BIAS ANALYSIS OF AN ALGEBRAIC SOLUTION FOR TDOA LOCALIZATION WITH SENSOR LOCATION ERRORS

A. Liu Yang, B. Fucheng Guo*, C. Le Yang†‡, D. Zhang Min*, E. Wenli Jiang**

* College of Electronic Science and Engineering, National University of Defense Technology, Changsha, Hunan, 410073, P. R. China

† School of Internet of Things (IoT) Engineering, Jiangnan University, Wuxi, Jiangsu

‡ Synergetic Innovation Center of Food Satety and Nutrition, Wuxi, Jiangsu, 214122, P. R. China

ABSTRACT

The nonlinearity inherent in the time difference of arrival (TDOA)-based source localization problem leads to biased source location estimates. The estimation bias of the closed-form TDOA-positioning technique,two-stage least squares(TSWLS) technology was established analytically in previous works. This paper extends the theoretical developments in the case where there exists no sensors location errors to investigate the bias of TDOA-positioning in the presence of sensor location errors. Specifically, the estimation bias of the algebraic two-stage TDOA localization algorithm proposed is derived. Simulations validate the obtained theoretical results. It is shown that different from the findings in previous works where the estimation bias of the two-stage solution mainly comes from its Stage-1 processing, both stages of the localization algorithm considered in this work can introduce significant estimation biases when sensor location errors are present.

Index Terms— time difference of arrival, bias analysis, source localization, sensor location error

1. INTRODUCTION

Source localization using time difference of arrival (TDOA) measurements has been extensively studied under diverse application scenarios such as radar [1] and wireless sensor networks [2]. The TDOA measurements are nonlinearly related to the source location and this could lead to the presence of bias in the source location estimate. When an approximately efficient TDOA localization algorithm is used and the TDOA noise is small, the localization bias would not make the localization mean square error (MSE) deviate evidently from the Cramer-Rao lower bound (CRLB). But the localization bias can limit the performance of e.g., the source tracking [3].

Due to the low computational complexity and the proven approximate efficiency, the two-stage localization algorithm [4] is perhaps one of the most popular TDOA positioning techniques. It estimates the source location and a nuisance parameter in its Stage-1 processing while Stage-2 refines the source location estimate. Both stages requires the use of closed-form weighted least squares (WLS) technique only. The original two-stage method was recently generalized to the scenario where the sensor locations are subject to random errors [5, 6]. The newly obtained two-stage solution inherits the advantages of being closed-form, not requiring proper initialization as in the iterative method and having lower computational load than e.g., the constrained total least squares (CTLS) algorithm [7] and the multidimensional scaling (MDS)-based method [8]. [9] examined the bias of two-stage WLS algorithm in the case where there is no sensor position errors. In [9], the estimation bias of the original two-stage algorithm [4] has been derived using the second-order perturbation analysis. It is found that the Stage-1 processing is the major source of the final localization bias in this case. As a result, the BiasRed and the BiasSub methods were proposed to mitigate the localization bias. In particular, the BiasSub technique reduces the localization bias via subtracting an estimated version of the bias from the localization result.

It is purpose of this paper to extend the bias analysis framework developed in [9] to the case where sensor location errors are present. Specifically, we shall derive the estimation bias of the two-stage solutions newly proposed for TDOA-positioning in the presence of sensor location errors. The obtained bias result can be used to establish bias mitigation techniques such as the BiasRed method in [9]. The theoretical developments will be verified by simulations. We shall also show that in the presence of sensor location errors, both processing stages of the considered localization algorithm would contribute significantly to the final estimation bias.

2. ALGORITHM OVERVIEW AND BIAS ANALYSIS

In this section, we shall first introduce the symbols and notations. We then summarize the two-stage algorithm in consideration and present the theoretical bias analysis.

978-0-9928626-3-3/15/$31.00 ©2015 IEEE

2.1. Symbols and Notations

M sensors are located at $\mathbf{s}_i^o = [x_i^o, y_i^o, z_i^o]^T, i = 1, 2..., M$. The sensor locations known to the localization algorithm are $\mathbf{s}_i = \mathbf{s}_i^o + \Delta\mathbf{s}_i$, where $\Delta\mathbf{s}_i$ is the location error of sensor i. Defining the sensor location vector as $\mathbf{s} = [\mathbf{s}_1^T, \mathbf{s}_2^T, ..., \mathbf{s}_M^T]^T$ and $\mathbf{s} = \mathbf{s}^o + \Delta s$, where $\mathbf{s}^o = [\mathbf{s}_1^{oT}, \mathbf{s}_2^{oT}, ..., \mathbf{s}_M^{oT}]^T$ and $\Delta\mathbf{s} = [\Delta\mathbf{s}_1^T, \Delta\mathbf{s}_2^T, ..., \Delta\mathbf{s}_M^T]^T$. $\Delta\mathbf{s}$ is assumed to be a zero-mean Gaussian vector with covariance matrix $\mathbf{Q}_s$.

