ANALYSIS OF THE HARMONICS CONTRIBUTION ON THE THREE-POINT INTERPOLATED DFT FREQUENCY ESTIMATOR

Daniel Belega[1], Dario Petri[2], Dominique Dallet[3]

[1]Department of Measurements and Optical Electronics, Politehnica University Timisoara, Romania, e-mail: daniel.belega@upt.ro
[2] Department of Industrial Engineering, University of Trento, Italy, e-mail: dario.petri@unitn.it
[3] IMS Laboratory, University of Bordeaux, Bordeaux INP, CNRS UMR5218, Talence, France, e-mail: dominique.dallet@ims-bordeaux.fr

ABSTRACT

In this paper the contribution of harmonics on frequency estimation obtained by the classical three-point interpolated Fourier method is investigated in the case when noisy and harmonically distorted complex sinusoids are analyzed. To this aim the expressions for the frequency estimation error due to harmonics and the approximated combined variance of the frequency estimator due to both harmonics and wideband noise are derived. Using the obtained expressions the contributions of each harmonic and wideband noise on the frequency estimation error are then compared. The accuracies of the derived expressions are verified through computer simulations.

Index Terms— complex sinusoid, error and statistical analysis, frequency estimation, harmonics, interpolated Fourier method

1. INTRODUCTION

In many engineering applications the frequency of a sinusoid need to be estimated accurately and in real-time. For this purpose Interpolated Discrete Fourier Transform (DFT) methods are commonly applied to the analyzed signal [1-10]. These methods provide accurate discrete frequency estimates and are very simple to understand and to implement. Essentially they are based on a two-step search procedure. In the first step (called coarse search) the peak location of the DFT spectrum of the analyzed signal is determined. That location corresponds to the rounded value of the signal discrete frequency expressed in bins. In the second step (called fine search) the interbin location is achieved by interpolating either the complex values or the module of the DFT samples related to the spectrum peak and its neighbors. The returned value represents the interbin location of the signal discrete frequency expressed in bins. The sum of the results returned by the two steps of the procedure represents the estimated discrete signal frequency.

In [5] Jacobsen and Kootsookos have suggested a three-point interpolated DFT estimator for complex sinusoids. The accuracy of that estimator, called JK estimator in the following, has been analyzed in the scientific literature only for noisy complex sinusoids. It has been shown that it exhibits a small bias [5,6]. However, in practice sinusoidal signals are often affected by harmonics, which have significant influence on the estimated frequency when the number of analyzed signal cycles is small. Unfortunately, the contribution of harmonics on the accuracy of the frequency estimates returned by the JK estimator (or the classical three-point interpolated DFT estimator) has not yet been analyzed in the scientific literature. This is the aim of this paper. For this purpose the expressions for the frequency estimation error due to harmonics, and the frequency estimator approximated combined variance due to both harmonics and wideband noise are derived. The obtained expressions allow the determination of the contribution of each harmonic to the frequency estimation error and its comparison with the effect of wideband noise. The derived theoretical results are verified by means of computer simulations.

2. ANALYSIS OF THE CONTRIBUTIONS OF THE HARMONICS AND WIDEBAND NOISE

The analyzed signal is a complex noisy and harmonically distorted discrete-time sinusoid, defined as:

$$y(m) = x(m) + e(m) = \sum_{h=1}^{N_h} A_h e^{j(2\pi h f m + \phi_h)} + e(m), \qquad m = 0, 1, \ldots, M-1 \tag{1}$$

where $x(\cdot)$ is the harmonically distorted sinusoid, $e(\cdot)$ is a complex additive white Gaussian noise of zero mean and variance σ^2, f is the signal discrete frequency, A_1, and ϕ_1 are the fundamental component amplitude and phase respectively, A_h, and ϕ_h are the amplitude and phase of the h-th harmonic, N_h is the maximum harmonic order, and M is the number of analyzed samples. Very often (1) is obtained by sampling a continuous-time signal of frequency f_{in} using a sampling rate f_s. In that case, the discrete frequency f can be expressed as:

