

# Spark under 2-D Fourier Sampling

Sampurna Biswas, Soura Dasgupta, Mathews Jacob and Raghuraman Mudumbai

**Abstract**—We consider the spark of submatrices of 2D-DFT matrices obtained by removing certain rows and relate it to the spark of associated 1D-DFT submatrices. A matrix has spark  $m$  if its smallest number of linearly dependent columns equals  $m$ . To recover an arbitrary  $k$ -sparse vector, the spark of an observation matrix must exceed  $2k$ . We consider how to choose the rows of the 2D-DFT matrix so that it is full spark, i.e. its spark equals one more than its row dimension. We consider submatrices resulting from two sets of sampling patterns in frequency space: On a straight line and on a rectangular grid. We show that in the latter case full spark is rarely obtainable, though vectors with certain sparsity patterns can still be recovered. In the former case we provide a necessary and sufficient condition for full spark, and show that lines with integer slopes cannot attain it.

**Index Terms**—Coprime sensing, full spark, compressed sensing, two dimensional, Fourier Sampling

## I. INTRODUCTION

This paper derives conditions under which submatrices of a two dimensional DFT matrix, obtained by picking a subset of its rows, have full spark. The spark of a matrix  $\mathbf{A} \in \mathbb{C}^{n \times m}$  is the smallest number of its columns that are linearly dependent. Such a matrix  $\mathbf{A}$  has full spark if  $\text{spark}(\mathbf{A}) = n + 1$ . The application rests in the recovery of sparse signals from their under-sampled linear measurements. Referred to as compressed sensing (CS), this approach has many applications, including MRI [3]–[5] and synthetic aperture radar [6]–[8].

In the general CS setting, the measurement process is modeled as

$$\mathbf{y} = \mathbf{A}\mathbf{x},$$

where  $\mathbf{x} \in \mathbb{C}^m$  is the sparse signal vector to be recovered from the observations  $\mathbf{y} \in \mathbb{C}^n$ . We say that a vector  $\mathbf{x}$  is  $k$ -sparse if it has at most  $k$  nonzero elements.

The observation matrix  $\mathbf{A}$ , is generally fat i.e.  $n < m$ . The condition  $\text{spark}(\mathbf{A}) > 2k$ , ensures that the unique recovery of a  $k$ -sparse  $\mathbf{x}$  is possible from  $\mathbf{y}$ . [9], [10]. Violation of this condition implies there are at least two distinct  $k$ -spark vectors  $\mathbf{z}$  and  $\mathbf{x}$  such that  $\mathbf{A}\mathbf{x} = \mathbf{A}\mathbf{z}$ . The seminal work of Candes and Tao and its extensions show that further restrictions on  $\mathbf{A}$  permit the recovery of  $\mathbf{x}$  through convex  $\ell_1$  optimization [9], [11], [12].

As the acquisition of each measurement comes with a penalty (e.g. acquisition time in MRI), it is beneficial to recover  $\mathbf{x}$  with as few measurements as possible. Thus it

This work is in part supported by US NSF grants EPS-1101284, ECCS-1150801, CNS-1329657, CCF-1302456, CCF-0844812, CCF-1116067, NIH IR21HL109710-01A1, ACS RSG-11-267-01-CCE, and ONR grant N00014-13-1-0202. All but the fifth author are with the Department of Electrical and Computer Engineering, University of Iowa, Iowa City, IA 52242. sampurnakgp@gmail.com, {dasgupta, rmudumbai}@engineering.uiowa.edu, {mathews-jacob}@uiowa.edu. Phone 319-335-5200. Fax 319-335-6028.

is desirable that one design a measurement matrix  $\mathbf{A}$  with the highest possible spark. Computing the spark of a large matrix is clearly intractable. Thus early work on assessing the spark of a matrix relied on certain lower bounds involving the notion of mutual coherence between the columns of  $\mathbf{A}$ , [9], [13]–[16]. As explained in Section II and Section III, in many imaging applications like MRI and synthetic aperture radar, the observation matrices are certain rows of 2-D DFT matrices, [19]. This paper provides some initial results on the spark of such submatrices.