TDOAs are measured with sensor 1 as the reference sensor. Define the $(M-1)\times 1$ TDOA measurement vector as $\mathbf{r} = [r_{21}, r_{31}, ..., r_{M1}]^T = \mathbf{r}^o + \mathbf{n}$, where $\mathbf{r}^o = [r_{21}^o, r_{31}^o, ..., r_{M1}^o]^T$ and $\mathbf{n} = [n_{21}, n_{31}, ..., n_{M1}]^T$ is the noise vector modeled as a zero-mean Gaussian vector with covariance matrix $\mathbf{Q}_t$. $\mathbf{n}$ is further assumed to be independent of the sensor location error vector $\Delta\mathbf{s}$.

The true TDOA measurement r_{i1}^o, $i = 2, 3, ...M$, is equal to

$$r_{i1}^o = r_i^o - r_1^o, r_i^o = \|\mathbf{u}^o - \mathbf{s}_i^o\| \tag{1}$$

where $\|*\|$ denotes the Euclidean distance and $\mathbf{u}^o = [x^o, y^o, z^o]^T$ is the source location to be identified.

2.2. Two-stage Algorithm

$Stage-1$: This stage estimates $\boldsymbol{\varphi}_1^o = [\mathbf{u}^{oT}, \hat{r}_1^o]^T$. The WLS solution is

$$\boldsymbol{\varphi}_1 = (\mathbf{G}_1^T\mathbf{W}_1\mathbf{G}_1)^{-1}\mathbf{G}_1^T\mathbf{W}_1\mathbf{h}_1 \tag{2}$$

where $\hat{r}_1^o = ||\mathbf{u}^o - \mathbf{s}_1||$. $\mathbf{W}_1$ is the weighting matrix equal to

$$\mathbf{W}_1 = (\mathbf{B}_1\mathbf{Q}_t\mathbf{B}_1^T + \mathbf{D}_1\mathbf{Q}_s\mathbf{D}_1^T)^{-1}. \tag{3}$$

$\mathbf{h}_1, \mathbf{G}_1, \mathbf{B}_1$ and $\mathbf{D}_1$ are defined in Section III.B of [6].

$Stage-2$: Stage-2 explores the relationship between $\hat{r}_1^o$ and $\mathbf{u}^o$ to obtain the final source location estimate. Specifically, we first find the WLS estimate of $\boldsymbol{\varphi}_2^o = (\mathbf{u}^o - \mathbf{s}_1) \odot (\mathbf{u}^o - \mathbf{s}_1)$ via

$$\boldsymbol{\varphi}_2 = (\mathbf{G}_2^T\mathbf{W}_2\mathbf{G}_2)^{-1}\mathbf{G}_2^T\mathbf{W}_2\mathbf{h}_2 \tag{4}$$

where $\mathbf{G}_2, \mathbf{W}_2$ and $\mathbf{h}_2$ are also defined in [6], and $\odot$ is the Hadamard product [10]. The final source location estimate is

$$\mathbf{u} = \Pi\sqrt{\boldsymbol{\varphi}_2} + \mathbf{s}_1 \tag{5}$$

$$\Pi = \text{diag}\Big\{\text{sgn}(\boldsymbol{\varphi}_1(1:3) - \mathbf{s}_1)\Big\} \tag{6}$$

where $\text{sgn}(*)$ is the signum function and it is used to avoid the sign ambiguity due to the square-root operation in (5).

2.3. Bias Analysis

We shall generalize the second-order perturbation analysis developed in [9] to derive the estimation bias of the two-stage localization algorithm presented in Section 2.2. The derivation starts with evaluating the estimation bias in the Stage-1 processing.

Bias in Stage -1: Subtracting $\boldsymbol{\varphi}_1^o$ from both sides of (2) yields the estimation error of Stage-1, which is equal to

$$\Delta\boldsymbol{\varphi}_1 = \boldsymbol{\varphi}_1 - \boldsymbol{\varphi}_1^o = (\mathbf{G}_1^T\mathbf{W}_1\mathbf{G}_1)^{-1}\mathbf{G}_1^T\mathbf{W}_1(\mathbf{h}_1 - \mathbf{G}_1\boldsymbol{\varphi}_1^o). \tag{7}$$