$$f = \frac{f_{in}}{f_s} = \frac{\nu}{M} = \frac{l + \delta}{M}, \tag{2}$$

where ν represents the discrete frequency expressed in bins (or the number of analyzed sinusoid cycles), l is the rounded value of ν, while δ ($-0.5 \le \delta < 0.5$) is the difference between ν and l. It is well known that $\delta = 0$ corresponds to coherent sampling, while noncoherent sampling (i.e. $\delta \neq 0$) usually occurs in practice due to the lack synchronization between the acquired continuous-time signal and the sampling rate [11].

The DFT of the signal $y(m)$ is:

$$\begin{aligned} Y(k) &= \sum_{m=0}^{M-1} y(m) e^{-j2\pi \frac{k}{M} m} = X(k) + E(k) \\ &= A_1 W(k-\nu) e^{j\phi_1} + \sum_{h=2}^{N_h} A_h W(k-\nu_h) e^{j\phi_h} + E(k), \end{aligned} \tag{3}$$

where $X(\cdot)$ and $E(\cdot)$ are the DFT of the signal $x(\cdot)$ and the noise $e(\cdot)$, respectively, and $W(\cdot)$ is the Discrete-Time Fourier Transform (DTFT) of the rectangular window $w(\cdot)$ of length M. For $\lambda/M << 1$, $W(\cdot)$ can be approximated as:

$$W(\lambda) = \frac{M \sin(\pi\lambda)}{\pi\lambda} e^{-j\pi\lambda}. \tag{4}$$

The integer value l of the number of analyzed cycles ν can be estimated as the location of the peak of $|Y(k)|$, $k = 1, 2, \ldots, M/2 - 1$. If the frequency signal-to-noise ratio is higher than a threshold of about 16-18 dB, it can be accurately obtained using a maximum search procedure applied to the discrete spectrum $|Y(k)|$, $k = 1, 2, \ldots, M/2 - 1$ [12].

Conversely, the fractional part δ of ν can be estimated by the JK estimator through the expression [5,6]:

$$\hat{\delta} = \mathrm{Re}\left\{ \frac{Y(l+1) - Y(l-1)}{Y(l-1) - 2Y(l) + Y(l+1)} \right\} \tag{5}$$

where $\mathrm{Re}\{\cdot\}$ denotes the real-value operator.

It is well known that the contribution of harmonics on the accuracy of the estimator $\hat{\delta}$ is high when ν is small [11], [12]. Also, we assume that $N_h \cdot f < f_s/2$, i.e. all the harmonics are inside the baseband $(0, f_s/2)$. This imply that the number of acquired cycles related to the h-th harmonic, ν_h, is equal to $\nu_h = h \cdot \nu = h \cdot (l + \delta)$, $h = 2, 3, \ldots, N_h$. In this case the estimator (5) is well approximated by the expression reported in the following proposition.

Proposition: For the complex noisy and harmonically distorted signal (1) the estimator $\hat{\delta}$ returned by (5) can be approximated as:

$$\begin{aligned} \hat{\delta} &\cong \delta + \sum_{h=2}^{N_h} (h-1) \frac{A_h}{A_1} \frac{\sin(\pi h \delta)}{\sin(\pi\delta)} \\ &\times \frac{\delta(\delta^2 - 1)(l+\delta)\cos(\Delta\phi_h + \pi(h-1)\delta)}{[(h-1)l + h\delta]\{[(h-1)l + h\delta]^2 - 1\}} \\ &+ \mathrm{Re}\left\{ \frac{(1-\delta)E(l-1) + 2\delta E(l) - (1+\delta)E(l+1)}{A_1[W(-\delta-1) - 2W(-\delta) + W(-\delta+1)]e^{j\phi_1}} \right\}, \end{aligned} \tag{6}$$

where $\Delta\phi_h = \phi_h - \phi_1$.

The proof of that proposition is given in the Appendix.

