TAYLOR-FOURIER SERIES ANALYSIS FOR FRACTIONAL ORDER SYSTEMS

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ABSTRACT

Dynamical systems describing a physical process with a dominant diffusion phenomenon require a large dimensional model due to their long memory. Without prior knowledge, it is however not straightforward to know if/whether one deals with a fractional order system or long memory effects.
Since the parametric modeling of a fractional system is very involved, we tackle the question whether fractional insight can be gathered in a non-parametric way.

In this paper we show that the classical Fourier basis leading to the Frequency Response Function (FRF) lacks fractional insight. Therefore, we introduce a Taylor-Fourier basis to obtain non-parametric insight in the fractional system. This analysis proposes a novel type of spectrum to visualize the spectral content of a fractional system: the Taylor-Fourier spectrum.

Index Terms—Non-parametric modeling, dynamic systems, fractional order systems, Taylor-Fourier basis, theory of frames

1. INTRODUCTION

Linear Time Invariant (LTI) dynamical systems are well studied and as such known to be fully described by their Impulse Response Function (IRF) or, their frequency domain equivalent, Frequency Response Function (FRF). Both serve as a non-parametric estimate of the studied dynamical system.

Diffusion phenomena complicate the dynamical modeling, particularly if diffusion is observed in non-homogeneous media. In [1] the Langevin diffusion equation was generalized to allow arbitrary diffusion processes:

$$m\frac{\partial^\alpha}{\partial t^\alpha}x = F(t) - \gamma\frac{d}{dt}x(t)$$

where m is the mass, $x(t)$ the one-dimensional displacement, $F(t)$ the external force, γ the viscosity parameter of the medium and $\frac{\partial^\alpha}{\partial t^\alpha}x$ denotes the fractional order derivative in Riemann-Liouville [2] or Caputo sense [3]. In physical systems one often discriminates between:

- anomalous or heterogeneous diffusion ($0 < \alpha < 1$),
- homogeneous diffusion ($\alpha = 1$),
- Lévy flight processes (($1 < \alpha < 2$)
- Richardson diffusion ($2 < \alpha$)

This work is financially supported by the Flemish research foundation – Fonds voor Wetenschappelijk Onderzoek Vlaanderen (FWO) under grant contract 1204614N and by Keysight Technologies under grant contract 3357.

See [4] for a very extensive overview.
In system identification and data driven modeling we wish to either parametrically or non-parametrically model the system's dynamics. This goal can be achieved by estimating the poles and zeros to understand the system's damping, resonances and frequency attenuation. One can also aim at describing the power spectrum of the observed signals to study the energy distribution over the frequency band or roll-off for instance. However, dealing with fractional systems complicates both modeling approaches. The parametric approach encounters difficulties to fit a classical rational form as model selection criteria exhibit over-parametrization due to the fractional system's long memory effects [2], [5]. Non-parametrically one cannot discriminate a fractional order system from an integer order system without prior knowledge by using the classical FRF, as will be illustrated in this paper.

Furthermore, we will propose a new non-parametric estimation method which extends the classical Fourier basis to the (fractional) Taylor-Fourier (TF) basis. The TF basis has already been proposed in power system analysis (see for instance [6] and [7]) to study amplitude modulation effects which present themselves as leakage when the power spectrum is computed through windowing. Now, we will study and apply this technique for fractional order systems.

2. FRACTIONAL ORDER LINEAR SYSTEMS

2.1 Partial fractional representation

We consider a linear time-invariant fractional order system $G(\omega)$ characterized by its partial fraction decomposition in terms of the system's complex conjugate poles p_k for $k \in \{1,2,\dots,K\}$ with fractional multiplicities $\{\alpha_1, \alpha_2, \dots, \alpha_K\}$. The frequency domain characteristic is given by:

$$G(\omega) = \frac{1}{2}\sum_{k=1}^{K}\frac{c_k}{(j\omega - p_k)^{\alpha_k}} + \frac{\bar{c}_k}{(j\omega - \bar{p}_k)^{\alpha_k}} \qquad (1)$$

where $\bar{a}$ denotes the complex conjugate of a and ω denotes the angular frequency or Fourier variable. Throughout this paper, we assume that the system (1) is commensurate such that there exists a $q \in \mathbb{R}$ such that $\alpha_k - 1 = kq$ for $k = 1, \dots, K - 1$.

