ASYMPTOTIC ANALYSIS OF A GLRT FOR DETECTION WITH LARGE SENSOR ARRAYS

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ABSTRACT

This paper addresses the performance analysis of two GLRT receivers in the case where the number of sensors M is of the same order of magnitude as the sample size N. In the asymptotic regime where M and N converge towards ∞ at the same rate, the corresponding asymptotic means and variances are characterized using large random matrix theory, and compared to the standard situation where $N \to +\infty$ and M is fixed. This asymptotic analysis allows to understand the behavior of the considered receivers, even for relatively small values of N and M.

Index Terms— Multichannel detection, asymptotic analysis, large random matrices

1. INTRODUCTION

Due to the spectacular development of sensor networks, it is common to be faced with multivariate signals of high dimension. Very often, the sample size that can be used in practice in order to perform statistical inference cannot be much larger than the signal dimension. In this context, it is well established that a number of fundamental existing statistical signal processing methods fail. It is therefore of crucial importance to revisit certain classical problems in the high-dimensional signals setting. Previous related works include e.g. [1] and [2] in the source localization using subspace method context, or [3], [4] in the context of unsupervised detection.

In the present paper, we address the problem of detecting a known signal corrupted by a temporally white, but possibly spatially correlated, additive complex Gaussian noise using a large sensor array. This is an important question that has been extensively addressed in the past, in the low dimensional signal context ([5], [6], [7]). Although our results can be used in more general situations, we focus on the detection of a synchronization sequence transmitted by a single transmitter in a multipath propagation channel. Using random matrix methods, we study the behaviour of the classical generalized likelihood test (GLRT) when the number of sensors M and the number of samples of the training sequence N are large and of the same order of magnitude. It is shown that the asymptotic behaviour of the relevant statistics is quite different from the standard situation where $N \to +\infty$ while M remains fixed, and that the performance predicted by the regime where M and N are large provide results that are reliable for realistic values of M and N. The large random matrix technics used in this paper are more or less classical (see e.g. [8], [9]), but our results are of interest in the field of statistical signal processing.

2. PRESENTATION OF THE PROBLEM

In the following, we assume that a single transmitter sends a synchronization sequence $(s_n)_{n=1,\ldots,N}$ through a channel with L paths, and that the corresponding signal is received on a receiver with M sensors. The received M-dimensional signal is denoted by $(\mathbf{y}_n)_{n=1,\ldots,N}$. When the transmitter and the receiver are perfectly synchronized, $\mathbf{y}_n$ is assumed to be given for each $n = 1, \ldots, N$ by

$$\mathbf{y}_n = \sum_{l=0}^{L-1} \mathbf{h}_l s_{n-l} + \mathbf{v}_n \tag{1}$$

where $\mathbf{v}_n$ is an additive complex Gaussian noise. Denoting by $\mathbf{H}$ the $M \times L$ matrix $\mathbf{H} = (\mathbf{h}_0, \ldots, \mathbf{h}_{L-1})$, $\mathbf{Y} = (\mathbf{y}_1, \ldots, \mathbf{y}_N)$ can be written as

$$\mathbf{Y} = \mathbf{H}\mathbf{S} + \mathbf{V} \tag{2}$$

where $\mathbf{V} = (\mathbf{v}_1, \ldots, \mathbf{v}_N)$ and where $\mathbf{S} = \begin{pmatrix} s_1 & \ldots & s_N \\ & \ldots & \\ s_{2-L} & \ldots & s_{N-L+1} \end{pmatrix}$ represents the signal matrix. We assume from now on that the size N of the training sequence satisfies $N > M + L$. In order to simplify the calculations and notations, we assume that the signal matrix is orthogonal, meaning that $\frac{\mathbf{S}\mathbf{S}^*}{N} = \mathbf{I}_L$. In this paper, we study the classical problem of testing the hypothesis H_1 characterized by (2) against the hypothesis H_0 defined by

$$\mathbf{Y} = \mathbf{V} \tag{3}$$

When the additive noise is complex Gaussian, temporally white, but possibly spatially correlated with unknown covariance matrix, it is well established (see e.g. [5]) that the generalized maximum likelihood test consists in comparing the statistics $\eta_{c,N}$ defined by

$$\eta_{c,N} = -\log\det\left[\mathbf{I}_L - (\mathbf{S}\mathbf{Y}^*/N)(\mathbf{Y}\mathbf{Y}^*/N)^{-1}(\mathbf{Y}\mathbf{S}^*/N)\right] \tag{4}$$

to a threshold. If the noise is known to be also spatially white, the synchronization statistics, normalized by M, becomes

$$\eta_{w,N} = \mathrm{Tr}\left((\mathbf{S}\mathbf{Y}^*/N)(\mathbf{Y}\mathbf{S}^*/N)\right)\left[\frac{1}{M}\,\mathrm{Tr}\left(\mathbf{Y}\mathbf{Y}^*/N\right)\right]^{-1} \tag{5}$$

As the size of the training sequence should be as small as possible, it is of practical interest to evaluate the behaviour of the two statistics when M and N are of the same order of magnitude.