From (6) it follows that the estimation error on δ can be expressed as:

$$\Delta_\delta = \hat{\delta} - \delta \cong \sum_{h=2}^{N_h} \varepsilon_h + \varepsilon_n, \tag{7}$$

where

$$\begin{aligned} \varepsilon_h &\stackrel{\Delta}{=} (h-1) \frac{A_h}{A_1} \frac{\sin(\pi h \delta)}{\sin(\pi\delta)} \\ &\times \frac{\delta(\delta^2 - 1)(l+\delta)\cos(\Delta\phi_h + \pi(h-1)\delta)}{[(h-1)l + h\delta]\{[(h-1)l + h\delta]^2 - 1\}}, \end{aligned} \tag{8}$$

and

$$\varepsilon_n \stackrel{\Delta}{=} \mathrm{Re}\left\{ \frac{(1-\delta)E(l-1) + 2\delta E(l) - (1+\delta)E(l+1)}{A_1[W(-\delta-1) - 2W(-\delta) + W(-\delta+1)]e^{j\phi_1}} \right\}. \tag{9}$$

The terms ε_h, $h = 2, 3, \ldots, N_h$ are due to the harmonics, while the term ε_n is due to wideband noise.

In the following we analyze both harmonically distorted sinusoid and noisy and harmonically distorted sinusoid, respectively.

a) *Harmonically distorted sinusoid*

In this case, from (7) it follows that the estimation error on δ is:

$$\Delta_\delta \cong \tilde{\varepsilon}_h = \sum_{h=2}^{N_h} \varepsilon_h, \tag{10}$$

where ε_h is given by (8).

For a given value of δ, (8) shows that:

- ε_h depends on the ratio A_h/A_1 between the harmonic and the fundamental amplitudes ;
- ε_h has a sine-wave like behavior with respect to the phase difference $\Delta\phi_h$;
- the error Δ_δ is null when coherent sampling occurs, i.e. $\delta = 0$;
- the error ε_h is null when $|\delta| = 1/h$ or $\Delta\phi_h + \pi(h-1)\delta = (2p+1)\pi/2$, where p is an integer;
- the error $|\varepsilon_h|$ decreases as h increases and/or l increases.

b) *Noisy and harmonically distorted sinusoid*

Since usually the sampling rate is asynchronous with the signal frequency, the phases of the fundamental component and harmonics vary randomly in subsequent acquisitions. Hence, in the following we model the phase differences $\Delta\phi_h$, $h = 1, 2, \ldots, N_h$ as uniform random variables. As a consequence, the terms ε_h can be modeled as random variables and the related contributions to the term $\tilde{\varepsilon}_h$ can be considered statistically independent of each other and the noise contribution ε_n. In fact they are due to different physical phenomena. Thus from (7) it follows that the approximated combined variance of the frequency estimator $\hat{\delta}$ can be expressed as [13]:

$$\sigma_{\hat{\delta}}^2 = \sigma_{\hat{\delta},h}^2 + \sigma_{\hat{\delta},n}^2 \cong \sum_{h=2}^{N_h} \rho_h + \sigma_{\hat{\delta},n}^2 \qquad (11)$$

where $\sigma_{\hat{\delta},h}^2 = \sum_{h=2}^{N_h} \rho_h$, is the contributeon due to the random variable $\tilde{\varepsilon}_h$, in which ρ_h represents the power of the error ε_h due to the h-th harmonic:

$$\rho_h \cong 0.5 \sum_{h=2}^{N_h} (h-1)^2 \left(\frac{A_h}{A_1}\right)^2 \frac{\sin^2(\pi h\delta)}{\sin^2(\pi\delta)} \times \frac{\delta^2(\delta^2-1)^2(l+\delta)^2}{[l(h-1)+h\delta]^2\{[l(h-1)+h\delta]^2-1\}^2}, \qquad (12)$$

and $\sigma_{\hat{\delta},n}^2$ is the variance contribution due to wideband noise, ant it is given by [14]:

$$\sigma_{\hat{\delta},n}^2 \cong \frac{\pi^2}{4} \frac{\delta^2(\delta^2-1)^2(3\delta^2+1)}{\sin^2(\pi\delta)} \frac{1}{M \cdot SNR}, \qquad (13)$$

in which $SNR = A_1^2/\sigma^2$ is the Signal-to-Noise Ratio.

