OVERSAMPLED BIPARTITE GRAPHS WITH CONTROLLED REDUNDANCY

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ABSTRACT

This paper extends our previous work on graph oversampling for graph signal processing. In the graph oversampling method, nodes are duplicated and edges are appended to construct oversampled graph Laplacian matrix. It can convert an arbitrary K-colorable graph into one bipartite graph which includes all edges of the original graph. Since it uses a coloring-based algorithm, performance of graph signal processing depends on the coloring results. In this paper, we present graph oversampling based on a few different graph bipartition methods which use maximum spanning tree and eigendecomposition. Furthermore, we consider the effective selection method of duplicated nodes. The performance of the oversampled graphs is compared through an experiment on graph signal denoising.

***Index Terms*—** Graph signal processing, graph bipartition, graph oversampling, graph wavelets, graph filter banks

1. INTRODUCTION

Graph signal processing has been developed as a useful tool for analysis of high-dimensional data [1], such as sensor and brain networks, traffic, learning, and images. Whereas signals of regular signal processing have very simple structures, those of graph signal processing are allowed to have complex irregular structures. Graph wavelet transforms can be used for analyzing, processing or compressing graph signals.

An important topic in graph wavelet transform is downsampling and upsampling. Similar to the aliasing of regular signal processing, the *spectral folding phenomenon* is occurred by downsampling in graph signal processing. In order to deal with this challenge, studies on decimated graph filter banks have focused on bipartite graphs and determined the perfect reconstruction conditions [2, 3]. Hence, the decimated transforms can only be applicable to bipartite graphs. There are some approaches for applying graph-based filter banks to arbitrary non-bipartite graphs. Coloring-based bipartition [2] decomposes a non-bipartite original graph into an edge-disjoint collection of bipartite subgraphs whose union is the original graph, and the transform is performed on each of these subgraphs. Recently, the maximum spanning tree (MST)-based bipartition was introduced [4]. In the method, one bipartite subgraph, which approximates the original graph so as to maximize the total weight of edges in the bipartite subgraph, is made and the transform is performed on the subgraph. Both approaches cannot utilize some edges in one-stage transform because a subgraph has only a part of edges of the original graph. Although the bipartition based on the polarity of the largest eigenvector [5] has been proposed as one of graph downsampling schemes, it has not been designed for a construction of bipartite graphs.

In our previous work [6, 7], we proposed graph oversampling, that yields oversampled graph Laplacian matrices as well as oversampled graph signals. We also introduced the graph oversampling method based on the coloring-based bipartition for non-bipartite graphs. It enables us to make one oversampled bipartite graph that includes all edges of the original non-bipartite graph. However, the oversampled graph has high redundancy and its performance obviously depends on the coloring results.

In this paper, we propose an effective graph oversampling method based on arbitrary bipartition algorithms. It can make the oversampled bipartite graph, which includes many edges in the original graph while controlling the redundancy, by appending the nodes which will affect the overall performance. We examine the performance of the oversampled graphs through a graph signal denoising experiment. On the basis of bipartite subgraphs made by the MST, by the approximate coloring [8], and by the coloring-based bipartition, the oversampled graph is constructed with various redundancies. From the experimental results, we discuss the effect of bipartition methods for graph oversampling.

The rest of this paper is organized as follows. In Section 2, we describe notations used in this paper and the decimated graph filter banks [2, 3, 6]. Section 3 presents three bipartition methods. Section 4 introduces the construction method of the oversampled graph with arbitrary redundancy. Section 5 shows the performance analysis of the oversampled graph based on the three bipartition schemes through denoising experiments. Section 6 concludes the paper.

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2. PRELIMINARIES

2.1. Graph Signals

A graph is represented as $\mathcal{G} = \{\mathcal{V}, \mathcal{E}\}$, where $\mathcal{V} = \{v_0, v_1, \dots, v_{N-1}\}$ and $\mathcal{E}$ denote sets of nodes and edges, respectively. The graph signal is defined as $\boldsymbol{f} \in \mathbb{R}^N$. We will only consider a finite undirected graph with no loops or multiple edges. The number of nodes is $N = |\mathcal{V}|$, unless otherwise specified. The (m, n)-th element of the adjacency matrix $\mathbf{A}$ is the weight of the edge between m and n if m and n are connected, and 0 otherwise. The degree matrix $\mathbf{D}$ is a diagonal matrix and its m-th diagonal element is $d_{mm} = \sum_n a_{mn}$. The unnormalized graph Laplacian matrix (GLM) is defined as $\mathbf{L} := \mathbf{D} - \mathbf{A}$ and the symmetric normalized GLM is $\boldsymbol{\mathcal{L}} := \mathbf{D}^{-1/2}\mathbf{L}\mathbf{D}^{-1/2}$. The symmetric normalized GLM has the property that its eigenvalues are within the interval $[0, 2]$, and we will use $\boldsymbol{\mathcal{L}}$ in this paper. The eigenvalues of $\boldsymbol{\mathcal{L}}$ are λ_i and ordered as: $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \dots \leq \lambda_{N-1} \leq 2$ without loss of generality. The eigenvector $\boldsymbol{u}_{\lambda_i}$ corresponds to λ_i and satisfies $\boldsymbol{\mathcal{L}}\boldsymbol{u}_{\lambda_i} = \lambda_i \boldsymbol{u}_{\lambda_i}$. The entire spectrum of $\mathcal{G}$ is defined by $\sigma(\mathcal{G}) := \{\lambda_0, \dots, \lambda_{N-1}\}$. The projection matrix for the eigenspace V_{λ_i} is $\mathbf{P}_{\lambda_i} = \sum_{\lambda=\lambda_i} \boldsymbol{u}_\lambda \boldsymbol{u}_\lambda^T$, where $\boldsymbol{u}_\lambda^T$ is the transpose of $\boldsymbol{u}_\lambda$. Let $h(\lambda_i)$ be the spectral kernel of filter $\mathbf{H}$. The spectral domain filter can be written as $\mathbf{H} = h(\boldsymbol{\mathcal{L}}) = \sum_{\lambda_i \in \sigma(\mathcal{G})} h(\lambda_i)\mathbf{P}_{\lambda_i}$. The spectral domain filtering of graph signals can be simply denoted as $\mathbf{H}\boldsymbol{f}$.

2.2. Two-Channel Graph Wavelet Filter Banks

A bipartite graph whose nodes can be decomposed into two disjoint sets L and H such that every edge connects a node in L to one in H can be represented as $\mathcal{G} = \{L, H, \mathcal{E}\}$. The downsampling function β_H of a bipartite graph is defined as

$$\beta_H(m) = \begin{cases} +1 & \text{if } m \in H, \\ -1 & \text{if } m \in L. \end{cases} \tag{1}$$

The diagonal downsampling matrix is $\mathbf{J}_H = \text{diag}\{\beta_H(m)\}$ and satisfies $\mathbf{J} = \mathbf{J}_H = -\mathbf{J}_L$. The downsampling-then-upsampling operation can be defined as $\mathbf{D}_{du,L} = \frac{1}{2}(\mathbf{I}_N + \mathbf{J}_L)$, $\mathbf{D}_{du,H} = \frac{1}{2}(\mathbf{I}_N + \mathbf{J}_H)$, where $\mathbf{I}_N$ is an $N \times N$ identity matrix.

$\mathbf{J}$ and $\mathbf{P}_{\lambda_i}$ are related as follows [2] (spectral folding phenomenon):

$$\mathbf{J}\mathbf{P}_{\lambda_i} = \mathbf{P}_{2-\lambda_i}\mathbf{J}. \tag{2}$$

The critically sampled filter banks decompose N input signals into $|L|$ lowpass coefficients and $|H|$ highpass coefficients, where $|L| + |H| = N$, as illustrated in Fig. 1. The overall transfer function can be written as

$$\begin{aligned} \mathbf{T} &= \frac{1}{2}\mathbf{G}_0(\mathbf{I} - \mathbf{J})\mathbf{H}_0 + \frac{1}{2}\mathbf{G}_1(\mathbf{I} + \mathbf{J})\mathbf{H}_1 \\ &= \frac{1}{2}(\mathbf{G}_0\mathbf{H}_0 + \mathbf{G}_1\mathbf{H}_1) + \frac{1}{2}(\mathbf{G}_1\mathbf{J}\mathbf{H}_1 - \mathbf{G}_0\mathbf{J}\mathbf{H}_0). \end{aligned} \tag{3}$$

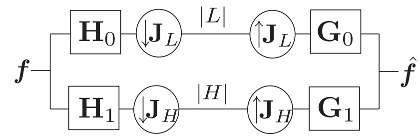

Fig. 1. Critically sampled two-channel graph filter bank.

The spectral folding term $\mathbf{G}_1\mathbf{J}\mathbf{H}_1 - \mathbf{G}_0\mathbf{J}\mathbf{H}_0$, arising from downsampling and upsampling, must be zero. In addition, $\mathbf{T} = \mathbf{I}_N$ must be satisfied for perfect reconstruction. Hence, the perfect reconstruction condition can be expressed as

$$\begin{aligned} g_0(\lambda)h_0(\lambda) + g_1(\lambda)h_1(\lambda) = 2, \\ -g_0(\lambda)h_0(2-\lambda) + g_1(\lambda)h_1(2-\lambda) = 0. \end{aligned} \tag{4}$$

There are two well-known perfect reconstruction filter sets: graph-QMF [2], which is the orthogonal transform and non-compact support, and graphBior [3], which is the biorthogonal transform and satisfies perfect reconstruction and compact support conditions.

2.3. M-Channel Graph Filter Banks

The authors proposed M-channel oversampled graph filter banks [6], where M is even and $M/2$ filters keep $|L|$ signals and other ones keep $|H|$ signals. Similar to the critically sampled case, the perfect reconstruction condition of the M-channel oversampled graph filter bank can be represented as

$$\sum_{k=0}^{M-1} g_k(\lambda)h_k(\lambda) = 2, \tag{5}$$

$$\sum_{k=0}^{M/2-1} g_k(\lambda)h_k(2-\lambda) - g_{k+M/2}(\lambda)h_{k+M/2}(2-\lambda) = 0. \tag{6}$$

for any $\lambda \in [0, 2]$. The design methods of perfect reconstruction filters are described in [6,9].

3. GRAPH BIPARTITIONS

The decimated graph filter bank is applicable only for bipartite graphs. We describe three bipartition methods to make a bipartite graph $\mathcal{G} = \{L_b, H_b, \mathcal{E}_b\}$ from the original non-bipartite graph $\mathcal{G} = \{\mathcal{V}, \mathcal{E}\}$. $\mathcal{E}_b \subset \mathcal{E}$ includes all the edges between sets L_b and H_b for all methods.

3.1. Maximum Spanning Tree-Based Bipartition

For the MST-based bipartition [4], the MST $\mathcal{T}$ is constructed by using Prim's algorithm [10]. The nodes are divided by the distance from the root node $r \in L$:

$$\begin{aligned} L_b :=& \{i \in \mathcal{V} : d_{\mathcal{T}}(i, r) \text{ is even}\}, \\ H_b :=& \{i \in \mathcal{V} : d_{\mathcal{T}}(i, r) \text{ is odd}\}. \end{aligned} \tag{7}$$

where $d_{\mathcal{T}}(i, r)$ is the number of edges of the shortest path in $\mathcal{T}$ from r to i.

3.2. Eigendecomposition-Based Bipartition

In graph theory, the approximate coloring partition has been known, which is based on the polarity of eigenvector $\overline{\boldsymbol{u}}_{\min}$ associated with the most negative eigenvalue of the graph adjacency matrix [8]:

$$\begin{aligned} L_b &:= \{i \in \mathcal{V} : \overline{u}_{\min}(i) \geq 0\}, \\ H_b &:= \{i \in \mathcal{V} : \overline{u}_{\min}(i) < 0\}. \end{aligned} \tag{8}$$

The bipartition can approximate the original graph so as to reduce the edges included in the same set of nodes.

3.3. Coloring-Based Bipartition

The Harary's algorithm [11] is the bipartition based on the graph coloring. A color is assigned to each node with the minimum number of colors so that no two adjacency nodes have the same color. An arbitrary K-colorable graph can be decomposed into an edge-disjoint collection of $\lfloor \log_2 K \rfloor$ bipartite subgraphs whose union is the original graph. One of bipartite subgraphs is determined as:

$$\begin{aligned} L_b &:= \{F_1, \ldots, F_{\lceil \frac{K}{2} \rceil}\}, \\ H_b &:= \{F_{\lceil \frac{K}{2} \rceil + 1}, \ldots, F_K\}, \end{aligned} \tag{9}$$

where the sets $F_1, \ldots, F_K$ contain the nodes assigned the same color.

4. GRAPH OVERSAMPLING

In this section, we describe the graph oversampling [6, 7], and propose the effective construction method of oversampled GLMs and oversampled graph signals while controlling the redundancy.

4.1. Oversampled Graph Laplacian Matrix

Fig. 2 shows an example of the transform using graph oversampling. By appending the nodes and the edges, the original bipartite graph $\mathcal{G} = \{L, H, \mathcal{E}\}$ is expanded to the oversampled bipartite graph $\widetilde{\mathcal{G}} = \{\widetilde{L}, \widetilde{H}, \widetilde{\mathcal{E}}\}$ that $\widetilde{L}$ and $\widetilde{H}$ includes L and H, respectively. The downsampling matrices $\mathbf{J}_{\widetilde{L}}$ and $\mathbf{J}_{\widetilde{H}}$ of the oversampled graph are defined by $\widetilde{L}$ and $\widetilde{H}$. The oversampled signal $\widetilde{\boldsymbol{f}}$ is written as

$$\widetilde{\boldsymbol{f}} = \begin{bmatrix} \boldsymbol{f} \\ \boldsymbol{f}' \end{bmatrix}, \tag{10}$$

where $\boldsymbol{f}'$ is the signal for additional nodes. The spectral domain filtering is performed based on the oversampled GLM.

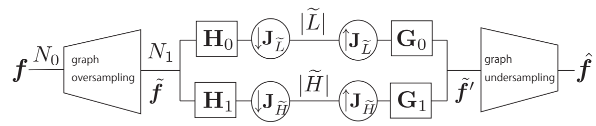

Fig. 2. Graph oversampling followed by the critically sampled graph filter bank.

Let $\mathbf{A}_0$ be an adjacency matrix of the original bipartite graph whose size is $N_0 \times N_0$. The normalized oversampled GLM $\widetilde{\boldsymbol{\mathcal{L}}}$ is $N_1 \times N_1$ ($N_1 > N_0$), and $N_1 - N_0$ is the number of the additional nodes. It is represented as

$$\widetilde{\boldsymbol{\mathcal{L}}} = \widetilde{\mathbf{D}}^{-1/2} \widetilde{\mathbf{L}} \widetilde{\mathbf{D}}^{-1/2} \tag{11}$$

where

$$\widetilde{\mathbf{L}} = \widetilde{\mathbf{D}} - \widetilde{\mathbf{A}} \tag{12}$$

$$\widetilde{\mathbf{A}} = \begin{bmatrix} \mathbf{A}_0 & \mathbf{A}_{01} \\ \mathbf{A}_{01}^T & \mathbf{0}_{N_1 - N_0} \end{bmatrix}, \tag{13}$$

in which $\widetilde{\mathbf{A}}$ is the oversampled adjacency matrix whose size is $N_1 \times N_1$ and $\widetilde{\mathbf{D}}$ is a degree matrix that normalizes the new GLM. Additionally, $\mathbf{A}_{01}$ contains information for the connection between the original nodes and appended ones. Note that nodes are appended so that $\widetilde{\boldsymbol{\mathcal{L}}}$ is still a bipartite graph. The filters in Fig. 2 can be represented as $\mathbf{H}_i = h_i(\widetilde{\boldsymbol{\mathcal{L}}})$ and $\mathbf{G}_i = g_i(\widetilde{\boldsymbol{\mathcal{L}}})$ for $i = 0, 1$.

4.2. Graph Oversampling Method

As described in Section 4.1, the appended nodes of the oversampled GLM can be arbitrarily connected to the original nodes, as long as the oversampled graph is bipartite. We consider an efficient way to construct such oversampled graphs. We assume that the appended nodes are only connected to nodes in $\widetilde{L}$ in order for the oversampled graph to be a bipartite graph and the number of lowpass coefficients to be the same as that of the critically sampled transform. The number N_a of additional nodes is determined depending on the desired redundancy. Hence, the oversampled graph is represented as $\widetilde{\mathcal{G}} = \{\widetilde{L} = L_b, \widetilde{H} = H_b \cup L'_b, \widetilde{\mathcal{E}}\}$ where $|L'_b| = N_a$. The following procedure describes the construction method of the oversampled graph from the original graph $\mathcal{G} = \{\mathcal{V}, \mathcal{E}\}$:

1. The foundation bipartite graph $\mathcal{G}_b = \{L_b, H_b, \mathcal{E}_b\}$ is generated from the original graph by using arbitrary bipartition method.
2. The remained subgraph is calculated as $\overline{\mathcal{G}} = \{\mathcal{V}, \mathcal{E} \setminus \mathcal{E}_b\}$. $\overline{\mathcal{G}}$ has two disjoint graphs: $\overline{\mathcal{G}}(L_b)$ and $\overline{\mathcal{G}}(H_b)$.
3. The appended nodes are selected according to the degrees of nodes in $\overline{\mathcal{G}}(L_b)$. The N_a nodes with the largest degrees are included in L'_b.
4. The nodes in L'_b are placed directly above each nodes in L_b of the foundation bipartite graph. The nodes in L'_b have

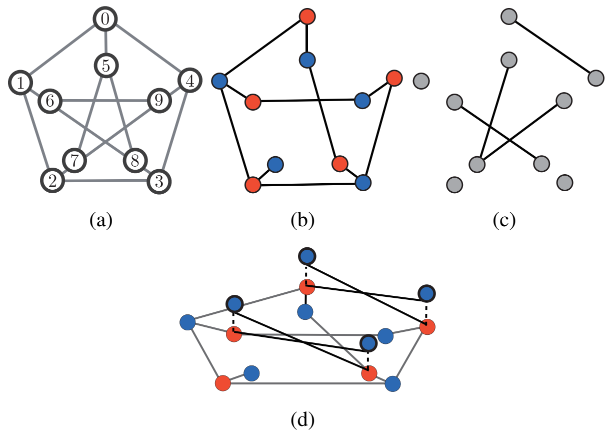

Fig. 3. Bipartite oversampled graph construction for Petersen graph. (a) Petersen graph. (b) Foundation bipartite graph. Red and blue nodes represent L_b and H_b, respectively. (c) Remained graph. (d) Oversampled bipartite graph. The dashed lines connect the nodes in L_b with the corresponding additional nodes. The gray lines are edges contained in foundation bipartite graph and the black lines are additional edges.

the same value as those in L_b. By letting L_b' be in $\widetilde{H}$, it can be connected freely with the nodes in L_b since they belong to $\widetilde{L}$.