To simplify the analysis made with random matrix theory, we study the limit distribution of both statistics with the assumption that the noise $(\mathbf{v}_n)_{n=1,\ldots,N}$ is temporally and spatially white, i.e. $\mathbb{E}(\mathbf{v}_m\mathbf{v}_n^*) = \sigma^2\mathbf{I}_M\delta(n-m)$. We want to be clear with that this assumption is only made when calculating the limit distributions. The two GLRT statistics remain optimal for their respective noise type.

We also mention that the resulting limit distribution of the statistics under each hypothesis does not, in general, allow us to obtain a reliable approximation of the performance of the test for small false alarm probabilities. For this, it is recommended to use large deviations theory (see e.g. [10]), which, in the large dimensional context, is not well developed. However, our results may be used to predict the detection probability corresponding to false alarm probabilities of the order of 0.1. This point is not developed in this paper.

In the following, we denote by $\mathbf{W}$ a $(N-L\times N)$ matrix for which the matrix $\mathbf{\Theta}=(\mathbf{W}^T,\frac{\mathbf{S}^T}{\sqrt{N}})^T$ is unitary and define the $(M\times N-L)$ and the $(M\times L)$ matrices $\mathbf{V}_1$ and $\mathbf{V}_2$ by

$$(\mathbf{V}_1,\mathbf{V}_2)=\mathbf{V}\mathbf{\Theta}^*=\left(\mathbf{V}\mathbf{W}^*,\mathbf{V}\frac{\mathbf{S}^*}{\sqrt{N}}\right) \tag{6}$$

It is clear that $\mathbf{V}_1$ and $\mathbf{V}_2$ are complex Gaussian random matrices with independent identically distributed entries of variance σ^2 , and that the entries of $\mathbf{V}_1$ and $\mathbf{V}_2$ are mutually independent. Moreover,

$$\frac{\mathbf{V}\mathbf{V}^*}{N}=\frac{\mathbf{V}_1\mathbf{V}_1^*}{N}+\frac{\mathbf{V}_2\mathbf{V}_2^*}{N} \tag{7}$$

Note that, as $N>M+L$, then matrix $\frac{\mathbf{V}_1\mathbf{V}_1^*}{N}$ is invertible a.s.

3. STANDARD ASYMPTOTIC ANALYSIS

In order to get a better understanding of the similarities and differences with the more complicated case where M and N converge towards $+\infty$ at the same rate, to be discussed in section 4, we first recall some standard results concerning the asymptotic distribution of the statistics under H_0 and H_1 when $N\to+\infty$ but when M remains fixed (see e.g. [7] where similar calculations are presented).

3.1. Asymptotic behaviour under hypothesis H_0

Under hypothesis H_0, $\eta_{w,N}$ can be written as

$$\eta_{w,N}=\mathrm{Tr}\left(\mathbf{V}_2\mathbf{V}_2^*/N\right)/\mathrm{Tr}\left(\mathbf{V}\mathbf{V}^*/MN\right) \tag{8}$$

$\mathrm{Tr}(\mathbf{V}\mathbf{V}^*/MN)$ converges a.s. towards σ^2. Therefore,

$$\eta_{w,N}=\frac{\mathrm{Tr}\left(\mathbf{V}_2\mathbf{V}_2^*/N\right)}{\sigma^2+o_P(1)}=\frac{\mathrm{Tr}\left(\mathbf{V}_2\mathbf{V}_2^*/N\right)}{\sigma^2}+o_P(1/N) \tag{9}$$

and since $\mathbf{V}_2$ is a $M\times L$ matrix, it appears immediately that the limit distribution of $N\eta_{w,N}$ is a χ^2 distribution with $2ML$ degrees of freedom. This implies that $\mathrm{E}(\eta_{w,N})\simeq L\frac{M}{N}$ and $\mathrm{Var}(\eta_{w,N})\simeq\frac{L}{N}\frac{M}{N}$. Similarly, $\eta_{c,N}$ can be written as

$$\eta_{c,N}=-\log\det\left(\mathbf{I}-\left(\mathbf{V}_2^*/\sqrt{N}\right)\left(\mathbf{V}\mathbf{V}^*/N\right)^{-1}\left(\mathbf{V}_2/\sqrt{N}\right)\right) \tag{10}$$

When $N\to+\infty$ and M remains fixed, $\mathbf{V}\mathbf{V}^*/N$ converges towards $\sigma^2\mathbf{I}$ and $\frac{1}{N}\mathbf{V}_2^*\left(\mathbf{V}\mathbf{V}^*/N\right)^{-1}\mathbf{V}_2$ converges towards the zero matrix. Therefore, $\eta_{c,N}$ converges a.s. towards 0. In order to characterize the limit distribution of $\eta_{c,N}$, we use the classical linearization procedure

$$\eta_{c,N}=\frac{1}{\sigma^2}\mathrm{Tr}\left(\mathbf{V}_2^*\mathbf{V}_2/N\right)+o_P(1/N)$$

from which it is clear that $\eta_{c,N}$ has the same limit distribution as $\eta_{w,N}$.