Fractional order systems are part of fractional order calculus and, as such, require fractional order derivatives. We use for the remainder of this paper fractional derivatives of Caputo-type, [3]. Note that the results established in this paper can be reproduced for fractional derivatives of the Riemann-Liouville type due to the commensurate orders.

Let $f(x)$ be a real-valued function which is at least n-times differentiable, then the (right) Caputo derivative of order $n-1<\alpha\leq n$ is defined as,

$$\frac{\partial^{\alpha}}{\partial x^{\alpha}}f(x)=\frac{1}{\Gamma(n-\alpha)}\int_{0}^{x}(x-t)^{n-\alpha-1}f(t)dt \qquad (2)$$

Note that for simplicity we selected the under-limit of the integral to coincide with the start of the observation window. From (1) and (2) we can compute the impulse response corresponding to the Laplace characteristic $G(s)$.

Proposition 1. For commensurate orders the frequency domain characteristic $G(s)$ holds the following impulse response

$$g(t)=\sum_{k=1}^{K}\operatorname{Re}\left(\frac{c_k}{\Gamma(\alpha_k)}t^{\alpha_k-1}e^{p_k t}\right) \qquad (3)$$

Furthermore, the impulse response in (3) is the solution of the following fractional order eigenvalue problem

$$\prod_{k=-K}^{K}\left(\frac{\partial}{\partial t}-p_i\right)^{\alpha_k}g(t)=0 \qquad (4)$$

where $\alpha_k=\alpha_{-k},\alpha_0=0,p_i=\bar{p}_{-i}$.

The proof of (3) is an immediate result of the Laplace transform for arbitrary derivatives, whereas (4) is established by applying the binomial series to the differential operator $\left(\frac{\partial}{\partial t}-p_i\right)^{\alpha_k}$.

2.2 Motivation and problem statement

As the Laplace transform represents an ordinary differential equation into a rational form, these meromorphic functions serve as universal approximators of a continuous-time, time-invariant dynamical system. As a result also fractional system (1) can be approximated arbitrarily well by a rational form of a sufficient high degree.

It has been observed in [5],[8] that systems with a diffusion component often require a high number of poles and zeros to allow an appropriate fit of the data. In [2] these diffusion systems were shown to exhibit chains of alternating poles and zeros as a function of their angular frequency which can be compressed into a pole of a fractional multiplicity as in (1).

The chains of alternating poles and zeros are the only indication that one is dealing with a fractional order system. This is illustrated in Fig. 1 where system (1) is used for $K=1,c_1=1,p_1=-45+480j$ and $\alpha=1.75$. The system was sampled at a rate of f_s =3450 Hz and the time series consisted of 2048 samples. The excitation signal was white Gaussian noise with a unit spectral density in the band up to 1.5 kHz. The example was modeled by rational form with 10 poles and 10 zeros as selected by an AIC model selection analysis. The parameter estimation of the rational form's coefficients followed a two-step procedure. Let $B(\omega),A(\omega)$ represent the rational form's polynomials of numerator and denominator respectively with coefficient vectors b,a, then in a first step the coefficients are identified through a Least Squares (LS) approach:

$$\underset{a,b}{\operatorname{argmin}}\sum_{k=1}^{\frac{N}{2}-1}|Y(k)A(\omega_k)-U(k)B(\omega_k)|^2 \qquad (5)$$

with ω_k the Fourier coefficients at bin k and $U(k),Y(k)$ are the respective discrete Fourier coefficients of the excitation signal and its response to the system respectively. In the next step, these LS estimates are applied as initial values to optimize the following nonlinear least squares problem

$$\underset{a,b}{\operatorname{argmin}}\sum_{k=1}^{\frac{N}{2}-1}\left|\frac{Y(k)}{U(k)}-\frac{B(\omega_k)}{A(\omega_k)}\right|^2 \qquad (6)$$

In Fig. 1 the solution of (5) and (6) is shown in blue and green, respectively, and the cross markers are the ratio $\frac{Y(k)}{U(k)}$.

From Fig. 1 it follows that based on the impulse response and the frequency response, one cannot see that one is dealing with a fractional order system. The FRF gives the impression of dealing with a second order system. Only the parametric modeling reveals the presence of poles and zeros which do not correspond to actual resonances.