For a given value of δ, remarks similar to those drawn from (8) can be derived also from (12). Moreover, (11) - (13) show that the contribution of harmonics becomes negligible as compared with the effect of noise when:

$$\sigma^2 >> \frac{2M}{\pi^2} \frac{(l+\delta)^2}{3\delta^2+1} \sum_{h=2}^{N_h} (h-1)^2 A_h^2 \times \frac{\sin^2(\pi h\delta)}{[l(h-1)+h\delta]^2\{[l(h-1)+h\delta]^2-1\}^2}. \qquad (14)$$

3. COMPUTER SIMULATIONS

The aim of this section is to verify through computer simulations the accuracies of expressions (10) and (11) and the statistical performance of the estimator $\hat{\delta}$ in the case of noisy and harmonically distorted sinusoids.

The amplitude of the fundamental component is assumed $A_1 = 1$. The analyzed signals contain the 2nd, 3rd, and 4th harmonics, with amplitudes in the ratios 4:2:1 in such a way that the related Total Harmonic Distortion ratio (THD) is equal to 10%. The number of analyzed samples is $M = 512$. The discrete frequency ν varies in the range [2.51, 12) bins with a step of 1/16. For each value of ν, 1000 records are considered by varying the phases of the fundamental component and harmonics at random.

a) *Harmonically distorted sinusoid*

Fig. 1 shows the maximum of the module of the frequency estimation error $|\Delta_\delta|_{max}$ returned by both (10) and simulations as a function of ν.

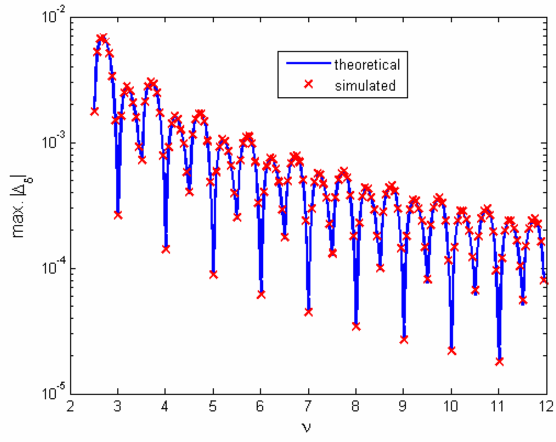

Fig. 1. Error $|\Delta_\delta|_{max}$ returned by (10) (solid line) and simulations (crosses) versus ν. Harmonically distorted signal with THD = 10%.

As we can see, there is a very good agreement between theoretical and simulation results. It is worth noticing that the error $|\Delta_\delta|_{max}$ is mainly due to the 2nd harmonic since $|\varepsilon_2|_{max} >> |\varepsilon_3|_{max}$ and $|\varepsilon_2|_{max} >> |\varepsilon_4|_{max}$. Also, Fig. 1 proves that the error contribution due to $\tilde{\varepsilon}_h$ becomes negligible when quasi-coherent sampling occurs, i.e. $\delta \cong 0$.

Fig. 2 shows the absolute value of the bias of the estimator $\hat{\delta}$ as a function of ν. It can be seen, for all the considered values of ν, the bias is negligible with respect to the maximum error reported in Fig.1, except when quasi-coherent sampling occurs.

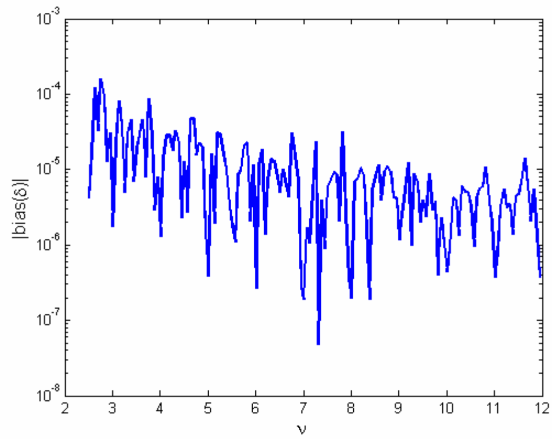

Fig. 2. Absolute value of the bias of the estimator $\hat{\delta}$ versus ν. Harmonically distorted signal with THD = 10%.

b) *Noisy and harmonically distorted sinusoid*

Fig. 3 shows the Mean Square Errors (MSE) of the estimator $\hat{\delta}$ and the approximated combined variance $\sigma_{\hat{\delta}}^2$ returned by (11) as a function of ν when the harmonically distorted sinewave considered above is corrupted by white Gaussian noise with zero mean and variance chosen to ensure a SNR = 40 dB. In addition, the MSE of the estimator $\hat{\delta}$ obtained in the case of harmonics free signal is reported in Fig. 3.

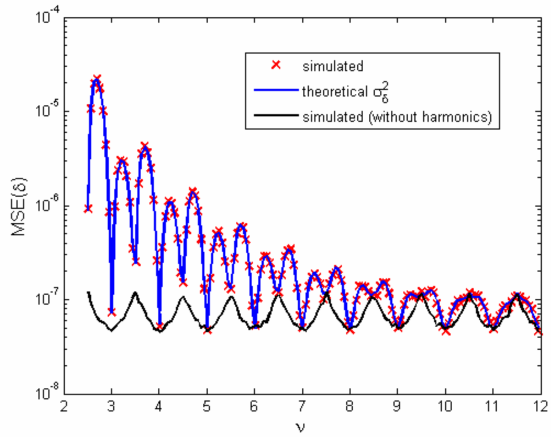

Fig. 3. MSE($\hat{\delta}$) returned by simulations (crosses), combined variance $\sigma^2_{\hat{\delta}}$ (11) (solid line). Harmonically distorted signal with *SNR* = 40 dB and THD = 10%. The MSE($\hat{\delta}$) returned by simulations in the case of harmonics free signal is also reported to show the effect of wideband noise.

Fig. 3 shows that the estimated frequency MSE and the combined variance $\sigma^2_{\hat{\delta}}$ are very close each other. This result holds for all the considered values of ν since the frequency estimation bias is negligible as compared to the standard deviation. Also, Fig. 3 shows that the estimated frequency MSE related to harmonically distorted sinusoids is significantly higher than the MSE due only to wideband noise when $\nu < 11$. Indeed, when the number of observed sinusoid cycles is small harmonics prevail. Conversely, when $\nu > 11$ the contribution of wideband noise prevails over harmonics. It is worth noticing that the value of ν above which the noise contribution prevails increases as *SNR* increases. Moreover, as in Fig. 1, Fig. 3 shows that the effect of harmonics is negligible when $\delta \cong 0$.

4. CONCLUSIONS

This paper focuses on the analysis of the contribution of harmonics on the frequency estimates returned by the three-point interpolated DFT method proposed in [5] and [6]. To this aim the expressions for the frequency estimation error due to harmonics and the frequency estimator approximated combined variance have been derived in the case of a noisy and harmonically distorted sinusoid. The accuracies of the derived expressions have been confirmed through computer simulations. It has been shown that the harmonics affect the frequency estimator when the number of analyzed sinusoid cycles is quite small. In these situations the frequency estimator MSE is almost equal to the related approximated combined variance. Moreover, the accuracy of the frequency estimator can be improved by using windowing or including harmonics in the signal model, but at the cost of an increase computational burden of the derived frequency estimators.

REFERENCES

[1] B.G. Quinn, "Estimating frequency by interpolation using Fourier coefficients," *IEEE Trans. Signal Process.* vol. 42, no. 5, pp. 1264-1268, May 1994.