3.2. Asymptotic behaviour under hypothesis H_1

When M is fixed while $N\to\infty$, the behaviour of $\eta_{w,N}$ under hypothesis H_1 is given by

Theorem 1 *$\eta_{w,N}$ converges almost surely towards $\overline{\eta}_{w,N}$ defined by*

$$\overline{\eta}_{w,N}=\frac{\mathrm{Tr}(\mathbf{H}\mathbf{H}^*)}{\frac{1}{M}\mathrm{Tr}(\mathbf{H}\mathbf{H}^*+\sigma^2\mathbf{I})}$$

Moreover, the asymptotic distribution of $\sqrt{N}(\eta_{w,N}-\overline{\eta}_{w,N})$ is a zero-mean Gaussian distribution with variance κ_1, where

$$\kappa_1=\frac{2\sigma^6\,\mathrm{Tr}(\mathbf{H}\mathbf{H}^*)+(\mathrm{Tr}(\mathbf{H}\mathbf{H}^*))^2\sigma^4/M}{\frac{1}{M^4}(\mathrm{Tr}(\mathbf{H}\mathbf{H}^*+\sigma^2\mathbf{I}))^4}$$

Under hypothesis H_1, $\eta_{w,N}$ is given by

$$\eta_{w,N}=\frac{\mathrm{Tr}\left[\left(\mathbf{V}_2/\sqrt{N}+\mathbf{H}\right)\left(\mathbf{V}_2/\sqrt{N}+\mathbf{H}\right)^*\right]}{\frac{1}{M}\mathrm{Tr}\left(\mathbf{Y}\mathbf{Y}^*/N\right)} \tag{11}$$

The denominator becomes

$$\begin{aligned}\frac{1}{M}\mathrm{Tr}\,\frac{\mathbf{Y}\mathbf{Y}^*}{N}=&\frac{1}{M}\mathrm{Tr}(\mathbf{H}\mathbf{H}^*+\sigma^2\mathbf{I})+\frac{1}{M}\mathrm{Tr}(\mathbf{H}\mathbf{V}_2^*)/\sqrt{N}+\\&+\frac{1}{M}\mathrm{Tr}(\mathbf{V}_2\mathbf{H}^*)/\sqrt{N}+\frac{1}{M}(\mathrm{Tr}\,\mathbf{V}_1\mathbf{V}_1^*/N-\sigma^2\mathbf{I})\\&+\frac{1}{M}\mathrm{Tr}(\mathbf{V}_2\mathbf{V}_2^*)/N\end{aligned}$$

When $N\to+\infty$ and M remains fixed, $\mathrm{Tr}(\mathbf{Y}\mathbf{Y}^*)/MN$ can be written as $(\mathrm{Tr}(\mathbf{H}\mathbf{H}^*+\sigma^2\mathbf{I}))/M+\xi$ where $\xi=\mathcal{O}_P(1/\sqrt{N})$. Note that $\mathrm{Tr}(\mathbf{V}_2\mathbf{V}_2^*)/N$ is $\mathcal{O}_P(1/N)$ and does not enter into the calculations in this asymptotic regime. Similarly, the numerator can be written as $\mathrm{Tr}(\mathbf{H}\mathbf{H}^*)+\beta$ where $\beta=\mathcal{O}_P(1/\sqrt{N})$. This gives us our expected value.

To calculate the variance, elementary manipulations show that

$$\begin{aligned}\eta_{w,N}=&\frac{\mathrm{Tr}(\mathbf{H}\mathbf{H}^*)}{\mathrm{Tr}(\mathbf{H}\mathbf{H}^*+\sigma^2\mathbf{I})/M}+\frac{1}{\mathrm{Tr}(\mathbf{H}\mathbf{H}^*+\sigma^2\mathbf{I})/M}\times\\&\times\left[\beta-\frac{\mathrm{Tr}(\mathbf{H}\mathbf{H}^*)}{\mathrm{Tr}(\mathbf{H}\mathbf{H}^*+\sigma^2\mathbf{I})/M}\xi\right]+\mathcal{O}_P(1/N)\end{aligned}$$

from which the variance is easily calculated to justify the Theorem.

We now specify the behaviour of $\eta_{c,N}$ in this asymptotic regime.