5. The edges between L_b' and L_b are appended by using the edge information of $\overline{\mathcal{G}}(L_b)$. As a result, all nodes can connect with L_b or L_b' while keeping the graph bipartite.
6. The isolated nodes in $\overline{\mathcal{G}}$ is removed.

For example, we construct the oversampled bipartite graph of the Petersen graph (Fig. 3(a)) with $N_a = |L_b|$. The foundation bipartite graph (Fig. 3(b)) is made by dividing the nodes according to the parity of the nodes, i.e. even-labeled nodes are in L_b and odd-labeled nodes are in H_b. Fig. 3(c) is the remained graph. In order to make the oversampled bipartite graph shown in Fig. 3(d), we place blue nodes right above the red ones of the foundation bipartite graph (Fig. 3(b)) and add edges by referring to the information about the edges of the remained subgraph (Fig. 3(c)). The additional blue nodes have the same values as the corresponding red nodes and are treated as $\boldsymbol{f}'$ in (10). Node v_2 is the red node in $\mathcal{G}_b$ and is isolated in $\overline{\mathcal{G}}$. Therefore, v_2 does not have a corresponding appended node.

5. EXPERIMENTAL RESULTS

We perform the experiment on graph signal denoising to examine the relationship between redundancy and performance gain for the oversampled graphs based on the different bipartition methods. Graph signals corrupted by white Gaussian noise are denoised by the simple hard thresholding. The oversampled graphs are constructed according to Section 4.2 with

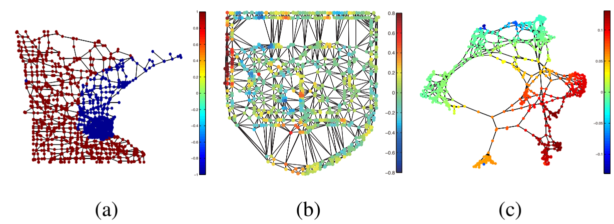

Fig. 4. Original signals. (a) *Minnesota Traffic Graph* ($N = 2642$). (b) *Yale Coat of Arms* ($N = 989$). (c) *Bus Graph* ($N = 685$).

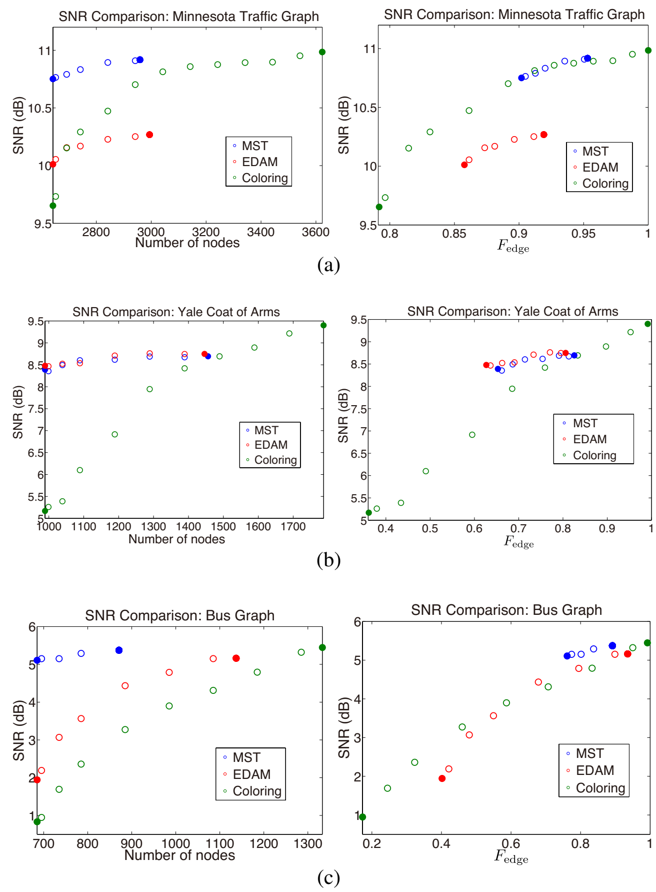

Fig. 5. SNR comparison. Filled circles indicate the results of the critically sampled graph and the oversampled graph with the maximum number of $|L_b'|$: (a) Signals on *Minnesota Traffic Graph* with $\sigma = 1/2$. (b) Signals on *Yale Coat of Arms* with $\sigma = 1/8$. (c) Signals on *Bus Graph* with $\sigma = 1/16$.

various set of redundancies. The foundation bipartite graphs are made by the MST, the eigendecomposition of adjacency matrix (EDAM) and the coloring-based algorithm. Then, we implement four-channel graph filter bank [9] derived from the oversampled linear phase perfect reconstruction filter bank [12] on these oversampled bipartite graphs. After the one-level decomposition of the corrupted input signal, the lowest

Table 1. Comparison of Bipartitions: $|L'_{\max}|$ is the maximum number of $|L'_b|$ and w_{total} is the total weight of the edges in $\mathcal{G}_b$.

Method	Original graph	MST	EDAM	Coloring
Minnesota Traffic Graph ($N = 2642$)				
$\|L_b\|$	–	1313	1335	1592
$\|L'_{\max}\|$	–	317	405	981
$\|\mathcal{E}_b\|$	3303	2979	2833	2615
Yale Coat of Arms ($N = 989$)				
$\|L_b\|$	–	488	489	798
$\|L'_{\max}\|$	–	457	467	798
$\|\mathcal{E}_b\|$	2701	1693	1764	976
Bus Graph ($N = 685$)				
$\|L_b\|$	–	351	452	648
$\|L'_{\max}\|$	–	186	495	648
$\|\mathcal{E}_b\|$	1282	976	515	203
w_{total}	185812	175117	113016	11696

frequency subband was kept, and the other high-frequency subbands are applied hard-thresholding with the threshold 3σ, where σ is the standard deviation of noise. The tested signals are on *Minnesota Traffic Graph*, *Yale Coat of Arms* and *Bus Graph* as shown in Fig. 4. The *Minnesota Traffic Graph* and the *Yale Coat of Arms* are unweighted graphs.

Table 1 compares the number of the lowest frequency coefficients $|L_b|$, the maximum number of $|L'_b|$,[1] the number of edges in the foundation graph and the total of edge weights in $\mathcal{G}_b$. Figure 5 shows SNRs plotted against the number of the nodes and the fraction of the number of the edges corresponding to that of the original graph: $F_{\text{edge}} = \frac{1}{2}(|\widetilde{\mathcal{E}}| + |\mathcal{E}_b|)/|\mathcal{E}|$. When we focus on the filled circles corresponding to the denoised results by using the foundation (critically sampled) graphs, we can observe that the performance of the graph bipartition depends on the underlying graph structure. From the comparison of the oversampled graphs, it can be seen that the SNRs are varied depending on the foundation graphs even if the oversampled graphs have the same number of nodes. However, if we see the number of edges, the performance has a relationship with F_{edge}, i.e., oversampled bipartite graphs which contains many edges in the original graph show relatively higher performances.

6. CONCLUSION

We proposed a new construction method of the oversampled graph utilizing the MST, the EDAM and the coloring-based bipartition with various redundancy. The proposed method can append many edges with desired redundancy. In the denoising experiment, we found that the bipartite graph having many edges in the original graph shows higher performance in many cases. It would be expected that constructing the foundation graph which maximizes the number of the isolated nodes in the remained graph leads high performance gain. It is an interesting open problem.

[1] Since the isolated nodes in $\overline{\mathcal{G}}$ is removed, the maximum number of $|L'_b|$ is less than $|L_b|$.

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