A NOVEL ALGORITHM FOR THE ESTIMATION OF THE PARAMETERS OF A REAL SINUSOID IN NOISE

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ABSTRACT

In this paper, we put forward a computationally efficient algorithm to estimate the frequency and complex amplitude of a real sinusoidal signal in additive Gaussian noise. The novel method extends an iterative frequency estimator for single complex exponentials that is based on interpolation on Fourier coefficients to the real case by incorporating an iterative leakage subtraction strategy. Simulation results are presented to verify that the proposed algorithm can obtain more accurate estimation than both time and frequency domain parameter estimators in the literature, and the estimation variance of the method sits on the Cramer-Rao lower bound with only a few iterations required.

***Index Terms*—** Frequency estimation, interpolation algorithm, Fourier coefficients, real sinusoid.

1. INTRODUCTION

The estimation of the parameters of a real sinusoidal signal in noise is a classical yet important research problem in many applications [1] [2]. The estimation problem in this paper is based on the signal model

$$x(n) = a\cos(2\pi f n + \phi) + w_r(n), \quad n = 0, \ldots, N-1, \tag{1}$$

with n representing the sample time index and N the number of samples. The parameters a, ϕ and f are the amplitude, initial phase and the normalised frequency $f \in (0, 0.5]$, which we try to estimate. The noise terms $w_r(n)$ are assumed to be real Gaussian with zero mean and variance σ^2. The signal to noise (SNR) is defined as $\rho = |a|^2/\sigma^2$.

Much work has been done on this problem and the available algorithms can be classified into time domain and frequency domain methods [3]. Traditional time-domain methods such as Prony's, Pisarenko's methods and the MUltiple SIgnal Classification (MUSIC) [2], are outperformed by later developed algorithms such as Estimation of Signal Parameters via Rotational Invariant Techniques (ESPRIT) [4] and Matrix Pencil [5]. They are based on the Singular Value Decomposition (SVD) operation to separate the signal and noise subspaces. Moreover, the recently proposed Weighted Least Squares (WLS) approach [6] is capable of achieving performance that attains the Cramer-Rao Lower Bound (CRLB) by employing an iterative minimisation procedure of the sum of the squared error between the ideal model and the noisy data. These are high resolution estimators that can achieve accurate estimates but at the same time are computationally complex due to the SVD operation and matrix inversion ($O(N^3)$). The frequency-domain estimators [1], on the other hand, are mostly based on interpolation of the DFT coefficients [7], [8]. Although they are computationally simpler, as they take advantage of the FFT implementation, they exhibit lower estimation accuracy due to the estimation bias caused by the spectral leakage [9]. In spite of windowing methods [1] [10] being proposed to remove the spectral leakage by pre-windowing the signal using a non-rectangular window, they achieve the reduction of the interference by sacrificing the estimation accuracy.

In this paper, we put forward a novel iterative algorithm that operates in the frequency domain yet can achieve unbiased and accurate parameter estimation with no windowing involved. At the heart of our algorithm is the estimator of Aboutanios and Mulgrew [8] [11] (the A&M algorithm), which we wrap in an iterative estimation strategy that incorporates leakage subtraction [12] to attain accurate parameter estimation. The computational cost of the novel algorithm is of the same order as the computational cost of the FFT operation ($O(N \log N)$), which is more efficient than the high resolution methods.

The rest of the paper is organised as follows. In Section 2, we review the original A&M algorithm and present the novel parameter estimator. The simulation results are shown in Section 3. Finally, conclusion is drawn in Section 4.

2. THE ESTIMATION ALGORITHM

Before putting forward the new estimation strategy, we review the original A&M algorithm for a single complex exponential.

