IMAGE REGULARIZATION WITH HIGHER-ORDER MORPHOLOGICAL GRADIENTS

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ABSTRACT

In this paper, we propose an image prior based on morphological image features for image recovery. The proposed prior is obtained as the sum of morphological gradient and its higher-order extensions. The morphological gradient is defined as the difference between dilation and erosion of an image and obtains a discretized modulus of gradient. In order to suppress artifacts appear in the recovered image, we introduce higher-order morphological gradients. The regularization problem with the proposed prior is reduced to a constrained minimization problem. In order to apply the subgradient method to this problem, we derive the subgradient of the proposed priors. We apply the proposed prior to image denoising and demonstrate that the proposed higher-order morphological gradient prior is capable to suppress staircase artifacts. Comparison with the total variation image prior is also demonstrated.

Index Terms— Image recovery, mathematical morphology, morphological gradient, image prior, regularization

1. INTRODUCTION

Image recovery is a problem that estimates the original image from the degraded observation. Since the inverse problem of the image recovery is usually ill-posed, the problem is treated as the regularization problem. In the regularization of the image recovery, the objective function that is defined as the addition of two terms, the fidelity term that is defined as a squared error and a prior term. For the prior term, the total variation (TV) prior [1–5] and its variants are widely applied. The standard TV prior is defined from by the sum of the modulus of gradients of intensity surface of the image. The gradients is derived from the differences of neighboring pixels, which correspond to directional derivatives. In the field of the mathematical morphology, the modulus of the gradients are obtained by morphological gradients [6, 7]. The morphological gradient is defined as the difference between the dilated and eroded images and is applied to edge detection [7]. The Morphological gradient is one of the image feature that is obtained by morphological operators. Recently, several image priors for image recovery that can be interpreted as morphological features have been proposed for resolving the regularization problem.

The total subset variational prior (TSV) [8] is defined from the difference between the maximum and minimum of a local clique. This difference can be interpreted as the morphological gradient. For the minimization, the objective function of the regularization is approximated as the continuously differentiable function by using the log-sum-exp function [9] that approximates the max and min functions [8]. The prior that includes the morphological feature can be easily computed, however, the minimization requires a quasi-Newton method with the approximation. In Ref. [10], morphologic regularization was proposed for achieving the super-resolution of images. In this regularization, the prior is defined as the difference between the closing [6] and opening [6]. The objective of this prior is to attenuate the irregular intensity variations that appear in the recovered high-resolution images. The minimization of the objective function is achieved with the convex optimization techniques without approximation. Due to the property of the morphology, the computational cost for the minimization is lower than other priors. However, the noise components that is invariant with opening and closing will be preserved in the recovered image. Therefore, this prior is applied to achieve image super resolution under the assumption that the variance of the noise is relatively small. In order to remedy this problem, soft-morphology is introduced in Ref. [11]. However, optimization for the soft-morphology still requires approximation with log-sum-exp functions and the quasi-Newton method.

In this paper, we propose the image prior includes an extension of the morphological gradient. The morphological gradient is extended to the higher-order morphological gradient that includes higher-order differentials. We propose the image prior that is defined as the weighted sum of the higher-order morphological gradients in order to suppress the artifacts that appear in the recovered image. In order to minimize the objective function, we apply the subgradient method [12]. Computation of the subgradient of the morphological gradient can be achieved during the erosion and dilation operation and requires low computational costs.

In the next section, we briefly explain the image regular-

This work was supported by JSPS KAKENHI Grant Number 23500210.

978-0-9928626-3-3/15/$31.00 ©2015 IEEE

ization. In Sect. 3, the higher-order extension of the morphological gradient is introduced for the image prior. In Sect. 4, the subgradient of the proposed prior are derived to apply the subgradient method for image denoising. Finally, the proposed prior is applied to image denoising and is comapred with the standard TV.

2. IMAGE REGULARIZATION WITH SMOOTHNESS PRIORS

The observed image $\mathbf{y}$ is assumed to be observed as $\mathbf{y} = \mathbf{Hw} + \mathbf{e}$, where $\mathbf{w}$ is the original image without degradation and $\mathbf{H}$ denotes a degradation process. $\mathbf{e}$ is the additive noise, which is supposed to be Gaussian in this paper. Assuming that the degradation process $\mathbf{H}$ is known, the estimation of the original image is obtained from the regularization as

$$\hat{\mathbf{w}} \in \arg\min_{\mathbf{f}} \frac{1}{2} \parallel \mathbf{y} - \mathbf{Hf} \parallel_2^2 + \lambda R(\mathbf{f}). \quad (1)$$

where $R(\mathbf{g})$ is the image prior term. λ is the regularization parameter. This problem can be transformed into a constraint minimization problem that is

$$\min_{\mathbf{f}} R(\mathbf{f}) \text{ subject to } \parallel \mathbf{y} - \mathbf{Hf} \parallel_2 \leq \sigma. \quad (2)$$

In the form of (1), the regularization parameter λ has to be specified. The optimum λ depends on both of the characteristics of the image and noise. If the standard deviation of the noise is known, $\mathbf{Hw}$ exists on the L_2 ball, of which radius and center coincide with the standard deviation and $\mathbf{y}$, respectively. Usually, σ in (2) is hence specified from the standard deviation of the noise. In this paper, we discuss the denoising problem in the form of (2) where σ corresponds to the standard deviation of the Gaussian noise and $\mathbf{H}$ is an identity.

In these regularization, the local smoothness of the recovered image is measured as the image prior term $R(\mathbf{f})$, which is the sum of the function of the intensities around the coordinate $\mathbf{x} \in \mathbb{Z}^2$ as

$$R(\mathbf{f}) = \sum_{\mathbf{x}\in\mathcal{I}} G\left(\{f_{\mathbf{x}+\mathbf{y}}\}_{\mathbf{y}\in\mathcal{C}}\right) \quad (3)$$

where $\mathcal{C}$ is a set of the coordinates that defines neighboring coordinates of $\mathbf{x}$. For the TV prior, G is defined as the modulus of the discretized gradient. In this paper, we introduce the morphological gradient, which can be computed by morphological operators.

3. MORPHOLOGICAL GRADIENTS

For two-dimensional continuous function $f(\mathbf{x})$, where $\mathbf{x} \in \mathbb{R}^2$ denotes a two-dimensional coordinate, the dilation and erosion $f(\mathbf{x})$ are respectively defined as

$$d_\rho \circ f(\mathbf{x}) = \sup_{\mathbf{y}\in\rho B_{\mathbf{x}}} f(\mathbf{y}) \text{ and } e_\rho \circ f(\mathbf{x}) = \inf_{\mathbf{y}\in\rho B_{\mathbf{x}}} f(\mathbf{y}). \quad (4)$$

The region $\rho B_{\mathbf{x}}$ is the circle with the radius ρ and the center $\mathbf{x}$ for two-dimensional functions. The morphological gradient [7] is defined as the limit of the difference between the dilation and the erosion as

$$g \circ f(\mathbf{x}) = \lim_{\rho\to 0} \frac{d_\rho \circ f(\mathbf{x}) - e_\rho \circ f(\mathbf{x})}{2\rho}. \quad (5)$$

The morphological gradient corresponds to the modulus of the gradient $|\nabla f(\mathbf{x})|$ for a differentiable function [7].

For a discrete image $\{f_{\mathbf{x}}\}_{\mathbf{x}\in\mathbb{Z}^2}$, the dilation and erosion are respectively defined as

$$D \circ f_{\mathbf{x}} = \max_{\mathbf{y}\in\mathcal{S}} f_{\mathbf{x}+\mathbf{y}} \text{ and } E \circ f_{\mathbf{x}} = \min_{\mathbf{y}\in\mathcal{S}} f_{\mathbf{x}+\mathbf{y}}. \quad (6)$$

$\mathcal{S}$ is a set of the coordinates and is referred to as a structuring element (SE). In general, the structuring element is defined as a subset of the three-dimensional space that is spanned by spatial and intensity axes. In this study, the intensity of the SE is omitted for discussion. The morphological gradient for discrete images is defined as the difference between the dilation and erosion as

$$M \circ f_{\mathbf{x}} = D \circ f_{\mathbf{x}} - E \circ f_{\mathbf{x}} \quad (7)$$

with the SE $\mathcal{S}$. For the discrete images, the property of the morphological gradient depends on the SE $\mathcal{S}$ of the pair of the dilation and erosion. In Ref. [7], several extensions of the basic morphological gradient are introduced for edge detection.

The morphological gradient (7) can be represented with discrete differentiations as

$$M \circ f_{\mathbf{x}} = \max_{y_1,y_2\in\mathcal{S}} |f_{\mathbf{x}+\mathbf{y_1}} - f_{\mathbf{x}+\mathbf{y_2}}|. \quad (8)$$

Therefore, the morphological gradient can be interpreted as theL_∞ norm of the absolute differences of possible pairs in the SE that are translated to the coordinate $\mathbf{x}$. On the other hand, the discretized gradient that is employed for the standard TV is defined as

$$G_{\mathbf{x}} = \sqrt{(f_{\mathbf{x}} - f_{\mathbf{x}-(\mathbf{1},\mathbf{0})})^2 + (f_{\mathbf{x}} - f_{\mathbf{x}-(\mathbf{0},\mathbf{1})})^2}. \quad (9)$$

In the regularization with TV, the TV prior term is defined as the sum of the discretized gradient over the entire image and is the L_1-L_2 norm of the intensity differences, which approximate directional derivatives. The sum of the morphological gradient corresponds to the L_1-L_∞ norm of the intensity differences.

By the way, one of problems of the image recovery with the TV prior is the staircase artifact that appears as undesirable intensity discontinuities in the recovered image. In order to avoid staircase artifact, the higher-order derivatives are included in the priors. In Ref. [3], the TV of the gradient are included in the prior. The higher-order derivatives are also employed in Ref. [4]. In order to remedy the staircase artifact, we extend the morphological gradient to include higher-order derivatives. We define the first-order morphological gradient as the basic morphological gradient in (7), $M_1 f_{\mathbf{x}} = M \circ f_{\mathbf{x}}$. The n-th order morphological gradient is defined by the composition of the morphological gradient as

$$M_n \circ f_{\mathbf{x}} = \underbrace{M \circ M \circ M \circ \cdots M \circ}_{n} f_{\mathbf{x}}. \tag{10}$$

Since the first-order morphological gradient approximates the modulus of the gradient $|\nabla f(\mathbf{x})|$, the second-order morphological gradient approximates $|\nabla|\nabla f(\mathbf{x})||$. In this study, we employ the second-order morphological gradient to suppress staircase effect that will occur in recovered images.

4. IMAGE REGULARIZATION WITH MORPHOLOGICAL GRADIENTS

We apply the morphological gradient and its higher-order extensions to the image prior. As the form of the prior in (3), the higher-order morphological gradient that includes the first to n-th morphological gradient is

$$R_n(\mathbf{f}) = \sum_{i=1}^{n} \sum_{\mathbf{x} \in \mathcal{I}} \alpha_i M_i \circ f_{\mathbf{x}}. \tag{11}$$

The constraint $\sum_{i=1}^{n} \alpha_i = 1$ is imposed on the weighting parameters $\{\alpha_i\}_{i=1}^{n}$ to ignore scale of the parameters. The dilation, negative of the erosion and morphological gradient are convex functions, since the max function is a convex function [9]. Therefore, the regularization (2) with the basic morphological gradient ($n = 1$) is a constrained convex minimization problem. In order to minimize the morphological gradient prior, we employ the projected subgradient method [12]. The projected subgradient method is a simple and classical method for convex minimization. The iteration of the projected subgradient method start with the initial guess of the recovered image $\mathbf{f}^{(0)}$. The update rule is as

$$\mathbf{f}^{(k+1)} = P_{\mathbf{y}}\left(\mathbf{f}^{(k)} - \beta_k \nabla R\left(\mathbf{f}^{(k)}\right)\right) \tag{12}$$

where $\nabla R(\mathbf{f})$ is the search direction at k-th iteration and is specified as any subgradient of the regularization function R. α_k is an monotonically decreasing sequence. In this study, we employ $\beta_k = \gamma/k$ where γ is a constant and obtained prior to the iteration. For image denoising, $P_{\mathbf{y}}$ is a projection operation onto the l_2-ball, of which center is specified as $\mathbf{y}$ and is implemented as

$$P_{\mathbf{y}}(\mathbf{f}) = \begin{cases} \mathbf{f} & \text{for } \| \mathbf{y} - \mathbf{f} \|_2 \leq \sigma \\ \sigma \frac{\mathbf{f}-\mathbf{y}}{\|\mathbf{f}-\mathbf{y}\|_2} + \mathbf{y} & \text{otherwise} \end{cases}. \tag{13}$$

In order to implement the projected subgradeint method, we derive the subgradient of the morphological gradient. The subdifferential of the dilation is derived from the subdifferential of the max function [9] and is

$$\frac{\partial D \circ f_{\mathbf{x}}}{\partial f_{\mathbf{z}}} = \begin{cases} 1, \text{ if } \mathbf{z} \in \mathcal{S}_{\mathbf{x}} \text{ and } \forall \mathbf{y} \in \mathcal{S}_{\mathbf{x}}, \mathbf{y} \neq \mathbf{z}, f_{\mathbf{z}} > f_{\mathbf{y}} \\ 0, \text{ if } \mathbf{z} \notin \mathcal{S}_{\mathbf{x}} \text{ or } f_{\mathbf{z}} < D \circ f_{\mathbf{x}} \\ \in [0,1] \text{ otherwise} \end{cases} \tag{14}$$

where $\mathcal{S}_{\mathbf{x}}$ is a set of coordinates that are supported by the SE that is translated to the coordinate $\mathbf{x}$. The subdifferential of the erosion is

$$\frac{\partial(-E \circ f_{\mathbf{x}})}{\partial f_{\mathbf{z}}} = \begin{cases} -1, \text{ if } \mathbf{z} \in \mathcal{S}_{\mathbf{x}} \text{ and } \forall \mathbf{y} \in \mathcal{S}_{\mathbf{x}}, \mathbf{y} \neq \mathbf{z}, f_{\mathbf{z}} < f_{\mathbf{y}} \\ 0, \text{ if } \mathbf{z} \notin \mathcal{S}_{\mathbf{x}} \text{ or } f_{\mathbf{z}} > E \circ f_{\mathbf{x}} \\ \in [-1,0], \text{ otherwise.} \end{cases} \tag{15}$$

For the update rule in (14), we choose the subgradients of the dilation and erosion as

$$\frac{\delta D \circ f_{\mathbf{x}}}{\delta f_{\mathbf{z}}} = \begin{cases} 1, \text{ if } \mathbf{z} \in \mathcal{S}_{\mathbf{x}} \text{ and } f_{\mathbf{z}} = D \circ f_{\mathbf{x}} \\ 0, \text{ if } \mathbf{z} \notin \mathcal{S}_{\mathbf{x}} \text{ or } f_{\mathbf{z}} < D \circ f_{\mathbf{x}} \end{cases} \tag{16}$$

and

$$\frac{\delta(-E \circ f_{\mathbf{x}})}{\delta f_{\mathbf{z}}} = \begin{cases} -1, \text{ if } \mathbf{z} \in \mathcal{S}_{\mathbf{x}} \text{ and } f_{\mathbf{z}} > E \circ f_{\mathbf{x}} \\ 0, \text{ if } \mathbf{z} \notin \mathcal{S}_{\mathbf{x}} \text{ or } f_{\mathbf{z}} > E \circ f_{\mathbf{x}}. \end{cases} \tag{17}$$

from the subdifferentials, respectively. With these subgradients, $\nabla R_1(\mathbf{f})$ for the iteration in (12) with the basic morphological gradient is obtained as

$$\frac{\delta R_1(\mathbf{f})}{\delta f_{\mathbf{z}}} = \sum_{\mathbf{x} \in \mathcal{I}} \frac{\delta D \circ f_{\mathbf{x}}}{\delta f_{\mathbf{z}}} + \frac{\delta(-E \circ f_{\mathbf{x}})}{\delta f_{\mathbf{z}}}. \tag{18}$$

In actual computation, the summation of (16) is given by counting the occurrence of the intensity $f_{\mathbf{z}}$ in the dilated image. The summation of (17) is also given by counting occurrence in the eroded image. Therefore, the computation of the search direction $\nabla R(\mathbf{f})$ can be achieved during the morphological operations.

For highr-order morphological gradients, $n > 1$, $R_n(\mathbf{f})$ is not convex. However, the proper search direction that is composed as

$$\frac{\delta R_n(\mathbf{f})}{\delta f_{\mathbf{z}}} = \sum_{i=1}^{n} \sum_{\mathbf{x} \in \mathcal{I}} \alpha_i \frac{\delta M_i \circ f_{\mathbf{x}}}{\delta f_{\mathbf{z}}} \tag{19}$$

is given, then the prior will converge to any saddle point. In order to compute $\nabla R_2(\mathbf{f})$, we use the chain rule. Use of the chain rule has been introduced in Ref. [10] for the morphologic regularization, of which prior consists of the opening and closing that are composite functions of dilation and erosion. The subgradient of the second-order morphological gradient is

$$\frac{\delta M_2 \circ f_{\mathbf{x}}}{\delta f_z} = \sum_{\mathbf{y} \in \mathcal{I}} \frac{\delta M_2 \circ f_{\mathbf{x}}}{\delta M \circ f_{\mathbf{y}}} \frac{\delta M \circ f_{\mathbf{y}}}{\delta f_{\mathbf{z}}}. \tag{20}$$

Table 1. Comparison of PSNRs of the denoised images.

Image	σ_G	TV	1st MG	2nd MG
Lena	10	33.29	33.50	33.95
	20	30.17	30.45	30.95
	30	28.29	28.62	29.15
Man	10	32.00	32.14	32.35
	20	28.86	29.01	29.24
	30	27.25	27.41	27.66
Boat	10	31.73	31.95	32.19
	20	28.50	28.73	29.03
	30	26.73	26.92	27.26
Barbara	10	30.01	30.22	30.72
	20	26.20	26.34	26.87
	30	24.56	24.64	25.07

Table 2. Comparison of SSIMs of the denoised images.

Image	σ_G	TV	1st MG	2nd MG
Lena	10	0.948	0.952	0.957
	20	0.900	0.909	0.917
	30	0.859	0.869	0.882
Man	10	0.941	0.945	0.949
	20	0.872	0.881	0.891
	30	0.817	0.828	0.843
Boat	10	0.943	0.948	0.951
	20	0.875	0.886	0.895
	30	0.817	0.830	0.843
Barbara	10	0.946	0.948	0.951
	20	0.874	0.876	0.885
	30	0.817	0.818	0.827

Fig. 1. Examples of original and noisy image. (Lena)

In this computation, $\delta M_2 \circ f_{\mathbf{x}} / \delta M \circ f_{\mathbf{y}}$ can be computed as same as the basic morphological gradient. In the case of the second-order, the search direction for the iteration can also be computed during the morphological operation with several multiplications for weighing. In this paper, we show only the subgradient for $n = 2$, however, iterative use of the chain rule yields the subgradient for any order.

5. EXAMPLES OF IMAGE DENOISING

In this section, we provide several examples of the image denoising with the morphological gradient priors. We employ four standard images, Lena, Man, Boat and Barbara, of which size is 512×512 pixels. The degraded images are generated by adding the Gaussian noise with the standard deviation σ_G to the original images. The SE of the dilation and the erosion is specified as 2×2 flat square SE. So, the set of $\mathcal{S}$ consists of four coordinates, $\mathcal{S} = \{(0,0),(1,0),(0,1),(1,1)\}$. For the iteration (12), we specify the initial image $\mathbf{f}^{(0)}$ as the degraded images for all examples. The constant γ is specified to be proportional to a noise level and is set to $\sigma_G / \max_{\mathbf{x}} |\delta R_n / \delta f_{\mathbf{x}}|$. The iteration is terminated when the relative change in the prior in (2) is less than a specified threshold. For the prior includes the second-order morphological gradient, the weighting parameters α_1 and α_2 have to be specified. We experimentally choose $\{\alpha_1, \alpha_2\} = \{0.75, 0.25\}$.

In each example, σ of the constraint in (15) is specified from the standard deviation of the noise. For comparison, the results obtained by the TV prior are also shown. We realize the TV denoising by using the Chambolle's algorithm in Ref. [2]. This algorithm achieves the regularization as the unconstrained minimization in (1). The regularization parameter λ for TV denoising is optimized to obtain $\| \mathbf{y} - \hat{\mathbf{w}} \|_2^2 \approx N\sigma_G^2$ where N denotes the number of pixels. The denoising results are evaluated in terms of the peak signal-to-noise ratio

(PSNR) and the structural similarity (SSIM) index [13]. In Table 1, the SNRs obtained by the TV, the first-order morphological gradient and the prior includes second-order morphological gradient are shown. As seen in Table 1, the large differences between the standard TV and the first-order morphological gradient cannot be observed. Both the TV and the morphological gradient prior are obtained form discretized versions of the modulus of the intensity gradient. Both methods hence obtain the similar results. In Fig. 1, the examples of the denoised image that are recovered from the degraded image with $\sigma_G = 20$ are shown. We see the staircase artifacts in both results obtained by the TV prior and the first-order morphological gradient prior. Comparing with result obtained with the first-order (Fig. 1(b)) and the second-order (Fig. 1(c)), the staircase effect are suppressed in the result obtained by the prior includes the second-order morphological gradient due to penalizing the second-order derivatives of the intensity. The advantage of the use of the second-order morphological gradient can also be seen the PSNRs (Table 1) and SSIMs (Table 2). The second-order morphological gradient improve the PSNRs in the range $0.4 - 0.8$ dB over the standard TV prior. In the cases of the lower noise level, the three methods obtains almost same SSIM index. Along with increment of the noise level, the second-order scheme improves the SSIM due to its capability to suppress the staircase artifacts.

6. CONCLUSIONS

In this paper, the image prior based on the higher-order morphological gradient is proposed. We extend the basic morphological gradient to the higher-order morphological gradients in order to suppress the staircase artifacts. For minimization of the objective function, we employ the projected subgradeint method. We show that the subgradient of the basic morphological gradients can be computed during the dilation and erosion without multiplications and other operations. This property will be utilized for the implementation of image recovery algorithms with low computational cost. In image denoising examples, we show that the denoising capability of the morphological gradient prior is comparable to the standard TV. The higher-order morphological gradient has an advantage for suppressing staircase artifacts.

In this paper, we only addressed the image denoising problem. Evaluation of the proposed prior for other image processing task and comparisons with other state of the arts methods are future topics. For the TV piror, the extensions on orientation of the gradient have been proposed in Ref. [5]. In our morphological prior, the properties of regularization depends on the shape of the SE. The use of the adaptive morphology [14], which employs spatial-varying SEs, and other morphological techniques for the regularization is also one of the future topic.

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