To evaluate $\mathbf{h}_1 - \mathbf{G}_1\boldsymbol{\varphi}_1^o$, we expand $r_1^o = ||\mathbf{u}^o - \mathbf{s}_1^o||$ around the noisy location of sensor 1, $\mathbf{s}_1$, using the Taylor-series (TS) expansion up to second-order terms. We arrive at

$$r_1^o \approx \hat{r}_1^o + \boldsymbol{\rho}_{\mathbf{u}^o,\mathbf{s}_1}^T\Delta\mathbf{s}_1 + \frac{1}{2}\Delta\mathbf{s}_1^T\mathbf{B}\Delta\mathbf{s}_1 \tag{8}$$

where $\mathbf{B} = \frac{1}{\hat{r}_1^o}(\mathbf{I}_{3\times 3} - \boldsymbol{\rho}_{\mathbf{u}^o,\mathbf{s}_1}\boldsymbol{\rho}_{\mathbf{u}^o,\mathbf{s}_1}^T)$. Using the definitions of $\mathbf{h}_1$ and $\mathbf{G}_1$, and applying (8), we arrive at

$$\begin{aligned}\Delta\boldsymbol{\varphi}_1 &= (\mathbf{G}_1^T\mathbf{W}_1\mathbf{G}_1)^{-1}\mathbf{G}_1^T\mathbf{W}_1(\mathbf{N}_t + \mathbf{N}_s)\\ \mathbf{N}_t &= \mathbf{B}_1\mathbf{n} + \mathbf{n}\odot\mathbf{n}\\ \mathbf{N}_s &= \mathbf{D}_1\Delta\mathbf{s} + \mathbf{r}^o\Delta\mathbf{s}_1^T\mathbf{B}\Delta\mathbf{s}_1 + \mathbf{N}_1\\ \mathbf{N}_1 &= \begin{bmatrix}\Delta\mathbf{s}_1^T\Delta\mathbf{s}_1 + \Delta\mathbf{s}_2^T\Delta\mathbf{s}_2\\ \Delta\mathbf{s}_1^T\Delta\mathbf{s}_1 + \Delta\mathbf{s}_3^T\Delta\mathbf{s}_3\\ \vdots\\ \Delta\mathbf{s}_1^T\Delta\mathbf{s}_1 + \Delta\mathbf{s}_M^T\Delta\mathbf{s}_M\end{bmatrix}\end{aligned} \tag{9}$$

where $\mathbf{N}_t$ and $\mathbf{N}_s$ denote the error terms due to TDOA measurement noise $\mathbf{n}$ and sensor location error $\Delta\mathbf{s}$, respectively.

Evaluating the expectation of $\Delta\boldsymbol{\varphi}_1$ in (9) gives the estimation bias of Stage-1. Let $\Delta\boldsymbol{\varphi}_{1,t} = (\mathbf{G}_1^T\mathbf{W}_1\mathbf{G}_1)^{-1}\mathbf{G}_1^T\mathbf{W}_1\mathbf{N}_t$ and $\Delta\boldsymbol{\varphi}_{1,s} = (\mathbf{G}_1^T\mathbf{W}_1\mathbf{G}_1)^{-1}\mathbf{G}_1^T\mathbf{W}_1\mathbf{N}_s$ such that $\Delta\boldsymbol{\varphi}_1 = \Delta\boldsymbol{\varphi}_{1,t} + \Delta\boldsymbol{\varphi}_{1,s}$. Taking expectation of $\Delta\boldsymbol{\varphi}_{1,t}$ yields [9]

$$\begin{aligned}E[\Delta\boldsymbol{\varphi}_{1,t}] &= \mathbf{H}_1(\mathbf{q}_t + 2\mathbf{Q}_t\mathbf{B}_1\mathbf{H}_1(4,:)^T)\\ &+ 2(\mathbf{G}_1^{oT}\mathbf{W}_1\mathbf{G}_1^o)^{-1}\begin{bmatrix}\mathbf{0}_{3\times 1}\\ tr(\mathbf{W}_1(\mathbf{G}_1^o\mathbf{H}_1 - \mathbf{I})\mathbf{B}_1\mathbf{Q}_t)\end{bmatrix}\end{aligned} \tag{10}$$

where

$$\mathbf{H}_1 = (\mathbf{G}_1^{oT}\mathbf{W}_1\mathbf{G}_1^o)^{-1}\mathbf{G}_1^{oT}\mathbf{W}_1 \tag{11}$$

$tr()$ denotes the matrix trace and $\mathbf{G}_1^o$ is the noise-free version of $\mathbf{G}_1$ (see the definition of $\mathbf{G}_1^o$ below). $\mathbf{H}_1(4,:)$ represents $4th$ row of the matrix $\mathbf{H}_1$. $\mathbf{q}_t$ is a column vector formed by the diagonal elements of $\mathbf{Q}_t$, the covariance matrix of the TDOA measurement noise vector $\mathbf{n}$ (see Section 2.1).

We proceed to evaluate $E[\Delta\boldsymbol{\varphi}_{1,s}]$ to find $E[\Delta\boldsymbol{\varphi}_1]$. For notation simplicity, let $\mathbf{P}_1$ be $\mathbf{G}_1^T\mathbf{W}_1\mathbf{G}_1$ and $\Delta\boldsymbol{\varphi}_{1,s}$ becomes

$$\Delta\boldsymbol{\varphi}_{1,s} = \mathbf{P}_1^{-1}\mathbf{G}_1^T\mathbf{W}_1\mathbf{N}_s. \tag{12}$$

We express $\mathbf{G}_1$ as

$$\mathbf{G}_1 = \mathbf{G}_1^o + \Delta\mathbf{G}_1 \tag{13}$$

where

$$\Delta\mathbf{G}_1 = -2\begin{bmatrix}\mathbf{A}_s & \mathbf{n}\end{bmatrix}, \mathbf{A}_s = \begin{bmatrix}(\Delta\mathbf{s}_2 - \Delta\mathbf{s}_1)^T \\ (\Delta\mathbf{s}_3 - \Delta\mathbf{s}_1)^T \\ \vdots \\ (\Delta\mathbf{s}_M - \Delta\mathbf{s}_1)^T\end{bmatrix}. \quad (14)$$

With the help of (13), $\mathbf{P}_1$ can be re-written as

$$\mathbf{P}_1 \approx \mathbf{P}_1^o + \Delta\mathbf{P}_1, \mathbf{P}_1^o = \mathbf{G}_1^{oT}\mathbf{W}_1\mathbf{G}_1^o \quad (15)$$

$$\Delta\mathbf{P}_1 = \mathbf{G}_1^{oT}\mathbf{W}_1\Delta\mathbf{G}_1 + \Delta\mathbf{G}_1^T\mathbf{W}_1\mathbf{G}_1^o \quad (16)$$

where $\Delta\mathbf{G}_1^T\mathbf{W}_1\Delta\mathbf{G}_1$ has been ignored because multiplying it with $\mathbf{N}_s$ would lead to the third and higher-order noise terms. Under small TDOA noise and small sensor location errors, we have from the Neumann expansion [10]

$$\begin{aligned}\mathbf{P}_1^{-1} &\approx (\mathbf{I} - \mathbf{P}_1^{o-1}\Delta\mathbf{P}_1)^{-1}\mathbf{P}_1^{o-1} \\ &\approx (\mathbf{I} - \mathbf{P}_1^{o-1}\Delta\mathbf{P}_1)\mathbf{P}_1^{o-1}.\end{aligned} \quad (17)$$

Using (17) and (13) in (12), and keeping up to second-order error terms lead to

$$\begin{aligned}\Delta\boldsymbol{\varphi}_{1,s} &\approx \mathbf{H}_1\mathbf{D}_1\Delta\mathbf{s} + \mathbf{H}_1\mathbf{N}_1 + \mathbf{H}_1\mathbf{r}^o\Delta\mathbf{s}_1^T\mathbf{B}\Delta\mathbf{s}_1 \\ &+ \mathbf{P}_1^{o-1}\Delta\mathbf{G}_1^T\mathbf{W}_1\mathbf{D}_1\Delta\mathbf{s} - \mathbf{P}_1^{o-1}\Delta\mathbf{P}_1\mathbf{H}_1\mathbf{D}_1\Delta\mathbf{s}\end{aligned} \quad (18)$$

where $\mathbf{N}_1$ is defined in (9). We shall evaluate the expectation of each summand on the right hand side of (18) to find $E[\Delta\boldsymbol{\varphi}_{1,s}]$. In particular, applying the fact that the sensor location error vector $\Delta\mathbf{s}$ has zero mean and a covariance matrix equal to $E[\Delta\mathbf{s}\Delta\mathbf{s}^T] = \mathbf{Q}_s$, we have

$$E[\mathbf{H}_1\mathbf{D}_1\Delta\mathbf{s}] = \mathbf{0} \quad (19)$$

$$E[\mathbf{H}_1\mathbf{r}^o\Delta\mathbf{s}_1^T\mathbf{B}\Delta\mathbf{s}_1] = \mathbf{H}_1\mathbf{r}^o tr(\mathbf{B}\cdot\mathbf{Q}_s(1:3,1:3)) \quad (20)$$

$$E[\mathbf{H}_1\mathbf{N}_1] = \mathbf{H}_1\begin{bmatrix}\mathbf{1}_{(M-1)\times 1} & \mathbf{I}_{M-1}\end{bmatrix}\cdot\mathbf{q}_s \quad (21)$$

where $\mathbf{q}_s$ is a column vector formed by the diagonal elements of $\mathbf{Q}_s$. Let $\mathbf{M}_2 = \mathbf{W}_1\mathbf{D}_1$, $m_j = 3(j-1)+1$ and $n_j = 3j$, $2 \le j \le M$. We have

$$E[\mathbf{P}_1^{o-1}\Delta\mathbf{G}_1^T\mathbf{W}_1\mathbf{D}_1\Delta\mathbf{s}] = -2\mathbf{P}_1^{o-1}\begin{bmatrix}\mathbf{A}_2 \\ 0\end{bmatrix} \quad (22)$$

$$\mathbf{A}_2 = \sum_{j=2}^{M}(\mathbf{Q}_s(m_j\!:\!n_j,:) - \mathbf{Q}_s(1:3,:))\mathbf{M}_2(j-1,:)^T. \quad (23)$$

Applying the definition $\mathbf{M}_3 = (\mathbf{H}_1\mathbf{D}_1)(1:3,:)$, we obtain

$$E[\mathbf{H}_1\Delta\mathbf{G}_1\mathbf{H}_1\mathbf{D}_1\Delta\mathbf{s}] = -2\mathbf{H}_1\mathbf{N}_3 \quad (24)$$

$$\mathbf{N}_3 = \begin{bmatrix} tr(\mathbf{M}_3(\mathbf{Q}_s(:,4:6) - \mathbf{Q}_s(:,1:3))) \\ \vdots \\ tr(\mathbf{M}_3(\mathbf{Q}_s(:,m_j:n_j) - \mathbf{Q}_s(:,1:3))) \\ \vdots \\ tr(\mathbf{M}_3(\mathbf{Q}_s(:,3(M-1)+1:3M) - \mathbf{Q}_s(:,1:3)))\end{bmatrix} \quad (25)$$

Finally, let $\mathbf{M}_4 = \mathbf{W}_1\mathbf{G}_1^o\mathbf{H}_1\mathbf{D}_1$ and it can be shown that

$$E[\mathbf{P}_1^{o-1}\Delta\mathbf{G}_1^T\mathbf{W}_1\mathbf{G}_1^o\mathbf{H}_1\mathbf{D}_1\Delta\mathbf{s}] = -2\mathbf{P}_1^{o-1}\begin{bmatrix}\mathbf{A}_3 \\ 0\end{bmatrix} \quad (26)$$

$$\mathbf{A}_3 = \sum_{j=2}^{M}(\mathbf{Q}_s(m_j\!:\!n_j,:) - \mathbf{Q}_s(1:3,:))\mathbf{M}_4(j-1,:)^T. \quad (27)$$

Taking expectation on both sides of (18) and substituting (19)-(21), (22), (24) and (26) yield the desired $E[\Delta\boldsymbol{\varphi}_{1,s}]$. Combining the obtained result with (10) gives the estimation bias from Stage-1 processing of the TDOA localization algorithm in Section 2.2, which is equal to

$$E[\Delta\boldsymbol{\varphi}_1] = E[\Delta\boldsymbol{\varphi}_{1,t}] + E[\Delta\boldsymbol{\varphi}_{1,s}]. \quad (28)$$

Bias in Stage−2: Similar to the derivation of (7), we subtract both sides of (4) by $\boldsymbol{\varphi}_2^o$ and use definitions of $\mathbf{h}_2$ and $\mathbf{G}_2$ to arrive at

$$\begin{aligned}\Delta\boldsymbol{\varphi}_2 &= (\mathbf{G}_2^T\mathbf{W}_2\mathbf{G}_2)^{-1}\mathbf{G}_2^T\mathbf{W}_2(\mathbf{h}_2 - \mathbf{G}_2\boldsymbol{\varphi}_2^o) \\ &= (\mathbf{G}_2^T\mathbf{W}_2\mathbf{G}_2)^{-1}\mathbf{G}_2^T\mathbf{W}_2(\mathbf{B}_2\Delta\boldsymbol{\varphi}_1 + \Delta\boldsymbol{\varphi}_1\odot\Delta\boldsymbol{\varphi}_1).\end{aligned} \quad (29)$$

The above functional form of the Stage-2 estimation error $\Delta\boldsymbol{\varphi}_2$ reflects the propagation of the estimation bias from Stage-1 to Stage-2 of the TDOA localization algorithm.

From [6], we know that $\mathbf{G}_2$ is a constant matrix but the matrices $\mathbf{B}_2$ and $\mathbf{W}_2$ contain noise, due to that $\boldsymbol{\varphi}_1$ and $\mathbf{G}_1$ are noise-contaminated. It is also worthwhile to point out that the location error in $\mathbf{s}_1$ has been taken into account when formulating the Stage-1 unknown vector $\boldsymbol{\varphi}_1^o = [\mathbf{u}^{oT}, \hat{r}_1^o]^T$. As a result, $\mathbf{s}_1$ should be considered *noise-free* when deriving the estimation bias of Stage-2 processing.

We shall follow the procedure in [9] to derive the bias of the Stage-2 processing, which is $E[\Delta\boldsymbol{\varphi}_2]$. In particular, it can be shown that

$$\begin{aligned}E[\Delta\boldsymbol{\varphi}_2] &\approx \mathbf{H}_2(\mathbf{c}_1 + \mathbf{B}_2^o E[\Delta\boldsymbol{\varphi}_1]) \\ &+ \mathbf{P}_2^{o-1}\mathbf{G}_2^T E[\Delta\mathbf{W}_2\mathbf{P}_3\mathbf{B}_2^o\Delta\boldsymbol{\varphi}_1]\end{aligned} \quad (30)$$

where

$$\mathbf{H}_2 = (\mathbf{G}_2^T\mathbf{W}_2^o\mathbf{G}_2)^{-1}\mathbf{G}_2^T\mathbf{W}_2^o \quad (31)$$

$$\mathbf{P}_3 = \mathbf{I} - \mathbf{G}_2\mathbf{H}_2, \mathbf{P}_2^o = \mathbf{G}_2^T\mathbf{W}_2^o\mathbf{G}_2 \quad (32)$$

and $\mathbf{c}_1$ is a column vector formed by the diagonal elements of $\mathbf{P}_1^{o-1}$. $\Delta\mathbf{W}_2$ is defined as

$$\Delta\mathbf{W}_2 \approx \mathbf{B}_2^{o-1}\Delta\mathbf{P}_1\mathbf{B}_2^{o-1} - \mathbf{B}_2^{o-1}\Delta\mathbf{B}_2\mathbf{W}_2^o - \mathbf{W}_2^o\Delta\mathbf{B}_2\mathbf{B}_2^{o-1} \quad (33)$$

such that $\mathbf{W}_2 = \mathbf{W}_2^o + \Delta\mathbf{W}_2$, where

$$\mathbf{W}_2^o = \mathbf{B}_2^{o-1}\mathbf{P}_1^o\mathbf{B}_2^{o-1}$$

$$\Delta\mathbf{B}_2 = 2\text{diag}\{\Delta\boldsymbol{\varphi}_1\}$$

and $\mathbf{B}_2^o = 2\text{diag}\left\{\boldsymbol{\varphi}_1^o - \begin{bmatrix}\mathbf{s}_1\\0\end{bmatrix}\right\}$.

We next evaluate $E[\Delta\mathbf{W}_2\mathbf{P}_3\mathbf{B}_2^o\Delta\boldsymbol{\varphi}_1]$ to find $E[\Delta\boldsymbol{\varphi}_2]$. For the clarity of the presentation, we express it as

$$E[\Delta\mathbf{W}_2\mathbf{P}_3\mathbf{B}_2^o\Delta\boldsymbol{\varphi}_1] = \alpha + \beta + \gamma. \quad (34)$$

Let $\mathbf{P}_4 = \mathbf{B}_2^{o-1}\mathbf{P}_3\mathbf{B}_2^o\mathbf{H}_1\mathbf{B}_1$. As a result, α can be shown to be equal to

$$\begin{aligned}\alpha &= E[\mathbf{B}_2^{o-1}\Delta\mathbf{P}_1\mathbf{B}_2^{o-1}\mathbf{P}_3\mathbf{B}_2^o\Delta\boldsymbol{\varphi}_1]\\ &= -2\mathbf{B}_2^{o-1}\left(\mathbf{A}_4 + \begin{bmatrix}\mathbf{A}_5\\ tr(\mathbf{W}_1\mathbf{G}_1^o\mathbf{P}_4\mathbf{Q}_t)\end{bmatrix}\right)\end{aligned} \quad (35)$$

where the matrices $\mathbf{A}_4$ and $\mathbf{A}_5$ are defined as

$$\begin{aligned}\mathbf{A}_4 &= E[\mathbf{G}_1^{oT}\mathbf{W}_1\Delta\mathbf{G}_1\mathbf{B}_2^{o-1}\mathbf{P}_3\mathbf{B}_2^o\Delta\boldsymbol{\varphi}_1]\\ &= \mathbf{G}_1^{oT}\mathbf{W}_1\mathbf{Q}_t\mathbf{P}_4(4,:)^T + \mathbf{G}_1^{oT}\mathbf{W}_1 \times\\ &\begin{bmatrix} tr(\mathbf{M}_5(\mathbf{Q}_s(:,4:6) - \mathbf{Q}_s(:,1:3)))\\ \vdots\\ tr(\mathbf{M}_5(\mathbf{Q}_s(:,m_j:n_j) - \mathbf{Q}_s(:,1:3)))\\ \vdots\\ tr(\mathbf{M}_5(\mathbf{Q}_s(:,3(M-1)+1:3M) - \mathbf{Q}_s(:,1:3)))\end{bmatrix}\end{aligned} \quad (36)$$

and

$$\mathbf{A}_5 = \sum_{j=2}^{M}(\mathbf{Q}_s(m_j:n_j,:) - \mathbf{Q}_s(1:3,:))\mathbf{M}_6(j-1,:)^T \quad (37)$$

Here, $\mathbf{M}_5 = (\mathbf{B}_2^{o-1}\mathbf{P}_3\mathbf{B}_2^o\mathbf{H}_1\mathbf{D}_1)(1:3,:)$ and $\mathbf{M}_6 = \mathbf{W}_1\mathbf{G}_1^o\mathbf{B}_2^{o-1}\mathbf{P}_3\mathbf{B}_2^o\mathbf{H}_1\mathbf{D}_1$. The definitions of m_j and n_j can be found above (22). β and γ are equal to [9]

$$\beta = -E[\mathbf{B}_2^{o-1}\Delta\mathbf{B}_2\mathbf{W}_2^o\mathbf{P}_3\mathbf{B}_2^o\Delta\boldsymbol{\varphi}_1] = -2\mathbf{B}_2^{o-1}\mathbf{p}_{5,W2} \quad (38)$$

$$\gamma = -E[\mathbf{W}_2^o\Delta\mathbf{B}_2\mathbf{B}_2^{o-1}\mathbf{P}_3\mathbf{B}_2^o\Delta\boldsymbol{\varphi}_1] = -2\mathbf{W}_2^o\mathbf{B}_2^{o-1}\mathbf{p}_5 \quad (39)$$

where $\mathbf{p}_{5,W2}$ and $\mathbf{p}_5$ are column vectors composed of the diagonal elements of $\mathbf{W}_2^o\mathbf{P}_5$ and $\mathbf{P}_5$, respectively. In particular, $\mathbf{P}_5$ is defined as

$$\mathbf{P}_5 = \mathbf{P}_3\mathbf{B}_2^o\mathbf{P}_1^o. \quad (40)$$

Putting (35), (38) and (39) into (34) and substituting the result back to (30) yield

$$E[\Delta\boldsymbol{\varphi}_2] = \mathbf{H}_2\left(\mathbf{c}_1 + \mathbf{B}_2 E[\Delta\boldsymbol{\varphi}_1] + \mathbf{W}_2^{o-1}(\alpha+\beta+\gamma)\right). \quad (41)$$

The estimation bias of the considered TDOA localization algorithm is, using (5),

$$E[\Delta\mathbf{u}] \approx \mathbf{B}_3^{o-1}(-\mathbf{c}_3 + E[\Delta\boldsymbol{\varphi}_2]). \quad (42)$$

Table 1. True sensor locations in meters

Sensor No	1	2	3	4	5	6
x_i^o	300	400	300	350	–100	200
y_i^o	100	150	500	200	–100	–300
z_i^o	150	100	200	100	–100	−200

$\mathbf{c}_3$ is a column vector formed by the diagonal elements of $\mathbf{C}_3$, which is the approximate covariance matrix of the localization algorithm output $\mathbf{u}$ equal to $\mathbf{B}_3^{o-1}(\mathbf{G}_2^T\mathbf{W}_2^o\mathbf{G}_2)^{-1}\mathbf{B}_3^{o-1}$. The diagonal matrix $\mathbf{B}_3^o$ is defined as $\mathbf{B}_3^o = 2\text{diag}\{\mathbf{u}^o - \mathbf{s}_1\}$. This completes the bias analysis for the algorithm presented in Section 2.2 for locating a source using TDOA measurements in the presence of sensor position errors.

3. SIMULATION

In this section, simulations are performed to corroborate the theoretical developments and gain more insights. The considered localization geometry is the same as [4]. There are $M = 6$ sensors whose true locations are summarized in table 1. The source is located at $\mathbf{u}^o = [285\ 483\ 209]^T$m, which is close to the sensor array. The covariance matrix of the TDOA measurement noise vector, $\mathbf{Q}_t$, is set to be equal to $\boldsymbol{\sigma}_t^2\mathbf{R}$, where $\boldsymbol{\sigma}_t^2 = 10^{-4}m^2$ is the noise variance and $\mathbf{R}$ is a matrix with all the diagonal elements being 1 and off-diagonal elements being 0.5 [4]. The covariance matrix of the sensor location error vector is $\mathbf{Q}_s = \sigma_s^2\mathbf{I}_{3M}$ and σ_s^2 has unit m^2 .

Monte Carlo simulations are conducted. In each ensemble run, the localization algorithm in Section 2.2 is applied to locate the source. Noisy TDOA measurements and erroneous sensor locations are produced by adding to the true values zero-mean Gaussian noise with covariance matrices $\mathbf{Q}_t$ and $\mathbf{Q}_s$. The localization MSE and the estimation bias from simulation are evaluated using $\text{MSE}(\hat{\mathbf{u}}) = \sum_{i=1}^{L}(\hat{\mathbf{u}}_i - \mathbf{u})^2/L$ and $\text{BIAS}(\hat{\mathbf{u}}) = \|\sum_{i=1}^{L}(\hat{\mathbf{u}}_i - \mathbf{u}^o)\|/L$, where $L = 10000$ is the total number of ensemble runs. Besides the two-stage method in consideration, we also simulate the BiasSub method that subtracts an estimated version of the localization bias from the localization result of the two-stage technique. The estimated localization bias is found via evaluating (42) by replacing the unknown true quantities $\mathbf{u}^o$ and $\mathbf{s}_i^o$ with the localization output $\mathbf{u}$ and the noisy sensor locations $\mathbf{s}_i$.

In Fig.1, we plot as function of the sensor location error variance σ_s^2, the localization CRLB and the localization MSEs of the two-stage method from Section 2.2 and its BiasSub-augmented version. It is found that the two localization techniques in consideration both attain the CRLB results approximately when σ_s^2 is small.

Fig.2 shows the localization biases of the two-stage method and the BiasSub technique. We observe that the

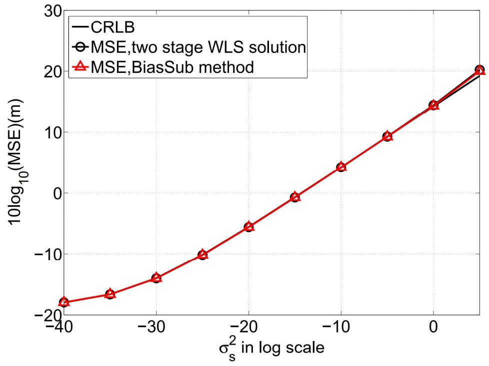

Fig. 1. Localization MSE as function of the sensor location error variance σ_s^2.

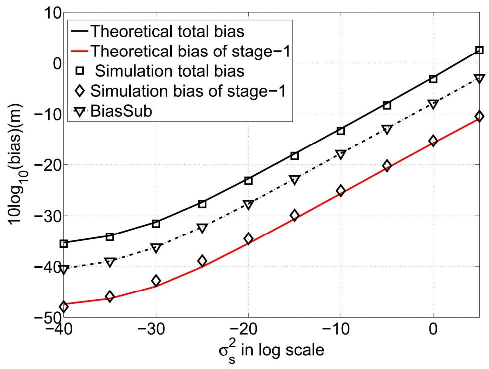

Fig. 2. Comparison of the final localization bias from (42) and the estimation bias of Stage-1 from (28).

theoretical bias results match the localization bias of the two-stage algorithm obtained via simulation very well. This justifies the validity of the bias analysis in Section 2.3. On the other hand, with BiasSub, the localization bias can be reduced significantly. We also examine the contribution of each processing stage to the final estimation bias of the two-stage algorithm. For this purpose, we include in the figure the theoretical localization bias of the Stage-1 processing (see (28)). Also included is the estimation bias of Stage-1 from simulation. It can be seen that the use of the Stage-2 processing increases greatly the localization bias of the two-stage method. This observation is different from the result in [9] where it was indicated that under precisely known sensor locations, Stage-1 of the two-stage localization method is the major source of the final estimation bias. Our findings might be explained by noting that the presence of sensor location errors increases the localization error of Stage-1, which propagates to the final localization result as estimation bias, due to the use of the nonlinear squaring and square root operations in Stage-2 of the two-stage method (see (4) and (5)).

4. CONCLUSION

This paper derived, via the use of the second-order perturbation analysis [9], the estimation bias of an algebraic two-stage algorithm [5, 6] that can localize a source using TDOA measurements in the presence of random sensor location errors. Computer simulations validated the theoretical results. It was demonstrated that different from the findings of [9], when the sensor locations are known imprecisely, both processing stages of the considered two-stage technique would contribute significantly to the final localization bias.

REFERENCES

[1] E. Weinstein, "Optimal source localization and tracking from passive array measurements," *IEEE Trans. Acoust., Speech, Signal Process.*, vol. 31, no. 1, pp. 69–71, Feb. 1982.

[2] G. Mao, B. Fidan, and B. Anderson, "Wireless sensor network localization techniques," *Computer networks*, vol. 51, no. 10, pp. 2529–2553, Jan. 2007.

[3] K. Dogancay, "Bias compensation for the bearings-only pseudolinear target track estimator," *IEEE Trans. Signal Process.*, vol. 54, pp. 59–68, Jan. 2006.

[4] Y. T. Chan and K. C. Ho, "A simple and efficient estimator for hyperbolic location," *IEEE Trans. Signal Process.*, vol. 42, no. 8, pp. 1905–1915, Aug. 1994.

[5] K. C. Ho and L. Yang, "On the use of a calibration emitter for source localization in the presence of sensor position uncertainty," *IEEE Trans. Signal Process.*, vol. 56, no. 12, pp. 5758–5772, Dec. 2008.

[6] M. Sun and K. C. Ho, "An asymptotically efficient estimator for TDOA and FDOA positioning of multiple disjoint sources in the presence of sensor location uncertainties," *IEEE Trans. Signal Process.*, vol. 59, no. 7, pp. 3434–3440, July 2011.

[7] H. G. Yu, G. Huang, and J. Gao, "Constrained total least-squares localisation algorithm using time difference of arrival and frequency difference of arrival measurements with sensor location uncertainties," *IET Radar, Sonar and Navigation*, vol. 6, no. 9, pp. 891–899, May 2012.

[8] H. W. Wei et al., "A novel weighted multidimensional scaling analysis for time-of-arrival-based mobile location," *IEEE Trans. Signal Process.*, vol. 56, pp. 3018–3022, July 2008.

[9] K. C. Ho, "Bias reduction for an explicit solution of source localization using TDOA," *IEEE Trans. Signal Process.*, vol. 60, no. 5, pp. 2101–2114, May 2012.

[10] T. K. Moon and W. C. Stirling, *Mathematical Methods and Algorithms for Signal Processing*, Prentice-Hall, Upper Saddle River, NJ, 2000.