[2] B.G. Quinn, "Estimation of frequency, amplitude, and phase from the DFT of a time series," *IEEE Trans. Signal Process.*, vol. 45, no. 3, pp. 814-817, Mar. 1997.

[3] E. Aboutanios and B. Mulgrew, "Iterative frequency estimation by interpolation on Fourier coefficients," *IEEE Trans. Signal Process.*, vol. 53, no. 4, pp. 1237-1241, Apr. 2005.

[4] Y. Liu, Z. Nie, Z. Zhao, and Q.H. Liu, "Generalization of iterative Fourier interpolation algorithm for single frequency estimation," *Digital Signal Process.*, vol. 21, no. 1, pp. 141-149, 2011.

[5] E. Jacobsen and P. Kootsookos, "Fast, accurate frequency estimators," *IEEE Signal Process. Mag.*, vol. 24, pp. 123-125, May 2007.

[6] C. Candan, "A method for fine resolution frequency estimation from three DFT samples", *IEEE Signal Process. Lett.*, vol. 18, no. 6, pp. 351-354, June 2011.

[7] C. Candan, "Analysis and further improvement of fine resolution frequency estimation method from three DFT samples," *IEEE Signal Process. Lett.*, vol. 20, no. 9, pp. 913-916, Sept. 2013.

[8] J-R Liao and S. Lo, "Analytical solutions for frequency estimators by interpolation of DFT coefficients," *Signal Process.*, vol. 100, pp. 93-100, 2014.

[9] D. Belega, D. Dallet, and D. Petri, "Statistical description of the sine-wave frequency estimator provided by the interpolated DFT method", *Measurement,* vol. 45, no.1, pp. 109-117, 2012.

[10] D. Belega and D. Petri, "Accuracy analysis of the multicycle synchrophasor estimator provided by the interpolated DFT algorithm", *IEEE Trans. Instrum. Meas.*, vol. 62, no. 5, pp. 942-953, May 2013.

[11] A. Ferrero and R. Ottoboni, "High-accuracy Fourier analysis based on synchronous sampling techniques," *IEEE Trans. Instrum. Meas.*, vol. 41 no. 6, pp. 780-785, Dec. 1992.

[12] C. Offelli and D. Petri, "Weigthing effect on the dicrete time Fourier transform of noisy signals," *IEEE Trans. Instrum. Meas.*, vol. 40, no. 6, pp. 972-981, Dec. 1991.

[13] Guide for the expression of Uncertainty in Measurements, International Organization for Standardization, Switzerland, 1993.

[14] D. Belega and D. Petri, "Frequency estimation by two-or three-point interpolated Fourier algorithms based on cosine windows," *Signal Process.*, vol. 117, pp. 115-125, Dec. 2015.

APPENDIX

Proof of the proposition

We denote by α the ratio in (5) as:

$$\alpha = \frac{Y(l+1) - Y(l-1)}{Y(l-1) - 2Y(l) + Y(l+1)}. \quad \text{(A.1)}$$

Using (3) and dividing both the numerator and the denominator of the above expression by $A_1[W(1-\delta) - 2W(-\delta) + W(-1-\delta)]e^{j\phi_1}$, after simple calculations we obtain the expression (A.2) given at the bottom of the page.

Since the module of the last two terms in the denominator of (A.2) is much less than 1 and the product of terms related to harmonics and/or wideband noise is negligible (with high probability) as compared with the others, the ratio α can be accurately approximated by the expression (A.3) given at the bottom of the page.

Using (4) the following equalities can be obtained:

$$W(1-\delta) - W(-1-\delta) = \frac{2M\sin(\pi\delta)}{\pi(\delta^2 - 1)} e^{j\pi\delta}, \quad \text{(A.4)}$$

$$W(1-\delta) - 2W(-\delta) + W(-1-\delta) = \frac{2M\sin(\pi\delta)}{\pi\delta(\delta^2 - 1)} e^{j\pi\delta}, \quad \text{(A.5)}$$

$$W[-(h-1)l - h\delta + 1] - W[-(h-1)l - h\delta - 1] = \frac{2M\sin(\pi h\delta)}{\pi\{[(h-1)l + h\delta]^2 - 1\}} e^{j\pi h\delta}, \quad \text{(A.6)}$$

$$W[-(h-1)l - h\delta + 1] - 2W[-(h-1)l - h\delta] + W[-(h-1)l - h\delta - 1] = \frac{2M\sin(\pi h\delta)}{\pi[(h-1)l + h\delta]\{[(h-1)l + h\delta]^2 - 1\}} e^{j\pi h\delta}. \quad \text{(A.7)}$$

By replacing (A.4) – (A.7) in (A.3), after some algebra the following equality can be derived:

$$\alpha \cong \delta + \sum_{h=2}^{N_h} (h-1) \frac{A_h}{A_1} \frac{\sin(\pi h\delta)}{\sin(\pi\delta)} \times \frac{\delta(\delta^2 - 1)(l+\delta)}{[(h-1)l + h\delta]\{[(h-1)l + h\delta]^2 - 1\}} e^{j(\phi_h - \phi_1 + \pi(h-1)\delta)} + \frac{(1-\delta)E(l-1) + 2\delta E(l) - (1+\delta)E(l+1)}{A_1[W(-\delta-1) - 2W(-\delta) + W(-\delta+1)]e^{j\phi_1}}. \quad \text{(A.8)}$$

By applying the real-value operator to (A.8) the expression (6) is finally achieved.

$$\alpha = \left\{ \frac{W(1-\delta) - W(-1-\delta)}{W(1-\delta) - 2W(-\delta) + W(-1-\delta)} + \sum_{h=2}^{N_h} \frac{A_h}{A_1} \frac{W[-(h-1)l - h\delta + 1] - W[-(h-1)l - h\delta - 1]}{W(1-\delta) - 2W(-\delta) + W(-1-\delta)} e^{j(\phi_h - \phi_1)} + \frac{E(l+1) - E(l-1)}{A_1[W(1-\delta) - 2W(-\delta) + W(-1-\delta)]e^{j\phi_1}} \right\}$$
$$\times \left\{ 1 + \sum_{h=2}^{N_h} \frac{A_h}{A_1} \frac{W[-(h-1)l - h\delta + 1] - 2W[-(h-1)l - h\delta] + W[-(h-1)l - h\delta - 1]}{W(1-\delta) - 2W(-\delta) + W(-1-\delta)} e^{j(\phi_h - \phi_1)} + \frac{E(l-1) - 2E(l) + E(l+1)}{A_1[W(1-\delta) - 2W(-\delta) + W(-1-\delta)]e^{j\phi_1}} \right\}^{-1}. \quad \text{(A.2)}$$

$$\alpha \cong \frac{W(1-\delta) - W(-1-\delta)}{W(1-\delta) - 2W(-\delta) + W(-1-\delta)} + \sum_{h=2}^{N_h} \frac{A_h}{A_1} \frac{W[-(h-1)l - h\delta + 1] - W[-(h-1)l - h\delta - 1]}{W(1-\delta) - 2W(-\delta) + W(-1-\delta)} e^{j(\phi_h - \phi_1)} -$$
$$- \delta \sum_{h=2}^{N_h} \frac{A_h}{A_1} \frac{W[-(h-1)l - h\delta + 1] - 2W[-(h-1)l - h\delta] + W[-(h-1)l - h\delta - 1]}{W(1-\delta) - 2W(-\delta) + W(-1-\delta)} e^{j(\phi_h - \phi_1)}$$
$$+ \frac{E(l+1) - E(l-1)}{A_1[W(1-\delta) - 2W(-\delta) + W(-1-\delta)]e^{j\phi_1}} - \delta \frac{E(l-1) - 2E(l) + E(l+1)}{A_1[W(1-\delta) - 2W(-\delta) + W(-1-\delta)]e^{j\phi_1}}. \quad \text{(A.3)}$$