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2.1. The A&M Algorithm

Table 1. The A&M Algorithm

Given	A complex exponential $x(n), n = 0, \dots, N-1$;
Find	$X(k) = \text{FFT}(x)$ and $Y(k) = \lvert X(k)\rvert^2$;
Find	$\hat{m} = \arg\max_k Y(k)$;
Set	$\hat{\delta} = 0$;
Loop	For i from 1 to 2, do
	(1) $X_{\pm} = \sum_{n=0}^{N-1} x(n) e^{-j\frac{2\pi}{N}(\hat{m}+\hat{\delta}\pm 0.5)n}$;
	(2) $\hat{\delta} = \hat{\delta} + \Re\{h_c\}$ or $\hat{\delta} = \hat{\delta} + \frac{N}{2\pi}\angle\hat{z}$,
	where $h_c = \frac{1}{2}\frac{X_+ + X_-}{X_+ - X_-}$, and
	$\hat{z} = \left[\cos\left(\frac{\pi}{N}\right) - 2jh_c \sin\left(\frac{\pi}{N}\right)\right]^{-1}$;
Find	$\hat{f} = \frac{\hat{m}+\hat{\delta}}{N}$.

From this point on, we use $\hat{\lambda}$ to denote the estimate of the parameter λ. The A&M algorithm estimates the frequency of the signal with the following model

$$x(n) = a_c e^{j2\pi nf} + w_c(n), \quad n = 0, \dots, N-1, \tag{2}$$

where a_c is the complex amplitude. The noise terms $w_c(n)$ are assumed to be complex Gaussian noise with zero mean and variance σ_c^2.

To estimate the frequency, the algorithm comprises two stages: the coarse estimation stage and the fine estimation stage. In the coarse estimation stage, the maximum bin of the periodogram is found to be the initial estimation of the frequency:

$$\hat{m} = \arg\max_k \left\{ \left| \sum_{n=0}^{N-1} x(n) e^{-j\frac{2\pi}{N}kn} \right|^2 \right\}. \tag{3}$$

As a result, the true frequency can then be represented by:

$$f = \frac{\hat{m}+\delta}{N}, \tag{4}$$

where $\delta \in [-0.5, 0.5]$ is the frequency residual. In the fine estimation stage, the coarse estimation is then refined by calculating δ using an estimator based on interpolation of Fourier coefficients on either side of the maximum bin at locations $(\hat{m} \pm 0.5)$. The noise-free interpolated coefficients are given by:

$$X_{\pm} = \sum_{n=0}^{N-1} x(n) e^{-j\frac{2\pi}{N}(\hat{m}\pm 0.5)n} = a_c \frac{1 + e^{j2\pi\delta}}{1 - e^{j\frac{2\pi}{N}(\delta \mp 0.5)}}. \tag{5}$$

By constructing the following estimator [11]:

$$z = e^{j\frac{2\pi}{N}\delta} = \left[\cos\left(\frac{\pi}{N}\right) - 2jh_c \sin\left(\frac{\pi}{N}\right)\right]^{-1}, \tag{6}$$

where

$$h_c = \frac{1}{2}\frac{X_+ + X_-}{X_+ - X_-}, \tag{7}$$

δ is obtained by:

$$\delta = \frac{N}{2\pi}\angle z, \tag{8}$$

where $\angle z$ is the angle of z. Eq.(8) is considered to be the exact version of the A&M estimator. Now by linearising the denominator of Eq.(5) using the approximation $1 - e^x \approx -x$, we have:

$$X_{\pm} \approx ja_c \frac{N}{2\pi}\frac{1 + e^{j2\pi\delta}}{\delta \mp 0.5}, \tag{9}$$

which results in:

$$\Re\{h_c\} \approx \delta, \tag{10}$$

where $\Re\{\bullet\}$ denotes the real part of $\bullet$. Eq.(10) is regarded as the approximate version of the A&M estimator and the real operation is necessary to reduce the estimation variance in the presence of noise [8].

For further improvement of accuracy, the fine estimation stage is performed for the second iteration, before which the estimated residual in the first iteration is removed from the maximum bin. After two iterations, the algorithm is capable of obtaining an asymptotically unbiased frequency estimate with the estimation variance only 1.0147 times the CRLB [8]. Finally, the estimation procedure of the A&M algorithm is summarised in Table 1.

2.2. The New Estimator

The real sinusoid shown in (1) can be represented as a sum of two complex exponentials:

$$\begin{aligned} x(n) &= a\frac{e^{j\phi}}{2}e^{j2\pi fn} + a\frac{e^{-j\phi}}{2}e^{-j2\pi fn} + w_r(n) \\ &= Ae^{j2\pi fn} + A^* e^{-j2\pi fn} + w_r(n) \\ &= s(n) + s^*(n) + w_r(n), \end{aligned} \tag{11}$$

where $A = ae^{j\phi}/2$.

The parameter estimation problem of a real sinusoid is then converted to the estimation problem of two complex exponentials $s(n)$ and $s^*(n)$. Intuitively, the estimation of frequency of $s(n)$, which is also the frequency of $x(n)$, can be performed by an iterative procedure involving removing interference caused by $s^*(n)$. In [13], the authors proposed an approach of converting the problem to the single-tone case by subtracting the component $s^*(n)$ before estimation. In this work, however, we utilise the idea presented in [12] and accomplish the estimation by only removing the leakage caused by $s^*(n)$ in an iterative fashion. However, as a real sinusoid is being considered, the leakage subtraction scheme in [12] is modified to accommodate the fact that the two exponentials have frequencies of opposite sign and complex conjugate amplitudes.

Let $\hat{m}$ be the maximum bin estimate of $s(n)$ and is assumed to be identical to the true value. Now the frequency of $s(n)$ is given by (4) with δ being the residual we need to estimate. We also denote $\hat{\delta}$ and $\hat{A}$ to be the estimates of δ and A obtained in the previous iteration. Then in each iteration, the noiseless interpolated Fourier coefficients at locations $\hat{m} + \hat{\delta} \pm 0.5$ are represented by:

$$\begin{aligned}\tilde{X}_{\pm} &= \sum_{n=0}^{N-1}[s(n) + s^*(n)]e^{-j\frac{2\pi}{N}(\hat{m}+\hat{\delta}\pm 0.5)n} \qquad (12)\\ &= A\frac{1 + e^{j2\pi(\delta-\hat{\delta})}}{1 - e^{j\frac{2\pi}{N}(\delta-\hat{\delta}\mp 0.5)}} + A^*\frac{1 + e^{-j2\pi(\delta+\hat{\delta})}}{1 - e^{-j\frac{2\pi}{N}(2\hat{m}+\delta+\hat{\delta}\pm 0.5)}}\\ &= S_{\pm} + \breve{S}_{\pm},\end{aligned}$$

where $S_{\pm}$ are the expected interpolated Fourier coefficients corresponding to $s(n)$ and $\breve{S}_{\pm}$ are the leakage terms introduced by $s^*(n)$. $\breve{S}_{\pm}$ can be estimated utilising $\hat{\delta}$ and $\hat{A}$:

$$\hat{\breve{S}}_{\pm} = \hat{A}^*\frac{1 + e^{-j4\pi\hat{\delta}}}{1 - e^{-j\frac{2\pi}{N}(2\hat{m}+2\hat{\delta}\pm 0.5)}}, \qquad (13)$$

and we can obtain the estimates of the expected coefficients by subtracting $\hat{\breve{S}}$ from $\tilde{X}$:

$$\hat{S} = \tilde{X}_{\pm} - \hat{\breve{S}}_{\pm}, \qquad (14)$$

which straightforwardly leads to the estimation function:

$$h_r = \frac{1}{2}\Re\left\{\frac{\hat{S}_+ + \hat{S}_-}{\hat{S}_+ - \hat{S}_-}\right\} \approx \delta - \hat{\delta}. \qquad (15)$$

Notice that here we use the approximate version of the A&M estimator as it is already capable of obtaining sufficiently accurate results. The estimation of A is obtained by

$$\begin{aligned}\hat{A} &= \frac{1}{N}\sum_{n=0}^{N-1}[x(n) - \hat{s}^*(n)]e^{-j\frac{2\pi}{N}(\hat{m}+\hat{\delta})n}\\ &= \frac{1}{N}\left(\sum_{n=0}^{N-1}x(n)e^{-j\frac{2\pi}{N}(\hat{m}+\hat{\delta})n} - \hat{A}^*\frac{1 - e^{-j4\pi\hat{\delta}}}{1 - e^{-j\frac{4\pi}{N}(\hat{m}+\hat{\delta})}}\right), \qquad (16)\end{aligned}$$

and we can obtain the estimation of a and ϕ from $\hat{A}$ by:

$$\hat{a} = 2|\hat{A}|, \quad \text{and} \quad \hat{\phi} = \angle\hat{A}. \qquad (17)$$

To conclude this section, the detailed procedure of the proposed real signal estimator is tabulated in Table 2.

3. SIMULATION RESULTS

In this section, we present the simulation results of the proposed frequency estimation algorithm to verify its performance. For all the results demonstrated below, we fix $N = 64$ and $a = 1$. 5,000 Monte Carlo runs were used for generating the figures in the section.

Table 2. The Proposed Estimator

Given	A real sinusoid $x(n), n = 0\dots, N-1$;
Find	$X(k) = \text{FFT}(x)$ and $Y(k) = \lvert X(k)\rvert^2$;
Find	$\hat{m} = \arg\max_k Y(k)$, if $\hat{m} \geq \frac{N}{2}, \hat{m} = N - \hat{m}$;
Set	$\hat{\delta} = 0$ and $\hat{A} = 0$;
Loop	For i from 1 to Q, do
	(1) $\tilde{X}_{\pm} = \sum_{n=0}^{N-1} x(n)e^{-j\frac{2\pi}{N}(\hat{m}+\hat{\delta}\pm 0.5)n}$;
	(2) $\hat{\breve{S}}_{\pm} = \hat{A}^*\frac{1 + e^{-j4\pi\hat{\delta}}}{1 - e^{-j\frac{2\pi}{N}(2\hat{m}+2\hat{\delta}\pm 0.5)}}$, and $\hat{S} = \tilde{X}_{\pm} - \hat{\breve{S}}$;
	(3) $\hat{\delta} = \hat{\delta} + \frac{1}{2}\Re\left\{\frac{\hat{S}_+ + \hat{S}_-}{\hat{S}_+ - \hat{S}_-}\right\}$;
	(4) $\hat{A} = \frac{1}{N}\left(\sum_{n=0}^{N-1} x(n)e^{-j\frac{2\pi}{N}(\hat{m}+\hat{\delta})n} - \hat{A}^*\frac{1 - e^{-j4\pi\hat{\delta}}}{1 - e^{-j\frac{4\pi}{N}(\hat{m}+\hat{\delta})}}\right)$;
Find	$\hat{f} = \frac{\hat{m}+\hat{\delta}}{N}$, $\hat{a} = 2\lvert\hat{A}\rvert$ and $\hat{\phi} = \angle\hat{A}$.

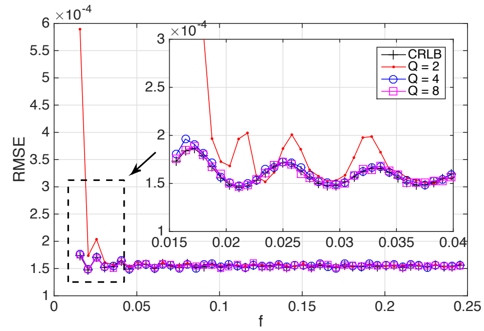
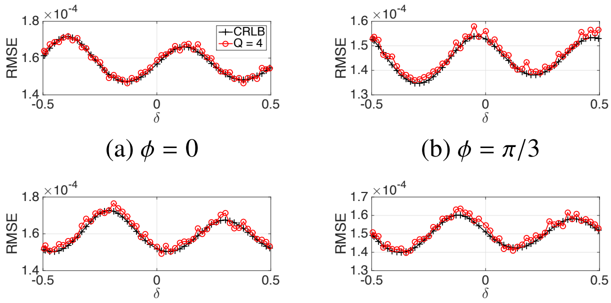


Fig. 1. RMSE of $\hat{f}$ versus f when $\phi = 0$ and SNR=20dB.

We firstly investigate the performance of the algorithm as a function of the frequency f. Due to the periodicity of the spectrum of a real signal, we are only interested in the results for $f \in [0, 0.25]$. As f move closer towards zero, the two complex components become closer to each other. On the other hand, when f shifts towards 0.25, the complex components approach to their largest separation. In this test we fix $\phi = 0$ and SNR = 20dB. Fig. 1 shows the Root Mean Square Error (RMSE) of $\hat{f}$ versus f from 0.0156 (which is $1/N$) to 0.25. For the sake of benchmarking the performance, the novel method is compared with the CRLB [14]. We can find that more iterations are needed when the frequency gets smaller. When $Q = 2$, the RMSE follows the CRLB at $f \geq 0.035$, while $Q = 8$ is sufficient for the RMSE to sit on the CRLB for all f.

In Fig. 2 we show the RMSE of $\hat{f}$ versus $\delta \in [-0.5, 0.5]$ for $\phi = 0, \pi/3, -2\pi/3$ and $\pi/2$. In this test we fix the maximum bin to be $m = 2$, set SNR = 20dB and implement the algorithm using $Q = 4$. We can find from the figure that the



(a) $\phi = 0$ (b) $\phi = \pi/3$

(c) $\phi = -2\pi/3$ (d) $\phi = \pi/2$

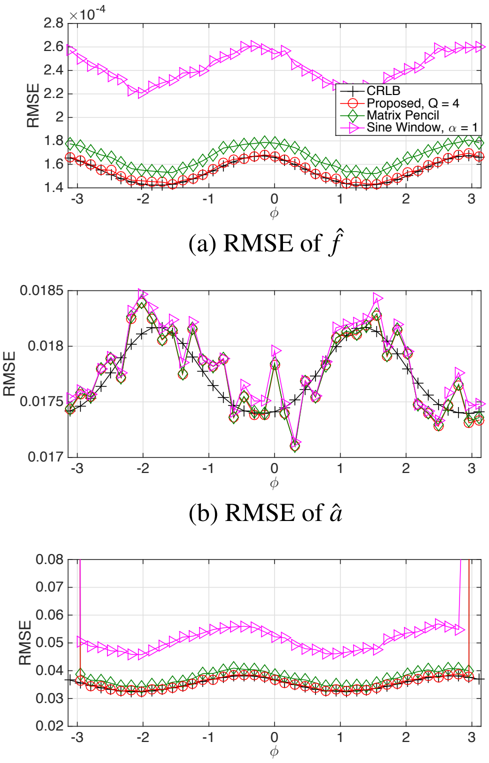
Fig. 2. RMSE of $\hat{f}$ obtained by the proposed method versus δ under various ϕ.

RMSE of $\hat{f}$ keeps tracking the CRLB for all ϕ. Fig. 3 shows RMSE of $\hat{f}$ versus ϕ when $f = 0.03$ and SNR = 20dB. Besides CRLB, we also plot the results obtained from the state-of-art time domain method Matrix Pencil [5] and the recently proposed $\sin^{\alpha}(n)$ windowing method [10] (the Sine Window method). For Matrix Pencil, the pencil parameter is set to $L = \lfloor N/3 \rfloor$ and for the Sine Window method, we set the window order to be $\alpha = 1$. We find that the proposed algorithm has the best performance that is extremely close to the CRLB from $\phi \in [-\pi, \pi]$ while different levels of gaps between the RMSE of $\hat{f}$ and $\hat{\phi}$ and the CRLB are exhibited by the other two methods. Notice that in Fig. 3(c) the RMSE of all methods exhibits a sharp jump near $\phi = \pm\pi$, which results from the phase wrapping effect.

Finally we examine the RMSE versus SNR. In this test the proposed method is still compared with CRLB, Matrix Pencil and the Sine Window methods. The processing parameters of the Matrix Pencil and the Sine Window methods are maintained the same as the previous test. Figs. 4 to 6 show RMSE of $\hat{f}$, $\hat{a}$ and $\hat{\phi}$ when $f = 0.1$ and $\phi = \pi/4$. It is observed that when $Q = 2$, the RMSE of the proposed method sits on the CRLB when SNR > 4dB. Meanwhile, the proposed method has considerably better performance than the Sine Window method and also slightly outperforms the Matrix Pencil for frequency and phase estimates when SNR $\geq$ 4dB. Fig. 7 shows the RMSE of $\hat{f}$ when $f = 0.02$ and $\phi = \pi/3$. As expected based on the results in Fig. 1, the proposed algorithm requires more iterations for unbiased performance and the RMSE sits on the CRLB when $Q = 4$. The RMSE plots of amplitude and phase estimates exhibit similar behaviour to the frequency estimate and therefore are not shown here due to space limitations.

4. CONCLUSION

We proposed in this paper an iterative algorithm for estimating the frequency, amplitude and phase of a real sinusoidal signal in additive Gaussian noise. The novel algorithm is extended from the frequency estimator for a single complex exponential, developed by Aboutanios and Mulgrew (the A&M algorithm), which is based on interpolation on Fourier coefficients. The novel method iteratively estimates the frequency by combining the interpolation function of the A&M algorithm with a leakage subtraction scheme. Simulation results verified that the proposed method is capable of obtaining RMSE that is extremely close to the CRLB.



(a) RMSE of $\hat{f}$

(b) RMSE of $\hat{a}$

(c) RMSE of $\hat{\phi}$

Fig. 3. RMSE of $\hat{f}$, $\hat{a}$ and $\hat{\phi}$ versus ϕ when SNR= 20dB.

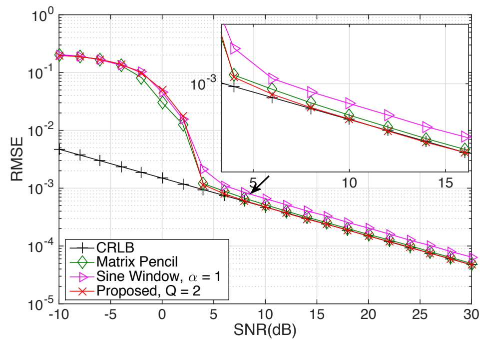


Fig. 4. RMSE of $\hat{f}$ versus SNR when $f = 0.1$ and $\phi = \pi/4$.

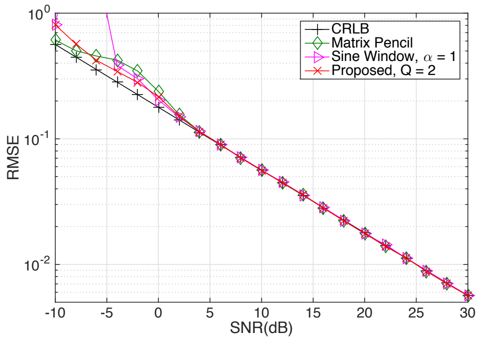


Fig. 5. RMSE of $\hat{a}$ versus SNR when $f = 0.1$ and $\phi = \pi/4$.

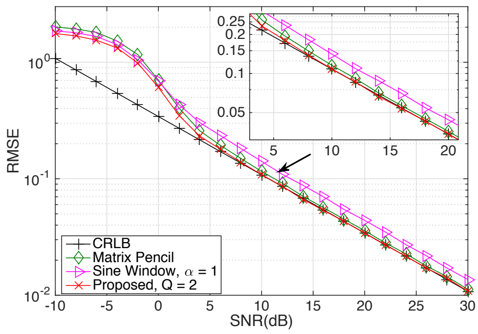


Fig. 6. RMSE of $\hat{\phi}$ versus SNR when $f = 0.1$ and $\phi = \pi/4$.

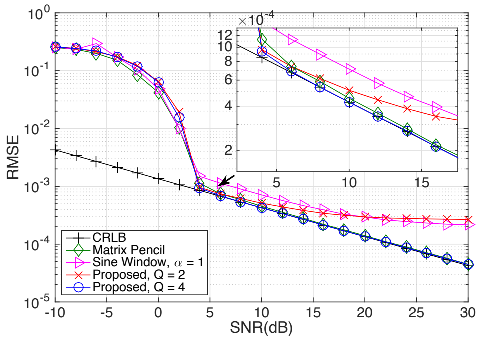


Fig. 7. RMSE of $\hat{f}$ versus SNR when $f = 0.02$ and $\phi = \pi/3$.

5. REFERENCES

[1] K. Duda, L.B. Magalas, M. Majewski, and T.P. Zielinski, "DFT-based estimation of damped oscillation parameters in low-frequency mechanical spectroscopy," *IEEE Transactions on Instrumentation and Measurement*, vol. 60, no. 11, pp. 3608 –3618, 2011.

[2] P. Stoica and R. L. Moses, *Introduction to Spectral Analysis*, Prentice Hall, Upper Saddle River, New Jersey, 1997.

[3] T.P. Zielinski and K. Duda, "Frequency and damping estimation methods - an overview," *Metrology and Measurement Systems*, vol. 18, no. 4, pp. 505–528, 2011.

[4] R. Roy and T. Kailath, "ESPRIT - Estimation of signal parameters via rotational invariance techniques," *IEEE Transactions on Acoustics, Speech, and Signal Processing*, vol. 37, no. 7, pp. 984–995, 1989.

[5] Y. Hua and T. K. Sarkar, "Matrix pencil method for estimating parameters of exponentially damped/undamped sinusoids in noise," *IEEE Transactions on Acoustics, Speech, and Signal Processing*, vol. 38, no. 5, pp. 814–824, 1990.

[6] H.C. So, K. W. Chan, Y.T. Chan, and K.C. Ho, "Linear prediction approach for efficient frequency estimation of multiple real sinusoids: algorithms and analyses," *IEEE Transactions on Signal Processing*, vol. 53, no. 7, pp. 2290–2305, July 2005.

[7] B.G. Quinn, "Estimation of frequency, amplitude, and phase from the DFT of a time series," *IEEE Transactions on Signal Processing*, vol. 45, no. 3, pp. 814–817, 1997.

[8] E. Aboutanios and B. Mulgrew, "Iterative frequency estimation by interpolation on Fourier coefficients," *IEEE Transactions on Signal Processing*, vol. 53, no. 4, pp. 1237–1242, 2005.

[9] F.J. Harris, "On the use of windows for harmonic analysis with the discrete Fourier transform," *Proceedings of the IEEE*, vol. 66, no. 1, pp. 51–83, 1978.

[10] K. Duda and S. Barczentewicz, "Interpolated DFT for $sin^{\alpha}(x)$ windows," *IEEE Transactions on Instrumentation and Measurement*, vol. 63, no. 4, pp. 754–760, 2014.

[11] E. Aboutanios, "Estimating the parameters of sinusoids and decaying sinusoids in noise," *IEEE Instrumentation and Measurement Magazine*, vol. 14, no. 2, pp. 8–14, 2011.

[12] S. Ye and E. Aboutanios, "An algorithm for the parameter estimation of multiple superimposed exponentials in noise," in *The 40th International Conference on Acoustic, Speech and Signal Processing (ICASSP)*, 2015.

[13] H. C. So and Y.T. Chan, "Short-time frequency estimation of a real sinusoid," in *IEICE Transactions on Fundamentals of Electronics, Communications and Computer Sciences*, 2005, vol. E88-A, No.9, pp. 2455–2459.

[14] S. M. Kay, "Fundamentals of statistical signal processing: Estimation theory," 1993.