. GENERATION OF A NON-STATIONARY MULTIFRACTAL TIME SERIES

The scaling property presented above has been given several denominations in the last thirty years. It has been called "non-stationary" multifractals [8], "non-conservative" multifractals [9], "multiaffine" field [10–12], and more recently, "multifractal random walk" [13, 14]. We shall use here the name "multifractal random walk", since it has gained popularity in the fields of continuous simulations and financial modelling using multifractal models.

The first proposals, in the 1990s, to generate such time series were discrete constructions [10, 11, 15–18]. They mostly consisted of taking the modelled non-stationary field (e.g. turbulent velocity) as a sum of a multiplicative and correlated positive field, and a random signed term. These models did not have straightforward continuous expressions. In 2001 a model was proposed, mainly with an objective of financial applications, called "multifractal random walk" [13]. It was mainly related to the proposal of [15]: the process is the discrete sum of a product of gaussian terms (to express the sign of the fluctuations) and of correlated lognormal terms.

The multifractal random walk was the continuous limit of this discrete sum. The objective was to model financial fluctuations, but this process had more general applications, and in all fields with all values of $H = \zeta(1)$ (see below).

The first fully continuous (log-ID) multifractal constructions were published in the late 1990s and early 2000s: there have been several proposals, belonging to two different families to generate positive multifractal measures or multifractal random walks using continuous stochastic processes. We consider here the latter construction.

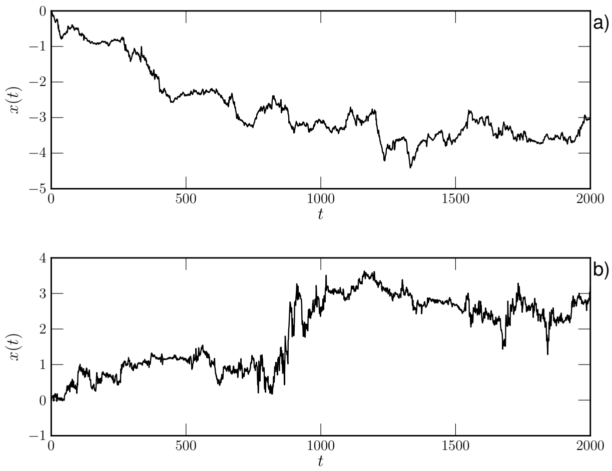


**Fig. 2**. Examples of moving average simulations of a multifractal process, with a lognormal multiplicative cascade with $\mu = 0.2$, and values of $h = 0.5$ (above), and $h = 0.3$ (below)

### 2.1. Moving average generation of a non-stationary multifractal time series

A few recent papers have considered the generation of a non-stationary multifractal time series (also called multiaffine or multifractal random walk), using a moving average representation of the form [19–21]:

$$X(t) = \int_0^t \epsilon(u) dY_h(u) \tag{6}$$

where $\epsilon$ is a kernel having multiplicative scaling properties and $Y_h(t)$ is a self-similar process of parameter $h$, independent of $\epsilon$ (meaning that $(Y_h(at))$ has the same distribution as $a^h(Y_h(t))$). When taking $Y_h$ as a fractional Brownian motion, the first point is to construct a stochastic integral with respect to fBm, and show that it is well defined and not diverging. This was done in [20] for $h > 1/2$.

Abry et al. [19] further explored the case $h > 1/2$ using fractional Wiener integrals. In such a situation, the process generated is shown to be converging and different from that previously produced. Let us note $K(q)$ the scaling moment function of $\epsilon$: at scale $\ell$ the moments of $\epsilon_\ell$ write

$$\langle \epsilon_\ell^q \rangle \sim \ell^{-K(q)} \tag{7}$$

and $\mu = K(2)$. The result of [19], with our notations, is the following (for $h > 1/2$ only):

$$\zeta_X(q) = \begin{cases} qh - K(q); & \mu \leq 2h - 1 \\ \frac{\mu+1}{2} q - K(q); & 2h < \mu + 1 \end{cases} \tag{8}$$

where $K(q)$ is the scaling exponent for the kernel function. Let us define here the Hurst exponent $H$ as:

$$H = \zeta_X(1) \tag{9}$$

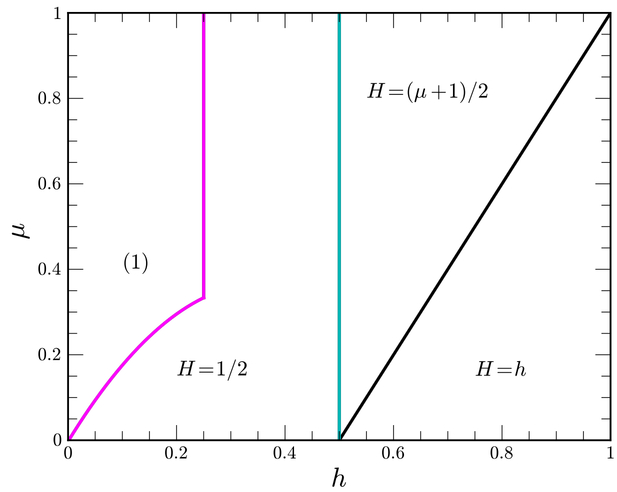


**Fig. 3**. Different values of $H = \zeta_X(1)$ obtained in a $h - \mu$ plane. The region (1) corresponds to a region with unknown value for $H$. The nonlinear (magenta) curve which is at the bottom left corresponds to the condition $K(1/2h) < 1/2h - 1$. The analytical expression of the curved part depends on the model.

When $\mu \leq 2h - 1$, we have $H = \zeta_X(1) = h$, whereas for $2h < \mu + 1$, $H = \zeta_X(1) = \frac{\mu+1}{2}$ and is not related to $h$. Examples of moving average simulations of a multifractal process, following Equation (6) with a lognormal multiplicative cascade with $\mu = 0.2$, and values of $H = 0.3$ and $H = 0.5$, are shown in Figure 2.

The case $h < 1/2$ has only been considered, up to now, in one paper. In the case $h < 1/2$, Perpete [21] has shown that the process is well defined and, using a different method, found the following scaling exponents:

$$\zeta_X(q) = \frac{q}{2} - K(q) \tag{10}$$

with the following conditions:

$$\begin{cases} 0 < h < 1/4 \;\&\; K(\frac{1}{2h}) < \frac{1}{2h} - 1 \\ 1/4 < h < 1/2 \end{cases} \tag{11}$$

This result is surprising since there is no $h$ dependence in the value of $\zeta_X(q)$. We have here in both cases $H = \zeta_X(1) = 1/2$.

The different $H$ values, obtained for all situations, are shown in Figure 3 in a $h - \mu$ plane. In the region (1), there is no result for the moment. The other zones correspond to different values of $H$, which is constant, or depends on $\mu$ or $h$.

### 2.2. The lognormal case

Continuous multifractal models are known to be log-ID (log-infinitely divisible), meaning that the logarithm of the process belongs to an ID law (Poisson, Gaussian, stable, etc.).

The generic continuous multifractal model is the lognormal model, which is often taken as illustration.

In the lognormal case, we have a quadratic expression:

$$K(q) = \frac{\mu}{2}(q^2 - q) \tag{12}$$

and the condition $K(\frac{1}{2h}) < \frac{1}{2h} - 1$ becomes $\mu < f(h)$ with

$$f(x) = \frac{2x(1-2x)}{1-x} \tag{13}$$

Since $0 < h < 1$ and $0 < \mu < 1$, we can plot in the $h - \mu$ plane the result in Figure 4, following the calculations of [19] and [21], given in Equations (8) and (11).

There are four zones in this $h - \mu$ quadrant: from left to right, there is no result in the zone left blank; the nonlinear curve is given by $f(x)$; for the next region we have $\zeta_X(q) = \frac{q}{2} - K(q)$; then $\zeta_X(q) = \frac{\mu+1}{2}q - K(q)$ and finally $\zeta_X(q) = qh - K(q)$.

### 2.3. Generalisation: moving average with the power $a > 0$ of a multiplicative cascade

There is a way to change the value of $\zeta_X(q)$, by taking a power $a > 0$ of the multifractal cascade in the stochastic integral:

$$X(t) = \int_0^t \epsilon^a(u)dY_h(u) \tag{14}$$

The exponent $\zeta_X(q)$ is then modified, as follows [19,21]:

$$\zeta_X(q) = \begin{cases} \frac{q}{2} - K(aq); & 0 < h < 1/4 \ \& \ K(\frac{1}{2h}) < \frac{1}{2h} - 1 \\ \frac{q}{2} - K(aq); & 1/4 < h < 1/2 \\ qh - K_2(q,a); & 1/2 < h \ \& \ K_2(2,a) \leq 2h - 1 \\ \frac{\mu+1}{2}q - K_2(q,a); & 1/2 < h \ \& \ 2h < K_2(2,a) + 1 \end{cases} \tag{15}$$

where $K_2(q,a) = K(aq) - qK(a)$. The introduction of this parameter $a$ thus gives a degree of freedom in the stochastic simulations and helps to explore more possibilities in the simulations.

Let us illustrate this for a lognormal process: in this case we have $K_2(q,a) = a^2K(q)$. The following quadratic form of the scaling moment function $\zeta(q)$ is a generic expression for a lognormal multifractal process:

$$\zeta_X(q) = qH - \frac{\mu'}{2}(q^2 - q) \tag{16}$$

This expression depends on 2 parameters: the first one is the classical Hurst index defined as $H = \zeta_X(1)$ and the second one is the intermittency parameter defined here as $\mu' = 2H - \zeta_X(2)$.

Our question here is: is it possible to retrieve Eq.(16), corresponding to a generic lognormal expression, through a stochastic moving average of the form Eq. (14)? And if yes, for which values of $h$, $a$ and $\mu$ (expressed using the desired parameters $H$ and $\mu'$)? This question is important since the lognormal case is a classical multifractal process, and stochastic simulations able to generate all the possible parameters are useful for many applications. For example for turbulent fields, $H$ is close to 1/3 and $\mu'$ is often smaller than 0.1.

Two cases are considered below, to partly answer this question.

#### 2.3.1. *Case $H < 1/2$*

We need to take $h > 1/2$; in this case the function write $\zeta_X(q) = \frac{q}{2} - K(aq)$ and does not depend on $h$. This gives the values:

$$a = \frac{\mu'}{2H - 1 + \mu'}; \quad \mu = \frac{\mu'}{a^2} \tag{17}$$

where we need to have $\mu' > 1 - 2H$. Thus all values are not accessible: the smaller $H$, the more the process must be intermittent (large $\mu'$).

#### 2.3.2. *Case $H > 1/2$*

Considering $a < 1$, the previous case with $h < 1/2$ will provide $H = 1/2 - K(a) > 1/2$. Hence Eq.(17) can still be used.

Or by taking $h > 1/2$, if $\mu' < 2H - 1$ we have $h = H$. In this case any $a$ and $\mu$ are possible with the condition $a^2\mu = \mu'$.

Finally if $\mu' \geq 2H - 1$, we have:

$$\mu = 2H - 1; \quad a = \sqrt{\frac{\mu'}{2H - 1}} \tag{18}$$

Thus in the case $H > 1/2$, the simulation is possible for all couple of values of $(H, \mu')$.

## 3. CONCLUSION

Multifractal random walks (or non stationary multi fractals, or multi-affine fields) have been obtained for a long time in many fields of applied and natural sciences. We have recalled here how to analyse such time series and to extract the nonlinear moment function $\zeta(q)$.

However, to generate such time series using continuous models has long been a challenge. We have reviewed here some recently obtained results, provided a new synthesis, and considered the situation of a desired quadratic moment function, for the lognormal case taken as generic and classical example of multifractal process.

Such quadratic expression is characterized by the parameters $H$ and $\mu'$ (Eq.(16)), and we have shown how to choose the parameters $h$, $a$ and $\mu$ in the generating expression given by Eq. (14), to retrieve this quadratic curve. Depending on the values of $H$ and $\mu'$, some values are still not accessible; for example for small $H$ and small $\mu$.

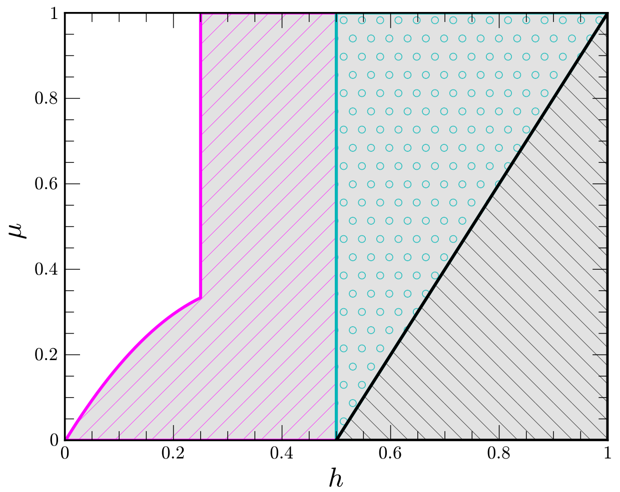


**Fig. 4**. The plane $h-\mu$ giving the value of the scaling exponent $\zeta_X(q)$ for a lognormal process. For the zone which is left in blank, there is no result for the moment. Increasing diagonal: $\zeta_X(q) = \frac{q}{2} - K(q)$; open dots: $\zeta_X(q) = \frac{\mu+1}{2}q - K(q)$ and decreasing diagonals $\zeta_X(q) = qh - K(q)$.

## REFERENCES


[1] Y.X. Huang, F.G. Schmitt, Z.M Lu, and Y.L. Liu, "An amplitude-frequency study of turbulent scaling intermittency using hilbert spectral analysis," *Europhys. Lett.*, vol. 84, pp. 40010, 2008.

[2] N.E. Huang, Z. Shen, S.R. Long, M.C. Wu, H.H. Shih, Q. Zheng, N.C. Yen, C.C. Tung, and H.H. Liu, "The empirical mode decomposition and the hilbert spectrum for nonlinear and non-stationary time series analysis," *Proc. R. Soc. London, Ser. A*, vol. 454, no. 1971, pp. 903–995, 1998.

[3] N.E. Huang, Z. Shen, and S.R. Long, "A new view of nonlinear water waves: The Hilbert Spectrum ," *Annu. Rev. Fluid Mech.*, vol. 31, no. 1, pp. 417–457, 1999.

[4] L. Cohen, *Time-frequency analysis*, Prentice Hall PTR Englewood Cliffs, NJ, 1995.

[5] P. Flandrin, *Time-frequency/time-scale analysis*, Academic Press, 1998.

[6] Y.X. Huang, F. G. Schmitt, Z.M. Lu, and Y.L. Liu, "Analyse de l'invariance d'échelle de séries temporelles par la décomposition modale empirique et l'analyse spectrale de hilbert," *Traitement du Signal*, vol. 25, pp. 481–492, 2008.

[7] Y.X. Huang, F.G. Schmitt, Z.M. Lu, P. Fougairolles, Y. Gagne, and Y.L. Liu, "Second-order structure function in fully developed turbulence," *Phys. Rev. E*, vol. 82, no. 2, pp. 026319, 2010.

[8] D. Schertzer, S. Lovejoy, F. G. Schmitt, Y. Chigirinskaya, and D. Marsan, "Multifractal cascade dynamics and turbulent intermittency," *Fractals*, vol. 5, no. 3, pp. 427–471, 1997.

[9] D. Schertzer and S. Lovejoy, "Physical modeling and analysis of rain and clouds by anisotropic scaling multiplicative processes," *J. Geophys. Res*, vol. 92, no. D8, pp. 9693–9714, 1987.

[10] T Vicsek and Albert-L Barabasi, "Multi-affine model for the velocity distribution in fully turbulent flows," *Journal of Physics A: Mathematical and General*, vol. 24, no. 15, pp. L845, 1991.

[11] Albert-László Barabási, Péter Szépfalusy, and Tamás Vicsek, "Multifractal spectra of multi-affine functions," *Physica A: Statistical Mechanics and its Applications*, vol. 178, no. 1, pp. 17–28, 1991.

[12] Yoshiki Kuramoto, Dorjsuren Battogtokh, and Hiroya Nakao, "Multiaffine chemical turbulence," *Physical review letters*, vol. 81, no. 16, pp. 3543, 1998.

[13] E. Bacry, J. Delour, and JF Muzy, "Multifractal random walk," *Phys. Rev. E*, vol. 64, no. 2, pp. 026103, 2001.

[14] F. G. Schmitt and L. Seuront, "Multifractal random walk in copepod behavior," *Physica A*, vol. 301, no. 1-4, pp. 375–396, 2001.

[15] Jens Eggers and Siegfried Grossmann, "Effect of dissipation fluctuations on anomalous velocity scaling in turbulence," *Physical Review A*, vol. 45, no. 4, pp. 2360, 1992.

[16] R Benzi, L Biferale, A Crisanti, G Paladin, M Vergassola, and A Vulpiani, "A random process for the construction of multiaffine fields," *Physica D: Nonlinear Phenomena*, vol. 65, no. 4, pp. 352–358, 1993.

[17] A Juneja, DP Lathrop, KR Sreenivasan, and G Stolovitzky, "Synthetic turbulence," *Physical Review E*, vol. 49, no. 6, pp. 5179, 1994.

[18] L. Biferale, G. Boffetta, A. Celani, A. Crisanti, and A. Vulpiani, "Mimicking a turbulent signal: Sequential multiaffine processes," *Phys. Rev. E*, vol. 57, pp. R6261–R6264, Jun 1998.

[19] Patrice Abry, Pierre Chainais, Laure Coutin, and Vladas Pipiras, "Multifractal random walks as fractional wiener integrals," *Information Theory, IEEE Transactions on*, vol. 55, no. 8, pp. 3825–3846, 2009.

[20] Carenne Ludena et al., "Lp-variations for multifractal fractional random walks," *The Annals of Applied Probability*, vol. 18, no. 3, pp. 1138–1163, 2008.

[21] Nicolas Perpete, "Construction of multifractal fractional randowm walks with hurst index smaller than 1/2," *Stochastics and Dynamics*, vol. 13, no. 04, 2013.