We note that the study of the spark of even 1-D DFT matrices is embryonic. Most advanced results are in [17] and in our own work in [18]. Motivated by developments in *coprime sampling*, [2], [18] in particular ties full spark submatrices of  $N \times N$ , 1-D DFT matrices to certain coprimeness conditions. We demonstrate in this paper that the spark of submatrices of 2-D DFT matrices obtained by certain specific types of sampling in frequency space (known as  $k$ -space) are related to the spark of associated 1-D DFT matrices. The sampling patterns considered are (i) on a line in  $k$ -space and (ii) on a rectangular grid. We observe that in the special case of sampling on a rectangular grid, the 2D-DFT matrix is in fact the Kronecker product of two 1-D DFT matrices. Compressive sensing in such a setting has been considered in [1], though to our knowledge no results are available for spark of such matrices.

Section III defines the sampling patterns we consider. Section IV recounts the results of [17] and a generalization of a result from [18]. Section V characterizes lines on  $k$ -space that result in full spark. One important conclusion is that while lines with rational slopes may lead to full spark observation matrices, lines with integer slopes never do. Section VI considers rectangular grids and shows that unless all rows of the 2-D DFT matrix are retained, full spark is impossible. It however, shows that such a matrix is still useful for the recovery of vectors with certain sparsity patterns. Section VII is the conclusion.

## II. BACKGROUND

The acquisition scheme in imaging schemes (e.g MRI, synthetic aperture radar) can be modeled as the 2-D Fourier samples of the continuous domain support limited object [19]:

$$\hat{\rho}[i] = \int_{\Omega} f(\mathbf{r}) \exp(-j2\pi \mathbf{k}_i^T \mathbf{r}) d\mathbf{r}; \mathbf{k}_i \in \mathbb{R}^2; i = 1, \dots, N. \quad (\text{II.1})$$

Here,  $\Omega \subseteq [0, 1] \times [0, 1]$  is the support of the object and  $\mathbf{k}_i$  is the  $i^{\text{th}}$  frequency point at which the acquisition is performed. The acquisition time is directly proportional to the total number of Fourier samples, specified by  $N$ . Since

the recovery of a continuous domain signal  $f(\mathbf{r})$  from finite measurements is ill-posed, a common practice in MRI is to model the continuous 2-D signal using a shift invariant representation [19]

$$f(\mathbf{r}) = \sum_{\mathbf{m}} \mathbf{c}[\mathbf{m}] \varphi\left(\frac{\mathbf{r}}{N} - \mathbf{m}\right). \quad (\text{II.2})$$

Here,  $\varphi$  are the voxel basis functions,  $\mathbf{c}[\mathbf{m}]$  are the weight coefficients and  $N$  is the sampling step. Note that higher values of  $N$  translate to higher resolution image models. Substituting the model in (II.1), one obtains, for suitable  $\hat{\varphi}(\mathbf{k}_i)$ , [19]

$$\begin{aligned} \hat{\rho}[i] &= \sum_{\substack{\mathbf{m} \\ \mathbf{k}_i \in \mathbb{R}^2; i=1, \dots, N}} c_{\mathbf{m}} \int_{\Omega} \varphi\left(\frac{\mathbf{r}}{N} - \mathbf{m}\right) \exp\left(-j2\pi\mathbf{k}_i^{\top} \mathbf{r}\right) d\mathbf{r}; \\ &= \hat{\varphi}(\mathbf{k}_i) \sum_{\substack{\mathbf{m} \\ \mathbf{k}_i \in \mathbb{R}^2; i=1, \dots, N}} c_{\mathbf{m}} \exp\left(-j2\pi\mathbf{k}_i^{\top} \mathbf{m}\right); \end{aligned}$$

If the voxel basis functions are sinc functions,  $\hat{\varphi}(\mathbf{k}_i) = 1$  within the bandwidth and is zero otherwise. If other models (e.g spline functions are used), the magnitude of  $\hat{\varphi}(\mathbf{k}_i)$  and  $\hat{\rho}[i]$  will decrease rapidly with increasing  $\mathbf{k}$ .

In MR imaging applications, the acquisition of the uniform Fourier samples on lines in  $\mathbf{k}$ -space is efficient. Specifically, the frequency encoding gradients along  $x$  and  $y$  directions are kept constant, while reading out the data at uniform intervals; the slope of the line is controlled by the ratio of the frequency encoding gradients [19]. This allows us to acquire a line of  $\mathbf{k}$ -space data in a single excitation. Phase encoding gradients can be used to shift the line from the origin. If Cartesian sampling is used, the data is acquired on equispaced lines in  $\mathbf{k}$ -space;  $N$  excitations are required to Nyquist sample the image. An alternative is radial sampling [19], where the slope of the lines are varied from excitation to excitation.

### III. NOTATION AND PRELIMINARIES

We define  $\mathbf{W}_N$  to be the  $N \times N$  1-D, DFT matrix with row and column indices taking values from the set  $\mathbb{Z}_N = \{0, 1, \dots, N-1\}$ . The  $il$ -th element of  $\mathbf{W}_N$  is  $e^{\frac{j2\pi il}{N}}$ . On the other hand, the 2-D DFT matrix has rows and columns indexed by 2-vectors  $\mathbf{k}$  and  $\mathbf{n}$ . There is one row per vector  $\mathbf{k}$ , and one column per vector  $\mathbf{n}$ . Throughout we assume that

$$\mathbf{n} \in \mathbb{Z}_N^2, \quad (\text{III.1})$$

i.e. it has precisely  $N^2$  columns. For a given  $\mathbf{k}$ , the corresponding row of the DFT-matrix will be denoted

$$\mathbf{W}^{(2)}(\mathbf{k}, \mathbf{n}) = \exp\left\{j\frac{2\pi\mathbf{k}^{\top}\mathbf{n}}{N}\right\}, \mathbf{n} \in \mathbb{Z}_N^2. \quad (\text{III.2})$$

The conventional 2-D DFT matrix [21] assumes that  $\mathbf{k} \in \mathbb{Z}_N^2$ , i.e. under (III.2) the matrix is in  $\mathbb{C}^{N^2 \times N^2}$ .

We are interested in the spark of a 2-D DFT matrices that have more columns than rows. The following special cases are of interest here.

- A rectangular subgrid: For some subsets  $S_i \subset \mathbb{Z}_N$  and  $\mathbf{D} = \text{diag}\{d_1, d_2\}$ ,  $d_i \in \mathbb{Z}_+$

$$\mathbf{W}^{(2)}(\mathbf{k}, \mathbf{n}) = \exp\left\{j\frac{2\pi\mathbf{k}^{\top}\mathbf{D}\mathbf{n}}{N}\right\}, \mathbf{k} \in S_1 \times S_2, \mathbf{n} \in \mathbb{Z}_N^2. \quad (\text{III.3})$$

Here  $\mathbb{Z}_+$  is the set of positive integers. We will denote the matrix thus formed  $\mathbf{W}^{(2)}(S, \mathbf{D})$ .

- A line with rational slope: Under (III.2), and  $L$  and  $M$  integers  $\mathbf{k} = [k_1, k_2]^{\top}$  are on a line:

$$K(L, M, S) = \{\mathbf{k} | Lk_1 - Mk_2 = 0, k_1 \in S \subset \mathbb{Z}_N\} \quad (\text{III.4})$$

We will denote matrix thus formed by  $\mathbf{W}^{(2)}(K(L, M, S))$ . Evidently  $K(L, M, S)$  has a slope  $L/M$  in  $\mathbf{k}$ -space.

Unlike the rectangular subgrid where each element of  $\mathbf{k}$  takes integer values, in (III.4) the second element of  $\mathbf{k}$  may be rational. Sampling in  $\mathbf{k}$ -space along a line such as (III.4) is quite common in MRI applications.

Define:

$$\mathbf{w}_N(l) = \left[1 \quad e^{\frac{j2\pi l}{N}} \quad \dots \quad e^{\frac{j2\pi l(N-1)}{N}}\right]. \quad (\text{III.5})$$

Define  $\mathbf{e}_1 = [1, 0]^{\top}$  and  $\mathbf{e}_2 = [0, 1]^{\top}$ . Then for suitable permutation matrices  $\mathbf{P}_i \in \mathbb{R}^{N^2 \times N^2}$  the row  $\mathbf{W}^{(2)}(\mathbf{k}, \mathbf{n})$  in (III.2) is expressible as

$$\begin{aligned} \mathbf{W}^{(2)}(\mathbf{k}, \mathbf{n}) &= \mathbf{w}_N(\mathbf{k}^{\top} \mathbf{e}_2) \otimes \mathbf{w}_N(\mathbf{k}^{\top} \mathbf{e}_1) \mathbf{P}_1 \\ &= \mathbf{w}_N(\mathbf{k}^{\top} \mathbf{e}_1) \otimes \mathbf{w}_N(\mathbf{k}^{\top} \mathbf{e}_2) \mathbf{P}_2 \end{aligned}$$

Define  $\mathbf{W}_N(S)$  to comprise the submatrix of  $\mathbf{W}_N$  formed by the rows indexed by  $S \in \mathbb{Z}_N$ . Then for permutation matrices  $\mathbf{P}_i, \mathbf{Q}_i$ ,

$$\begin{aligned} \mathbf{W}^{(2)}(S, \mathbf{D}) &= \mathbf{P}_1 \mathbf{W}_N(d_1 S) \otimes \mathbf{W}_N(d_2 S) \mathbf{Q}_1 \\ &= \mathbf{P}_2 \mathbf{W}_N(d_2 S) \otimes \mathbf{W}_N(d_1 S) \mathbf{Q}_2 \end{aligned} \quad (\text{III.6})$$

Here for a scalar  $d$ , if  $S = \{s_1, \dots, s_n\}$  then  $dS = \{ds_1, \dots, ds_n\}$ .

### IV. SPARK OF SUBMATRICES OF $\mathbf{W}_N$

We observe that even the results on the spark of submatrices  $\mathbf{W}_N(S)$  defined in the previous section are few, [17], [18]. Here we recount a few that are used in later sections. We also provide a new result that to our knowledge is new.

Suppose for some  $i$  and  $l$ ,  $S = \{i, i+1, \dots, i+l-1\}$  i.e.  $S$  contains consecutive elements of  $\mathbb{Z}_N$ . Define

$$z_n = e^{j\frac{2\pi n}{N}} \quad (\text{IV.1})$$

Then the matrix comprising any  $l$  columns of  $\mathbf{W}_N(S)$ , indexed by the integers  $i_1, \dots, i_l$  can be expressed as

$$\begin{bmatrix} 1 & 1 & \dots & 1 \\ z_{i_1} & z_{i_2} & \dots & z_{i_l} \\ \vdots & \vdots & \ddots & \vdots \\ z_{i_1}^{l-1} & z_{i_2}^{l-1} & \dots & z_{i_l}^{l-1} \end{bmatrix} \text{diag}\{z_{i_1}^i, z_{i_2}^i, \dots, z_{i_l}^i\}.$$

and being a product of a Vandermonde and a nonsingular diagonal matrix, is thus nonsingular for distinct  $z_{i_n}$ . Consequently, such a  $\mathbf{W}_N(S)$  has full spark. Examples where non-consecutive rows cause spark to be lost can be found in [17] and [18]. An exception is provided by a classical result that can be traced back to Chebotarëv, (see [20]).

*Theorem 4.1:* Suppose  $N$  is prime. Then for all  $S \subset \mathbb{Z}_N$ ,  $\mathbf{W}_N(S)$  has full spark.

The results of [17] below provide a way to generate additional sets  $\bar{S}$ , beyond those comprising consecutive integers, for which  $\mathbf{W}_N(\bar{S})$  has full spark.

*Theorem 4.2:* Suppose for some  $S \subset \mathbb{Z}_N$ ,  $\mathbf{W}_N(S)$  has full spark. Then so does:

- (i)  $\mathbf{W}_N((S+i) \bmod N)$  for all  $i \in \mathbb{Z}_N$ .
- (ii)  $\mathbf{W}_N(MS)$  for all  $M$  that is coprime with  $N$ .
- (iii)  $\mathbf{W}_N(\mathbb{Z}_N \setminus S, N)$ .

We next provide Theorem 4.3 that generalizes a result in [18]. It requires the following Lemma from [21].

*Lemma 4.1:* Consider integers  $1 \leq M < N$ ,  $M$  and  $N$  not coprime. Then there exists  $1 \leq n < N$  such that  $N$  divides  $Mn$ .

Using this lemma we now prove the following theorem.

*Theorem 4.3:* For integer  $1 < M < N$ ,  $S \in \mathbb{Z}_N$ , with  $|S| > 1$  suppose  $\mathbf{W}(S)$  has full spark. Then  $\mathbf{W}_N(MS)$  has full spark if and only if  $M$  and  $N$  are coprime. Otherwise  $\text{spark}(\mathbf{W}_N(MS)) = 2$ .

The corresponding result in [18] assumes that  $S$  contains consecutive integers.

## V. SPARK ON A LINE WITH RATIONAL SLOPE

We now consider the special case when the  $k$ -space samples lie on a line with possible noninteger slopes. Specifically, with  $K(L, M, S)$  as in (III.4) and  $L$  and  $M$  integers we consider submatrices whose rows are indexed by  $\mathbf{k} \in K(L, M, S)$  and  $\mathbf{n} \in \mathbb{Z}_N^2$ . Observe that with  $\mathbf{k} = [k_1, k_2]^T$ , under (III.4) one has:

$$\begin{aligned} \mathbf{W}^{(2)}(\mathbf{k}, \mathbf{n}) &= \exp\left(j \frac{2\pi k_1}{NM} (Mn_1 + Ln_2)\right) \\ &= \exp\left(j \frac{2\pi k_1 [M, L] \mathbf{n}}{NM}\right). \end{aligned} \quad (\text{V.1})$$

For the moment we make the following simplifying assumption, which we will relax later.

*Assumption 5.1:* The integers  $M$  and  $L$  in (V.1) are coprime. Under this assumption we have the following lemma.

*Lemma 5.1:* Suppose under Assumption 5.1 and  $k = 1$ , there exist distinct  $\mathbf{n}_i \in \mathbb{Z}_N^2$  such that:

$$k[M, L] \mathbf{n}_1 \bmod MN = k[M, L] \mathbf{n}_2 \bmod MN \quad (\text{V.2})$$

Then:

- (i)  $M < N$ , and
- (ii) (V.2) holds for all  $k \in \mathbb{Z}$ .

Further, if  $M < N$  then there exist distinct  $\mathbf{n}_i \in \mathbb{Z}_N^2$  such that (V.2) holds for  $k = 1$ .

The next lemma establishes a sufficient condition for a full spark  $\mathbf{W}^{(2)}(k(L, M, S))$ .

*Lemma 5.2:* Suppose Assumption 5.1 holds,  $M \geq N$  and for some  $N > l \geq 2$ ,  $S = \mathbb{Z}_l$ . Then  $\mathbf{W}^{(2)}(K(L, M, S))$  has full spark. Further if  $M < N$ , then  $\mathbf{W}^{(2)}(K(L, M, S))$  has spark 2.

In fact using, Theorem 4.2 and 4.3 a stronger result can be obtained.