Theorem 2 *$\eta_{c,N}$ converges almost surely towards $\overline{\eta}_{c,N}$ defined by*

$$\overline{\eta}_{c,N}=\log\det\left(\mathbf{I}+\frac{\mathbf{H}^*\mathbf{H}}{\sigma^2}\right)$$

Moreover, the asymptotic distribution of $\sqrt{N}(\eta_{c,N}-\overline{\eta}_{c,N})$ is a zero-mean Gaussian distribution with variance κ_2, where

$$\kappa_2=\mathrm{Tr}\left[\mathbf{I}-\left(\mathbf{I}+\frac{\mathbf{H}^*\mathbf{H}}{\sigma^2}\right)^{-2}\right] \tag{12}$$

It is easily seen that

$$\eta_{c,N}=\log\det\left(\mathbf{I}+\frac{\mathbf{H}^*\mathbf{H}}{\sigma^2}\right)+o_P(1)$$

Using the linearization procedure, we obtain immediately that

$$\eta_{c,N}=\log\det\left(\mathbf{I}+\frac{\mathbf{H}^*\mathbf{H}}{\sigma^2}\right)+\mathrm{Tr}\left[\left(\mathbf{I}+\frac{\mathbf{H}^*\mathbf{H}}{\sigma^2}\right)\mathbf{\Delta}_N\right]+\mathcal{O}_P(1/N)$$

where matrix $\mathbf{\Delta}_N$ is given by

$$\mathbf{\Delta}_N = -\mathbf{H}^*(\mathbf{H}\mathbf{H}^* + \sigma^2\mathbf{I})^{-1}\mathbf{\Gamma}_N(\mathbf{H}\mathbf{H}^* + \sigma^2\mathbf{I})^{-1}\mathbf{H} + (\mathbf{V}_2^*/\sqrt{N})(\mathbf{H}\mathbf{H}^* + \sigma^2\mathbf{I})^{-1}\mathbf{H} + \mathbf{H}^*(\mathbf{H}\mathbf{H}^* + \sigma^2\mathbf{I})^{-1}(\mathbf{V}_2/\sqrt{N})$$

with $\mathbf{\Gamma}_N = \mathbf{Y}\mathbf{Y}^*/N - \left(\mathbf{H}\mathbf{H}^* + \sigma^2\mathbf{I}\right)$. The Theorem follows from the observation that $\|\mathbf{Y}\mathbf{Y}^*/N - (\mathbf{H}\mathbf{H}^* + \sigma^2\mathbf{I})\| \to 0$ a.s. when $N \to \infty$ and M is fixed, and from standard calculations.

4. ASYMPTOTIC ANALYSIS WHEN M AND N CONVERGE TOWARDS ∞ AT THE SAME RATE

In this section, we assume that M and N converge towards $+\infty$ in such a way that $c_N = \frac{M}{N}$ converges towards c, where $0 < c < 1$ while the number of paths L remains fixed. This hypothesis is of course consistent with the condition $N > M + L$. This asymptotic analysis differs deeply from the analysis in section 3. In particular, it is no longer true that the empirical covariance matrix $\mathbf{Y}\mathbf{Y}^*/N$ converges in the spectral norm sense towards its mathematical expectation. This, of course, is due to the fact that the number of entries of this $M \times M$ matrix is of the same order of magnitude than the MN available scalar observations. We also note that for any deterministic $M \times M$ matrix $\mathbf{A}$, the $L \times L$ matrix $\frac{1}{N}\mathbf{V}_2^*\mathbf{A}\mathbf{V}_2$ converges towards 0 when $N \to +\infty$ and M remains fixed, while this does not hold when M and N are of the same order of magnitude.

As M is growing, we have to specify how the power of the useful signal component $\mathbf{HS}$ is normalized. In the following, we assume that the norms of vectors $(\mathbf{h}_l)_{l=0,\ldots,L-1}$ remain bounded when the number of antennas M increases. This implies that the signal to noise ratio at the output of the matched filter $\mathbf{S}^*\mathbf{Y}/\sqrt{N}$, i.e. $\frac{1}{L}\mathrm{Tr}(\mathbf{H}^*\mathbf{H})/\sigma^2$, is a $\mathcal{O}(1)$ term in our asymptotic regime. We however mention that the received signal to noise ratio $\frac{1}{N}\mathrm{Tr}(\mathbf{H}\mathbf{S}\mathbf{S}^*\mathbf{H}^*/M\sigma^2)$ converges towards 0 at rate $\frac{1}{M}$ when M increases. Before studying the behaviour of the statistics, we first review some useful results [8],[11],[9].

4.1. Background material.

Proposition 1 *We consider an $(M \times N)$ random matrix $\mathbf{X}$ with variance σ^2 i.i.d. complex Gaussian entries. