GENERALIZED RICCI CURVATURE BASED SAMPLING AND RECONSTRUCTION OF IMAGES

A. Shu Lin, B. Zhongxuan Luo*, C.Jielin Zhang*, D. Emil Saucan†**

* Dalian University of Technology
School of Mathematical Sciences
Dalian 116024, Liaoning, China

† Technion - Israel Institute of Technology
Department of Mathematics
Technion 32000, Haifa, Israel

ABSTRACT

We introduce a novel method of image sampling based on viewing grayscale images as manifolds with density, and sampling them according to the generalized Ricci curvature introduced by Bakry, Emery and Ledoux. A variation of this approach, due to Morgan and his students is also considered. This new paradigm generalizes ideas and results that are by now common in Imaging and Graphics. We apply the new algorithm to natural and range images, as well as cartoons and show that the proposed method produces results similar to those obtained by employing more standard approaches. Furthermore, we show that our approach extends naturally to other types of images, in particular to MRI and CT, where its potential applications are maximal, as well as to meshes.

Index Terms— Weighted manifolds, generalized Ricci curvature, image sampling and reconstruction.

1. INTRODUCTION

Motivated both by the classical Shannon Sampling theorem, as well as by practical considerations, the sampling and reconstruction of images represents a central problem in Imaging and in related fields and has represented, in consequence, an important research topic during the last few decades (see e.g. [1] and the bibliography therein). Since the publication of [2], considering images as surfaces and higher dimensional manifolds, embedded in some ambient space (usually $\mathbb{R}^n$), has become a common paradigm. Using this framework, sampling and reconstruction results were proven for meshes (Graphics) [3, 4], images [5], as well as more general signals [6].

Weighted manifolds arise naturally and frequently in Imaging. Such weights (or densities) may appear as uncertainties intrinsic to the acquiring of the image (for instance in Ultrasonography), in modeling textures and, in a variety of instances, as ad hoc tools employed at various stages of the implementation of a variety of tasks, such as smoothing, (elastic) registration, warping, segmentation, etc. (see e.g. [7]). Moreover, in the context of Medical Imaging, densities appear at even a more basic, intrinsic level: Indeed, the density of many types of MRI images equals the very proton density. Therefore, modeling such images by manifolds with density represents, as far as the physical acquiring process is concerned, the proper approach and, as such, one can expect more accurate results.

However, a more commonly encountered type of image represents the simplest and most natural example of image as a weighted manifold, namely grayscale images. Indeed, such an image can be viewed as nothing more then the graph of a distribution (the grayscale) over a very basic type manifold (a rectangle). It is this very "toy example" that represent the subject of study of this paper. Other types of naturally arising distributions over manifolds appear in Imaging (certain types of textures) and in Graphics (such as luminosity over a surface/mesh). While experiments with meshes are currently in progress, we defer the more interesting – but also far more complicated – case of medical images to further study and show, as already stated above, the feasibility of this approach for grayscale images – natural and range images, and also for cartoon type images.

2. MANIFOLDS WITH DENSITY

In solving many physical, as well as purely mathematical problems, one is conducted, in many instances, to consider *manifolds with density* (see [8]) that is Riemannian manifolds M^n, endowed, in addition, with a smooth, positive density function $\Psi = \Psi(x)$, that induces weighted n- and $(n-1)$-volumes, e.g. in the classical cases $n = 2$ and $n = 3$, volume, area and length. More precisely, the volume, area and length elements dV, dA, ds of the weighted manifold (M^n, Ψ) are given by:

$$dV = \Psi dV_0, dA = \Psi dA_0, ds = \Psi ds_0\,,$$

where dV_0 represents the natural (Riemannian) volume element of M^n, etc. Usually density functions of the type $\Psi(x) = e^{-\varphi(x)}$ are considered. (However, more general density functions can, and have been, studied – see [8].) Note that

Research partly supported by the Natural Science Foundation of China Grants No.61033012, No.11171052 and No.61432003.

prescribing a density on a manifold is not equivalent to a conformal scaling of the metric by a factor $\lambda(x)$, because such conformal deformations produce scaling of, for instance, the area and volume, by different powers of λ.

To generalize Gaussian curvature to the setting of manifolds with density, one has first to make appeal to the works of Bakry, Emery and Ledoux [9, 10], where the following generalization of *Ricci curvature* (see [11] for a deep, but short presentation of the classical notion) is introduced:

$$\mathrm{Ric}_\varphi = \mathrm{Ric} + \mathrm{Hess}\varphi\,, \tag{1}$$

(where Hess denotes the Hessian matrix). It is important for us to note that, for surfaces, Ricci curvature reduces, essentially, to Gaussian curvature K, more precisely $K = \frac{1}{2}\mathrm{Ric}$. Another generalization of Gaussian curvature for weighted surfaces is due to Corwin et al. [12], namely:

$$K_\varphi = K + \Delta\varphi\,, \tag{2}$$

where $\Delta\varphi$ denotes the Laplacian of φ. It should be stressed that this represents a natural generalization: Not only does it reduce to the usual Gaussian curvature (up to a multiplicative constant) for $\varphi \equiv \mathrm{const.}$, it also satisfies a generalized Gauss-Bonnet Theorem. Note that, unlike Morgan [8], but following other authors, we adopted the "+" convention for the sign of the Hessian and Laplacian commonly accepted in Geometry, since this is more intuitive and befits better the context of imaging where grayscale values are always positive.

As we have already noted above, for certain types of medical images, the distribution φ arises quite naturally. For range and ultrasound images, a proper choice of the distribution is the one suggested in [13]. However, for grayscale images a simpler approach is possible, since any such image can be viewed as a distribution over a grid (of pixels). It is precisely this remark that we exploit in the sequel.

3. IMPLEMENTATION AND ALGORITHMS

3.1. Sampling of weighted manifolds

As already noted in the introduction, the problem of sampling of images, meshes and more general signals has been studied in detail and with a variety of methods. However, all these works have in common the fact that say, the image, is sampled using curvature, and more specifically, *Gaussian curvature*, in a combination will some *extrinsic* curvature resulting from the specific embedding considered (see the discussion above). As it was shown in [14] Gaussian (or *sectional*) curvature by itself is still an excellent sampling tool, since it allows for the faithful reconstruction of a given manifold, from topological point of view. In fact, Grove and Petersen also proved in [14] a stronger result, namely that Ricci curvature (which is a weaker invariant than Gaussian curvature) also suffices to topologically reconstruct the manifold (up to *homotopy type*, to be more precise). This result was extended to manifolds with density (weighted manifolds) in [15], where the role of Ricci curvature is played by a *generalized Ricci curvature*, that includes, as a special case, the one given by (1). This approach was shown in [16] to be a proper sampling method, allowing for good reconstruction of the manifold from the metric, not just topological, viewpoint.

It is precisely this fact that we exploit in the present article, taking advantage of the fact noted above that, in dimension 2, hence for grayscale surfaces, Ricci curvature essentially coincides with Gaussian curvature, an invariant that is easy to compute by numerical methods.

3.2. Curvature and Laplacian computation

The first term in the right-hand of (1) is easy to compute since, for images, the base manifold is a flat one (being a planar rectangle), thus its Gaussian curvature is identically 0, hence so is its Ricci curvature.

Computation of the Hessian is based on the method proposed in [17]. More precisely, we used the following formula for the Hessian $H_\varphi(p) = H_\varphi(x, y)$ of the function φ at the pixel $p = p(x, y)$.

$$\begin{aligned} H_\varphi(x,y) = \varphi(x,y)\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix} + \varphi(x_1+1,x_2)\begin{bmatrix}-2 & -1\\ -1 & 0\end{bmatrix} \\ +\varphi(x_1{+}2,x_2)\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix} + \varphi(x_1{+}1,x_2{+}1)\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix} \\ +\varphi(x_1,x_2{+}1)\begin{bmatrix}0 & -1\\ -1 & -2\end{bmatrix} + \varphi(x_1,x_2{+}2)\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix} \\ +\ldots\,; \end{aligned} \tag{3}$$

the relevant pixels, hence the remaining terms being determined by symmetry.

In the implementation of (2) the Laplacian used was the one provided by the standard Matlab package.

3.3. Implementation

In the papers mentioned above [14–16], the sampling density is theoretically, essentially, equal to $1/|\max \mathrm{Ric}_\varphi|$. (This is a generalization of the similar sampling criterion using Gaussian/sectional curvature for surfaces [5] and higher dimensional manifolds [14].) For practical reasons, in this paper we implemented a variation of this basic idea based upon [18]. The correlation between two points $I(x_i, y_i), I(x_j, y_j)$ on the grayscale surface of the image $I = I_\varphi$ used in [18] is given by

$$D[I(x_i,y_i), I(x_j,y_j)] = \sigma^2\left(1 + k\frac{||H(i)| - |H(j)||}{||H(i)| + |H(j)||}\right)e^{-\lambda d_{ij}}\,, \tag{4}$$

where d_{ij} denotes the (Euclidean) distance between the pixels $p_i = p_i(x_i, y_i)$ and $p_j = p_j(x_j, y_j)$, i.e. $d_{ij} = \sqrt{(x_i - x_j)^2 + (y_i - y_j)^2}$, and where k is an empirically determined parameter, $1 \leq k \leq 10$. (For instance, for the

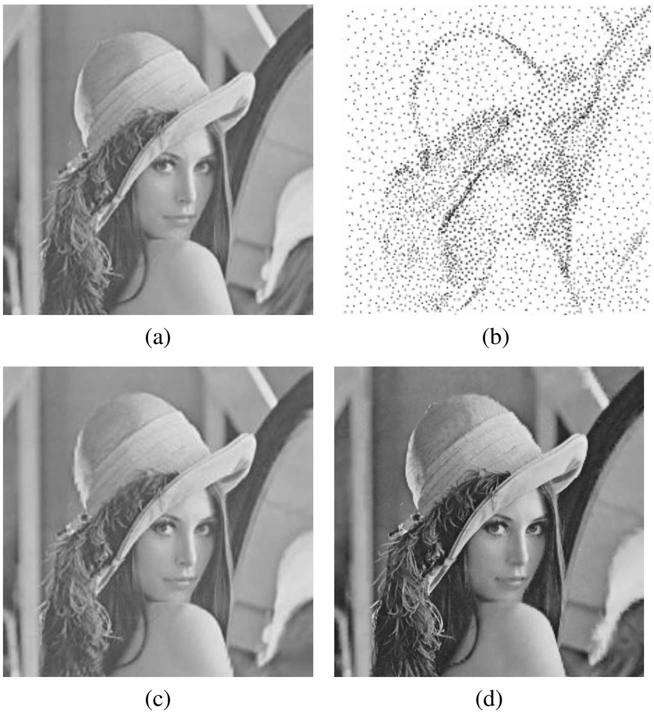


Fig. 1. Sampling and reconstruction of “Lenna”: (a) Original image; (b) Image sampled using 4096 points; (c) Generalized Ricci curvature based reconstruction (using 16000 sampling points); (d) Piecewise-linear reconstruction (cf. [5]).

case of “Lenna”, the best experimentally found value of this parameter is $k = 5$.)

Equation (2) suggests another way of computing the curvature of a weighted surface providing the sampling density required, where Gaussian curvature is used, instead of Ricci curvature, (as initially theoretically considered in [14], as well as in the applied context of [5]).

4. EXPERIMENTAL RESULTS

We tested our proposed method on a number of images – natural images, both standard test images (“Lenna”, “Cameraman”) and a self-acquired one; 3 common range images and a cartoon-type image (denoted henceforth as “White”). As it is illustrated in Figure 1, even without further improvements (see discussion in Section 5 below) the suggested approach allows for the reconstruction of the image with accuracy approaching that of the more classical method employed in [5]. More precisely, while the later allows for better sampling of highly curved regions, our method allows for a more faithful reconstruction of more uniform regions, producing less artifacts.

The proposed method produces better results on images having flatter grayscale surface, such as range images – see Figure 2 and Table 1. This is a consequence of the geometric fact that, in regions with higher curvature (i.e. with higher “jumps” in the grayscale level) the density of sampling points has to be proportionally increased. This behavior differentiates natural images from range images and from cartoons – see Figure 3. It is also the reason why “Lenna”, whose grayscale surface displays the highly convoluted zone corresponding to the feathers, has a lower compression rate than the other images. (Cartoons have flat corresponding surfaces, however they also display intrinsic noise along certain edges, where high sampling density is generated, thus lowering the overall compression rate. For instance the compression rate for “White” is only 3.05.) Additional experiments were performed based on (2), i.e. using the Laplacian instead of the Hessian. As can be seen in Figure 4, there is no specific advantage of one approach over the other: There is little difference between the results for range images, while for nat-

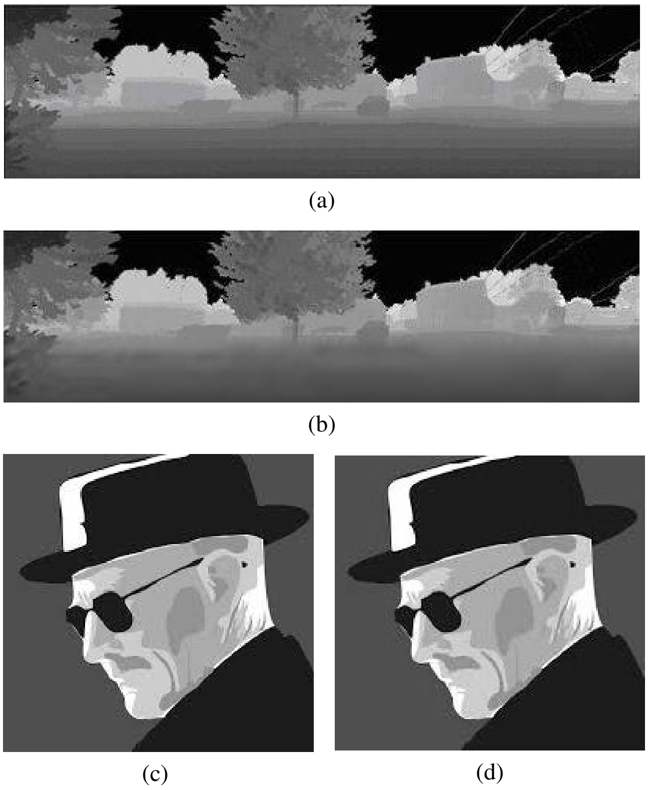


Fig. 2. Our method produces overall good results for types of images whose grayscale surface is flat, such as range images – (a) original image and (b) reconstructed image, and cartoons – (c) and (d), original, respective reconstructed image. The algorithm may fail at some of the very sharp edges.

Lenna	Dog	Cameraman	Range1	Range2	Range3
3.18	3.42	3.55	4.14	4.23	4.35

Table 1. Compression Rate measured as (number of sampling points)/(total number of points in the original image). Note that range images display an overall better compression rate.

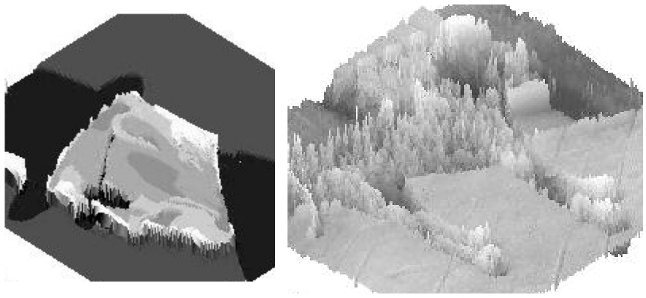


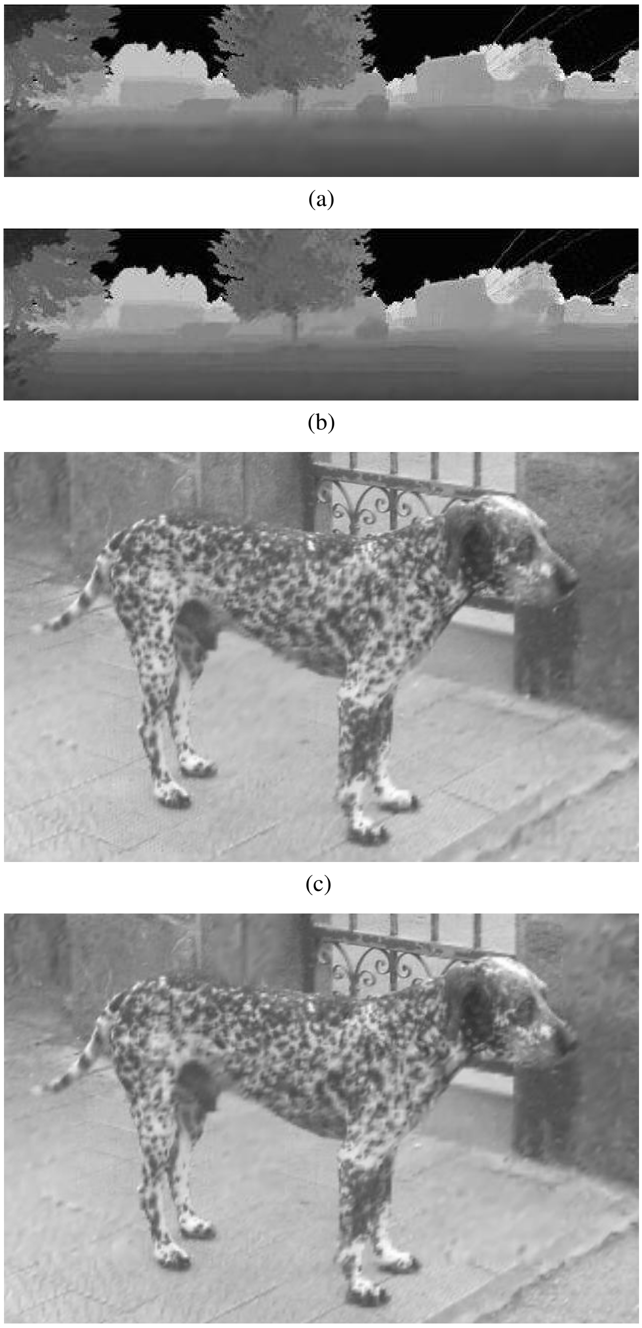
Fig. 3. Our method produces better results for types of images whose grayscale surface is generally flat, such as range images and cartoons (left), as opposed to natural images (right), which display more regions of high curvature.

ural images the Laplacian produces less artifacts along sharp edges, at the price of a somewhat lesser separation of small details (as indeed expected from a smoothing operator). However, these results show, perhaps against conventional intuition, that the classical Laplacian provides a satisfactory sampling density, at least for grayscale images.

5. CONCLUSION AND FUTURE WORK

As indicated already in the introduction, the present paper represents solely a first step in using the generalized Ricci curvature for Imaging purposes. Clearly, there are many directions along which the current research can be extended. We list below the most important and promising ones:

- The most important application of the generalized Ricci curvature is, as already mentioned, to medical images, such as MRI images. Since, for such images, the densities arise in an intrinsic manner and since, moreover, the raw, geometric data is itself 3-dimensional, the challenge – as well as the potential benefit – of the applying the generalized Ricci curvature are greatly enhanced. However, it is premature to infer from here that MRI images behave more like range images, rather than like natural images.
- Relevant 3-dimensional manifolds appear less trivially in the context of video. Here the 2-dimensional basic images (i.e. frames) evolve along a third dimension (time). If the data itself is 3-dimensional (e.g. if acquired, as above, by medical imaging techniques), the ensuing manifold will be 4-dimensional. The interest for such manifolds is not just theoretical, as a possible setting for the implementation of our proposed method: Since most landmarks criteria used in classical Graphics and Imaging are curvature based, one can naturally use the generalized Ricci curvature to determine landmarks on images that are acquired or produced by means of intrinsic densities.
- Another natural application of the sampling method introduced herein is in the field of Graphics. Here the densities arise as luminosity, degrees of shading, etc. Moreover,



(a)

(b)

(c)

(d)

Fig. 4. Hessian vs. the Laplacian reconstruction: There is no significant difference between the results for range images – see (a) for the Hessian based, and (b) for the Laplacian based reconstruction. For natural images the Laplacian (c) produces less artifacts along sharp edges, at the price of a somewhat lesser separation of small details rendered by the Hessian (d).

while in the present paper we restricted ourself to images, experiments on meshes where also performed and a detailed exposition of the results is currently in preparation. Moreover, similar densities, but also non-trivial ones, describing various texture properties, are also encountered in Imaging and Pattern Recognition.

- It is intuitively clear that if sampling points on a given manifold are chosen with a high enough sampling density, it is possible to reconstruct a "nice enough" function defined on the considered manifold. The observation above has been formally stated and proven in [19], [20] where the sampling density is given, again, by curvature. It is only natural to extend this result to weighted manifolds, using the generalized Ricci curvature. This direction of study, both on the theoretical and the experimental levels, represents work in progress.

  Apart from this new directions of study, a number of technical improvements of our proposed method should also be explored:

- In the experiments presented above, no smoothing was applied to the grayscale surface. However, since the differential operators appearing in (1) and (2) are originally devised in a smooth setting, it is interesting to also verify the method on smoothened data, e.g. by the application of various (smoothing) filters, e.g. the basic Gaussian filter.
- The choice we made for the computational versions of the Hessian and Laplacian are, by now means, unique. Therefore, further experiments with other modes of computing Hessian and the Laplacian would be interesting. This is especially true for the case of meshes (i.e. Graphics), where a discrete, well established, version of the Laplacian exists, as does for the Gaussian curvature.
- For compact manifolds (such as appearing in Graphics) use the more precise sampling density provided in [14] and extended to weighted manifolds in [15].

REFERENCES

[1] M. Unser, "Sampling - 50 years after shannon," *Proceedings of the IEEE*, vol. 88, no. 4, pp. 569–587, 2000.

[2] N. Sochen, R. Kimmel, and R. Malladi, "A general framework for low level vision," *Image Processing, IEEE Transactions on*, vol. 7, no. 3, pp. 310–318, 1998.

[3] G. Leibon and D. Letscher, "Delaunay triangulations and voronoi diagrams for riemannian manifolds," in *Proceedings of the Sixteenth Annual Symposium on Computational Geometry*.

[4] S.W. Cheng, T.K. Dey, and E.A. Ramos, "Manifold reconstruction from point samples," in *Proc. ACM-SIAM Sympos. Discrete Algorithms*.

[5] E. Saucan, E. Appleboim, and Y.Y. Zeevi, "Sampling and reconstruction of surfaces and higher dimensional manifolds," *Journal of Mathematical Imaging and Vision*, vol. 30, no. 1, pp. 105–123, 2008.

[6] E. Saucan, E. Appleboim, and Y.Y. Zeevi, "Geometric approach to sampling and communication," *Sampl. Theory Signal and Image Process.*, vol. 11, no. 1, pp. 1–24, 2012.

[7] S. Angenent, E. Pichon, and A. Tannenbaum, "Mathematical methods in medical image processing," *Bulletin of the American Mathematical Society*, vol. 43, no. 3, pp. 365–396, 2006.

[8] F. Morgan, "Manifolds with density," *Notices Amer. Math. Soc.*, vol. 52, pp. 853–858, 2005.

[9] D. Bakry and M. Émery, "Diffusions hypercontractives," in *Séminaire de Probabilités XIX 1983/4, LNM* **1123**, 1985, pp. 177–206.

[10] D. Bakry and M. Ledoux, "Lévy-gromov's isoperimetric inequality for an infinite dimensional diffusion generator," *Invent. Math.*, vol. 123, pp. 259–281, 1996.

[11] M. Berger, *A Panoramic View of Riemannian Geometry*, Springer-Verlag, Berlin, 2003.

[12] I. Corwin, N. Hoffman, S. Hurder, V. Šešum, and Xu Y., "Differential geometry of manifolds with density," *Rose Hulman Undergraduate Journal of Mathematics*, vol. 7, no. 1, pp. 1–15, 2006.

[13] J. Huang, A.B. Lee, and D. Mumford, "Statistics of range images," in *Proc. of the CVPR 2000*, 2000, pp. 1324–1331.

[14] K. Grove and P. Petersen, "Bounding homotopy types by geometry," *Ann. of Math.*, vol. 128, pp. 195–206, 1988.

[15] E. Saucan, "Curvature based triangulation of metric measure spaces," *Contemporary Mathematics*, vol. 554, pp. 207–227, 2011.

[16] E. Saucan, "A simple sampling method for metric measure spaces," *preprint*, vol. arXiv:1103.3843v1 [cs.IT], 2011.

[17] S. Moriguch and K. Murota, "On discrete hessian and convex extensibility," *Journal of the Operations Research Society of Japan*, vol. 6, pp. 1305–1315, 1997.

[18] Y. Eldar, M. Lindenbaum, M. Porat, and Y. Y. Zeevi, "The farthest point strategy for progressive image sampling," *IEEE Transactions on Image Processing*, vol. 55, pp. 48–62, 2012.

[19] I. Pesenson, "A sampling theorem on homeogeneous manifolds," *Trans. Amer. Math. Soc.*, vol. 352, no. 9, pp. 4257–4269, 2000.

[20] I. Pesenson, "Sampling in paley-wiener spaces on combinatorial graphs," *Trans. Amer. Math. Soc.*, vol. 352, pp. 5603–5627, 2008.