*Theorem 5.1:* Suppose for some integer  $N$ , the set  $S$  is such that  $\mathbf{W}_{MN}(S)$  has full spark and Assumption 5.1 holds. Then for given integer  $p$ ,  $\mathbf{W}^{(2)}(K(L, M, pS))$  has full spark iff  $p$  and  $MN$  are coprime and  $M \geq N$ .

In summary, this result has two features. First it translates the spark condition for a 2-D DFT matrix in to one for a 1-D DFT matrix. Second it demands that for a line of slope  $L/M$  in  $k$ -space, with  $L$  and  $M$  coprime,  $M$  cannot be smaller than  $N$ . This precludes integer slopes, or for that matter a  $45^\circ$  line. Though of course one can sample on lines that are arbitrarily close in their slope to any given line.

To understand how Assumption 5.1 can be relaxed suppose  $M$  and  $L$  have greatest common divisor (gcd)  $p$ . Define

$$\bar{M} = \frac{M}{p} \quad \text{and} \quad \bar{L} = \frac{L}{p} \quad (\text{V.3})$$

Then  $\bar{M}$  and  $\bar{L}$  are coprime and

$$\frac{2\pi k_1}{NM} (Mn_1 + Ln_2) = \frac{2\pi k_1}{N\bar{M}} (\bar{M}n_1 + \bar{L}n_2).$$

Then all one needs to do is to replace  $M$  by  $\bar{M}$  in both Lemma 5.2 and Theorem 5.1.

## VI. SPARK ON A RECTANGULAR SUBGRID

We now turn to the setting of rectangular sampling in  $k$ -space and first provide a seemingly negative result. Later in the section we show that in fact there are vectors with particular sparsity patterns that can be recovered through rectangular sampling.

First observe from (III.6), that to within permutations of rows and columns the matrix in (III.3) is the Kronecker product of submatrices of a 1-D DFT matrix. In general, barring pathologies the rank of a Kronecker product exceeds that of the factors. Theorem 6.1 below shows that this is not the case for spark of Kronecker products.

*Theorem 6.1:* Suppose  $\mathbf{A} \in \mathbb{C}^{n_A \times m_A}$  and  $\mathbf{B} \in \mathbb{C}^{n_B \times m_B}$ . Then the following hold.

- (i) If  $n_B < m_B$  then,
 
$$\text{spark}(\mathbf{A} \otimes \mathbf{B}) \leq \text{spark}(\mathbf{B}).$$
- (ii) If  $n_A < m_A$  then,
 
$$\text{spark}(\mathbf{A} \otimes \mathbf{B}) \leq \text{spark}(\mathbf{A}).$$
- (iii) If  $n_B < m_B$ , and  $\text{spark}(\mathbf{A}) \geq \text{spark}(\mathbf{B})$ 

$$\text{spark}(\mathbf{A} \otimes \mathbf{B}) = \text{spark}(\mathbf{B}).$$

Thus rectangular sampling in  $k$ -space does not offer benefits in terms of improved spark. In fact in light of Theorem 4.3 should either  $d_1$  or  $d_2$  in (III.6) share a common factor with  $N$ , then spark of the 2-D DFT matrix will be just two.

Yet there are specialized settings where Kronecker products carry increased benefits. Specifically, Kronecker products exhibit certain advantages in recovery of vectors that are *jointly sparse*. Thus consider

$$\mathbf{X} = [\mathbf{x}_1 \quad \mathbf{x}_2 \quad \cdots \quad \mathbf{x}_m] \in \mathbb{C}^{n \times m}. \quad (\text{VI.1})$$

We call the set of vectors  $\mathbf{x}_i$  jointly  $k$ -sparse if  $\mathbf{X}$  has at most  $k$  nonzero rows. Now, with  $\mathbf{A} \in \mathbb{C}^{n_A \times m}$  and  $\mathbf{B} \in \mathbb{C}^{n_B \times n}$ , consider the observations:

$$\mathbf{y} = (\mathbf{A} \otimes \mathbf{B}) \text{vec}(\mathbf{X}), \quad (\text{VI.2})$$

where

$$\text{vec}(\mathbf{X}) = \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \\ \vdots \\ \mathbf{x}_m \end{bmatrix}.$$

Observe  $\text{vec}(\mathbf{X})$  is  $mk$ -sparse. Thus, on the face of it unique recovery of  $\mathbf{X}$  from  $\mathbf{y}$  requires:

$$\text{spark}(\mathbf{A} \otimes \mathbf{B}) > 2mk. \quad (\text{VI.3})$$

Theorem 6.1 would then suggest the requirement,

$$\min \{\text{spark}(\mathbf{A}), \text{spark}(\mathbf{B})\} > 2mk. \quad (\text{VI.4})$$

We now argue that (VI.4) is in fact too conservative.

Indeed with

$$\mathbf{y} = \begin{bmatrix} \mathbf{y}_1 \\ \mathbf{y}_2 \\ \vdots \\ \mathbf{y}_{n_A} \end{bmatrix},$$

(VI.2) reduces to:

$$\mathbf{y}_i = \mathbf{B} \left( \sum_{l=1}^m a_{il} \mathbf{x}_l \right), \quad \forall i \in S_{n_A}. \quad (\text{VI.5})$$

As  $\mathbf{X}$  can have at most  $k$  nonzero rows, for all  $i \in S_{n_A}$ ,  $\mathbf{q}_i = \sum_{l=1}^m a_{il} \mathbf{x}_l$  is  $k$ -sparse. Thus  $\text{spark}(\mathbf{B}) > 2k$  suffices to uniquely determine all the  $\mathbf{q}_i$ , and in the process determining the joint support of the columns of  $\mathbf{X}$ . Finally,  $\text{spark}(\mathbf{A}) > m$  suffices to extract the  $\mathbf{x}_i$  from the  $\mathbf{q}_i$ . Thus far smaller values for  $\text{spark}(\mathbf{A})$  and  $\text{spark}(\mathbf{B})$  than that suggested by (VI.4) suffices to resolve this joint sparsity problem.

## VII. CONCLUSION

We have considered the spark of submatrices of 2-D DFT matrices under two specific types of sampling in  $k$ -space. In both cases we show that spark is related to the spark of certain related 1-D DFT matrices. The first sampling pattern, we consider involves a single line in  $k$ -space and provide a necessary and sufficient conditions for full spark. One interesting conclusion is that lines with integer slopes preclude full spark. The second type of sampling is on rectangular grid. We show that full spark is in general impossible in such a setting. However, we also provide conditions under which such matrices can recover jointly sparse vectors.

An open problem of substantial benefit to MRI is as follows: Instead of sampling on a single line in  $k$ -space suppose we sampled over several lines. How to extend the results of Section V in such a setting? We are currently working towards resolving this problem, possibly through the aid of Theorem 5.1.

## REFERENCES

- [1] M. Duarte, and R. Baraniuk, "Kronecker compressive sensing," in *Image Processing, IEEE Transactions*, vol. 21, no. 2, pp. 494–504, 2012.
- [2] P. P. Vaidyanathan, and P. Pal, "Sparse Sensing With Co-Prime Samplers and Arrays," in *IEEE Trans. on Signal Processing*, vol. 59, pp. 573–586, 2011.
- [3] M. Lustig, D. L. Donoho, J. M. Santos, and J. M. Pauly, "Compressed sensing MRI," *IEEE Signal Processing Magazine*, vol. 25, no. 2, pp. 72–82, 2008.
- [4] R. Chartrand, "Fast algorithms for nonconvex compressive sensing: MRI reconstruction from very few data," in *IEEE International Symposium on Biomedical Imaging: From Nano to Macro, 2009. ISBI'09.* IEEE, 2009, pp. 262–265.
- [5] U. Gampfer, P. Boesiger, and S. Kozerke, "Compressed sensing in dynamic MRI," *Magnetic Resonance in Medicine*, vol. 59, no. 2, pp. 365–373, 2008.
- [6] R. Baraniuk and P. Steeghs, "Compressive radar imaging," in *Radar Conference, 2007 IEEE.* IEEE, 2007, pp. 128–133.
- [7] V. M. Patel, G. R. Easley, D. M. Healy Jr, and R. Chellappa, "Compressed synthetic aperture radar," *IEEE Journal of Selected Topics in Signal Processing*, vol. 4, no. 2, pp. 244–254, 2010.
- [8] L. Zhang, M. Xing, C.-W. Qiu, J. Li, J. Sheng, Y. Li, and Z. Bao, "Resolution enhancement for inversed synthetic aperture radar imaging under low snr via improved compressive sensing," *IEEE Transactions on Geoscience and Remote Sensing*, vol. 48, no. 10, pp. 3824–3838, 2010.
- [9] D. L. Donoho and M. Elad, "Optimally sparse representation in general (nonorthogonal) dictionaries via  $l_1$  minimization," *Proceedings of the National Academy of Sciences*, vol. 100, no. 5, pp. 2197–2202, 2003.
- [10] P. Feng and Y. Bresler, "Spectrum-blind minimum-rate sampling and reconstruction of multiband signals," in *IEEE International Conference on Acoustics, Speech, and Signal Processing, 1996. ICASSP-96. Conference Proceedings.*, 1996, vol. 3. IEEE, 1996, pp. 1688–1691.
- [11] E. J. Candes and T. Tao, "Near-optimal signal recovery from random projections: Universal encoding strategies?" *Information Theory, IEEE Transactions on*, vol. 52, no. 12, pp. 5406–5425, 2006.
- [12] E. J. Candès, J. Romberg, and T. Tao, "Robust uncertainty principles: Exact signal reconstruction from highly incomplete frequency information," *Information Theory, IEEE Transactions on*, vol. 52, no. 2, pp. 489–509, 2006.
- [13] T. Strohmer and R. W. Heath Jr, "Grassmannian frames with applications to coding and communication," *Applied and computational harmonic analysis*, vol. 14, no. 3, pp. 257–275, 2003.
- [14] A. S. Bandeira, M. Fickus, D. G. Mixon, and P. Wong, "The road to deterministic matrices with the restricted isometry property," *Journal of Fourier Analysis and Applications*, vol. 19, no. 6, pp. 1123–1149, 2013.
- [15] P. Xia, S. Zhou, and G. B. Giannakis, "Achieving the Welch bound with difference sets," *IEEE Transactions on Information Theory*, vol. 51, no. 5, pp. 1900–1907, 2005.
- [16] N. Y. Yu and Y. Li, "Deterministic construction of fourier-based compressed sensing matrices using an almost difference set," *EURASIP Journal on Advances in Signal Processing*, vol. 2013, no. 1, pp. 1–14, 2013.
- [17] B. Alexeev, J. Cahill, and D. G. Mixon, "Full spark frames," *Journal of Fourier Analysis and Applications*, vol. 18, no. 6, pp. 1167–1194, 2012.
- [18] H. K. Achanta, S. Biswas, S. Dasgupta, M. Jacob, B. N. Dasgupta and R. Mudumbai, "Coprime Conditions for Fourier Sampling for Sparse Recovery," in *Proceedings of the Eighth Sensor Array and Multichannel Signal Processing Workshop*, A Coruna, Spain, June 2014.
- [19] Z.P. Liang and P. C. Lauterbur, *Principles of Magnetic Resonance Imaging: A Signal Processing Perspective*, Wiley-IEEE Press, 1999.
- [20] H. Steinhagen, P. abd Lenstra, "Chebotarëv and his density theorem," *Math. Intelligencer*, vol. 18, pp. 26–37, 1996.
- [21] P. P. Vaidyanathan, *Multirate Systems and Filter Banks*. Prentice Hall, 1993.