Since the FRF is clearly not a good tool, we wish to study an alternative technique which allows inspecting the fractional order properties in a non-parametric way

The lack of fractional insight of the classical FRF is due to the impulse response (3) of a fractional system. The impulse response no longer consists of damped exponentials but holds a polynomial term. As a result, a basis of the form $t^m e^{j\omega_0 kt}$ with ω_0 the fundamental (angular) frequency seems more natural. In the next section, we study how Taylor-Fourier expansion can be obtained for (3).

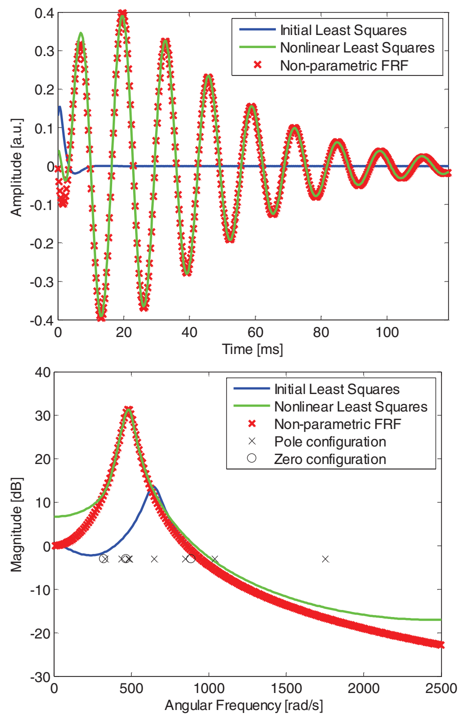

Fig. 1. An integer-order approximation of (1) with model order 13/14: Time domain (top), Freq. domain (bottom)

3. TAYLOR-FOURIER ANALYSIS

3.1 Taylor-Fourier basis

Consider the set of impulse responses $g(t)$ on the compact time horizon $t \in [0, T]$ which are band-limited such that the frequency characteristic $G(\omega)$ has a compact support $\omega \in [0, \pi f_s]$. It is well known that these impulse responses represent the band-limited square-integrable functions denoted by $\mathcal{L}_2^{\pi f_s}([0,T])$. We denote the Hilbert space of square integrable functions by $\mathcal{L}_2([0,T])$.The following is a classical result from functional analysis.

Proposition 2. The set $\mathcal{B} = \left\{ e^{2\pi j \frac{k}{N} f_s t} \middle| k \in \mathbb{Z}, N \in \mathbb{N}_0 \right\}$ forms a complete orthogonal set in $\mathcal{L}_2^{\pi f_s}([0,T])$.

The proof can be found in [9]. Due to the impulse response (3), we consider the set for $\alpha \in \mathbb{Q}^+$

$$\mathcal{B}_\alpha = \left\{ t^\alpha e^{2\pi j \frac{k}{N} f_s t} \middle| k \in \mathbb{Z}, N \in \mathbb{N}_0 \right\} \quad (7)$$

Proposition 2 together with the observation that $t^\alpha \notin \mathcal{L}_2^{\pi f_s}([0,T])$, since these are not band-limited, leads to the following intermediate result:

Lemma 1. The set (7) forms a complete set in $\mathbb{V}_\alpha$ consisting of the functions $f(t)$ on $[0,T]$ such that $\frac{f(t)}{t^\alpha} \in \mathcal{L}_2^{\pi f_s}([0,T])$. Moreover $\mathbb{V}_\alpha$ is a complete (Hilbert) subspace of $\mathcal{L}_2([0,T])$.

The proof is a trivial consequence of Proposition 2. Consider next the set $\mathcal{P}_q^m$ consisting of the fractional polynomials $c_0 + c_1 t^q + \cdots + c_m t^{qm}$. This leads to the following generalization of Lemma 1.

Proposition 3. The union $\cup_{l=0}^m \mathcal{B}_{lq}$ of Taylor-Fourier functions forms a dense set in $\mathbb{V}_q^m$ consisting of the functions $f(t)$ on $[0,T]$ such that $\frac{f(t)}{Q(t)} \in \mathcal{L}_2^{\pi f_s}([0,T])$ with $Q(t) \in \mathcal{P}_q^m$. Moreover $\mathbb{V}_\alpha^m$ is a complete (Hilbert) subspace of $\mathcal{L}_2([0,T])$.

The proof follows by noting that $\mathbb{V}_q^m$ is the direct sum of the vector spaces $\mathbb{V}_{ql}$ for $l = 0, \dots, m$. This result implies that the union $\cup_{l=0}^m \mathcal{B}_{lq}$ is a Schauder basis such that for each function in $\mathbb{V}_q^m$ a unique Taylor-Fourier expansion exists. The next subsection describes how this expansion can be computed.

3.2 Taylor-Fourier series expansion

The problem with the obtained Schauder basis $\cup_{l=0}^m \mathcal{B}_{lq}$ is that its elements are not orthogonal but oblique. As a result, we must show that this basis is a Riesz basis to use its good properties. In [10], a Riesz basis is defined as a Schauder basis $f_1, f_2, f_3, \dots$ such that there exist a constant C with the property that or $f = \sum_{n=1}^\infty c_n f_n$ it holds that for each $M \geq 0$

$$\left\| \sum_{n=1}^{M} c_n f_n \right\|^2 \leq C \sum_{n=1}^{M} |c_n|^2 \quad (8)$$

Note that sequences which are no Schauder basis but satisfy (Bessel's) inequality (8) are called Bessel sequences in functional analysis.

Proposition 4. The Schauder basis $\cup_{l=0}^m \mathcal{B}_{lq}$ is a Riesz basis of $\mathbb{V}_q^m$ for $q \geq 1$.

The proof is found in [11]. The theory of Riesz bases is well known in the literature of wavelets, particularly frames. We refer to [12] and [13] for the interested reader in this material. Establishing the result in (8) allows defining the following linear, bounded and invertible operator (see [10], p. 185 eq. 2) on the Hilbert space $\mathbb{V}_\alpha^m$ which holds the key to construct the Taylor-Fourier series.

$$F: \mathbb{V}_q^m \to \mathbb{V}_q^m : f \mapsto \sum_{l=0}^{m} \sum_{k=-\infty}^{\infty} < f(t), f_{l,k}(t) > f_{l,k}(t) \quad (9)$$

with $f_{l,k}(t) = t^{lq} e^{2\pi j \frac{k}{N} f_s t}$ and $< f, g > = \frac{1}{T} \int_0^T f(t) g(t) dt$

The most important consequence of a Riesz basis is that the operator F in (9) maps a Riesz basis to a Riesz basis which forms a bi-orthogonal sequence: sequences f_n, g_n with $n \in \mathbb{Z}$ are called bi-orthogonal if

$$< f_n, g_m > = 0 \ \text{ for } n \neq m$$

This leads to the most important theorem on Riesz bases applied to $\cup_{l=0}^m \mathcal{B}_{l\alpha}$ establishing the Taylor-Fourier series.

Proposition 5. Let $g(t) \in \mathbb{V}_q^m$ such that $\tilde{g}(t)$ is its dual with the property $g = F(\tilde{g})$ then the following Taylor-Fourier series holds

$$g(t) = \sum_{l=0}^{m} \sum_{k=-\infty}^{\infty} < \tilde{g}(t), f_{l,k}(t) > f_{l,k}(t) \quad (10)$$

The general proof is found in [10] (p. 186, Lemma 5). Note that the classical Fourier basis is orthogonal to itself such that the Fourier basis is equal to its dual. The dual of the Taylor-Fourier basis is not directly accessible so we need a computational trick to obtain (10).

4. DIGITAL COMPUTATION OF THE TF ANALYSIS

Computers cannot handle a countable basis $\cup_{l=0}^m \mathcal{B}_{lq}$ so we restrict the analysis to the finite dimensional subspace where the Taylor-Fourier analysis is restricted to

$$g(t_n) = \sum_{l=0}^{m} \sum_{k=-\frac{N}{2}+1}^{\frac{N}{2}-1} c_{l,k} t_n^{lq} e^{2\pi j \frac{k}{N} f_s t_n} \quad (11)$$

where t_n represent the sampled time instants and $c_{l,k}$ the unknown TF coefficients. It is clear from the linear combination in (11) that finding the unknown TF coefficients is a regression problem. As a result, we define the regression matrix X with elements given by

$$[X]_{n,k} = t_n^{\lfloor \frac{k}{N+1} \rfloor q} \exp\left(2\pi j \frac{k - \left\lfloor \frac{k}{N+1} \right\rfloor (N+1) - \frac{N}{2}}{N} f_s t_n \right)$$

where $\lfloor a \rfloor$ denotes rounding the number a to the nearest integer towards $-\infty$ or the floor operator. The next proposition transforms Proposition 5 into a numerical optimization problem.

Proposition 6. Under the conditions of Proposition 5, we find that $c_{l,k} =< \tilde{g}(t), f_{l,k}(t) >$ is the solution of

$$\underset{c}{\operatorname{argmin}} \|c\|_2 \ \text{ subject that (11) holds} \quad (12)$$

The proof can be found in [11]. The solution (12) is very noise sensitive due to the constraint (11). In practice, we may expect that the measured sampled impulse response computed from the FRF is noisy. Hence, we solve the Ridge regression problem which relaxes the constraint (11). As a result, we solve

$$\underset{c}{\operatorname{argmin}} \|c\|_2 \ \text{ subject to } \|g - Xc\|_2 \leq \sigma\sqrt{N} \tag{13}$$

where g denotes the vector containing the samples $g(t_n)$ and σ the standard deviation of the measurement noise. It is well known that the Ridge regression allows an analytical solution through a weighted least squares approach [14].

5. NUMERICAL EXAMPLES

In this section, we consider two examples. One is an extension of the toy example used in Fig. 1 to the case $K = 3$ with different fractional orders. The second example is a known fractional system "Randle's model for electrical batteries".

5.1 Fractional order pole model

Consider the following fractional system of type (1) with $K = 3$, $c_i = 9\ \forall i = 1,2,3$, $p_1 = -5 + 2\pi j.75$, $p_2 = -10 + 2\pi j.150$ and $p_3 = -10 + 2\pi j.300$ and corresponding fractional multiplicities $\alpha_1 = 1.25$, $\alpha_2 = 1.5$ and $\alpha_3 = 1.75$. Note that the resonances are harmonically related. The system is commensurate with $q = 0.25$.

The system is excited by a band-limited zero-mean white Gaussian noise sequence in the band [0,5]kHz with a unit standard deviation. As a result, the excitation and its response to the fractional system is sampled at a rate $f_s = 10$kHz satisfying Nyquist's criterion. A total record of 2^{12} samples were gathered.

The selected basis for the TF analysis was w.r.t. equation (11) equal to $N = 1024$ and $m = 3$. The optimization (13) was performed with $\sigma = 3.5 \times 10^{-4}$ (this number was established to obtain the best fit). Once the TF coefficients have been estimated through (13), we extract from (11) the various terms per Fourier basis given by

$$g_k(t_n) = e^{2\pi j \frac{k}{N} f_s t_n} \sum_{l=0}^{m} c_{l,k} t_n^{lq} \tag{14}$$

where the sum represents the exponential and fractional polynomial damping. For each function in (14) we compute its Fourier spectrum through the discrete Fourier transform and plot its magnitude spectrum as a function of the Fourier basis index k as illustrated in Fig. 4. The plot

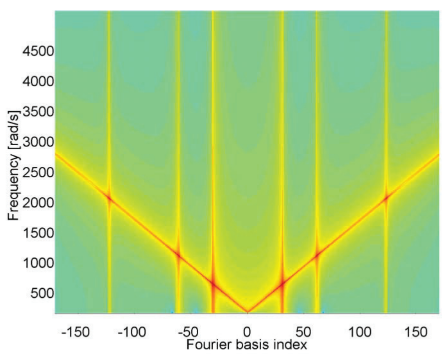

Fig. 4. Taylor-Fourier spectrum for $q = 0.25$

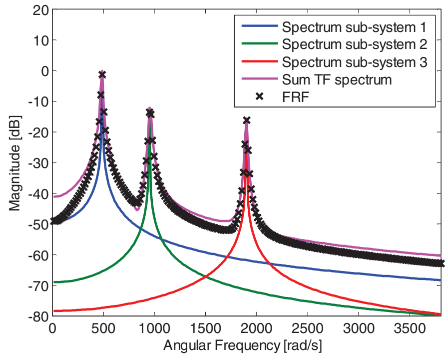

Fig. 2. Significant vertical lines in the TF spectrum

reveals three sub-systems corresponding to different basis functions. The intersections with the diagonal line shows the resonance frequency of each sub-system as can be read from the vertical axis. The actual Fourier spectrum can be computed from Fig. 4 as the sum over all Fourier basis indices which sum all subsystems. Since the TF spectrum is a sparse representation as a function of the Fourier basis index, we can extract these basis indices corresponding to the three vertical lines w.r.t. the positive side. These three lines can be redrawn in a magnitude plot where the corresponding vertical axis in the TF spectrum becomes the horizontal axis and the intensity of the TF spectrum the vertical axis. In Fig. 2 the three significant vertical lines of the TF spectrum are shown each corresponding to a fractional subsystem. The sum of the TF spectrum over the different Fourier basis functions (Magenta curve) reveals a good match to the obtained FRF (cross markers).

On top of that, we know that the sub-systems are of fractional order as we selected $q = 0.25$. It is not possible to know the exact fractional order of the sub-systems present but the commensurate order can be derived. Indeed, selecting an incorrect fractional order q renders the TF spectrum completely noisy. This can be explained by the optimization (13): minimizing the squares of the coefficients implies a sparse representation. The sparse representation is only feasible if the correct TF basis is selected, otherwise the coefficients are fit but none are really significant. Note however that the sum of the TF spectrum over the Fourier basis functions still results in the same magenta curve as in Fig. 2 due to the constraint in the optimization. The subsystems however lose their meaning and significance. It remains an open question how the commensurate order q can be selected automatically.

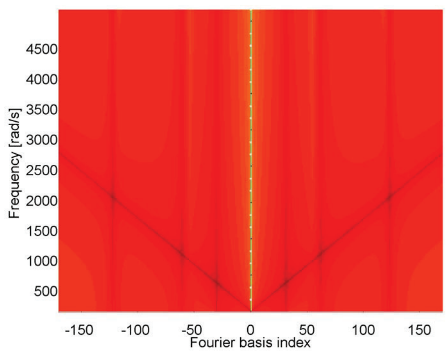

Fig. 3. Taylor-Fourier spectrum for $q = 0$

5.2 Randle's circuit

Randle's circuit describes the impedance of an electrical battery and is often used in electrical impedance spectroscopy. The fractional diffusion component is due to the constant phase element replacing the double layer capacitance in the circuit. Randle's transfer function is given by

$$G(s) = \frac{R_1 R_2 C s^{\alpha+1} + (R_1 + R_2) s^{\alpha} + \sigma C R_1 s + \sigma}{R_2 C s^{\alpha+1} + s^{\alpha} + \sigma C s} \quad (15)$$

with common values (see [15]): $R_1 = 20\Omega, R_2 = 100\Omega, C = 25\mu F, \sigma = \frac{300\Omega}{\sqrt{s}}, \alpha = 0.5$. We simulated (15) with a band-limited Gaussian noise voltage excitation in the band [0,1]Hz. The signals were sampled at a rate of $f_s = 2$Hz and a total record of 2^{12} was considered.

Note that the system (15) is not of the type (1), nor does the commensurate order satisfy the condition set by Proposition 4. Thus this example is challenging and reveals the robustness of the proposed methodology to deviating scenarios. The TF spectrum with a commensurate order of $q = 0.5$ was computed in Fig. 5. The spectrum reveals a significant contribution spread around Fourier basis $n = 100$. This is in agreement with the FRF revealing a resonance around Fourier bin 100. Due to the violation of fractional type (1) the TF spectrum is less sparse. Nevertheless, the TF spectrum for $q = 0$ reveals only noise without even a clear significant region. Thus we conclude that the TF analysis provides fractional insight in a broad class of fractional systems.

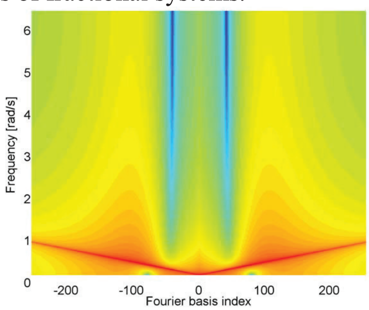

Fig. 5. The TF spectrum for Randle's circuit with $q = 0.5$.

6. CONCLUSION

In this paper, we revisited Taylor-Fourier analysis known from power system analysis. We conducted a thorough study of its foundations in functional analysis to apply this analysis with a fractional order basis to fractional order systems. We introduced the novel TF spectrum which provides fractional insight in the behavior of the system studied. The TF spectrum exhibits interesting properties both from a mathematical viewpoint and a computational viewpoint as fractional insight is obtained without parametrically modeling the fractional system's transfer function.

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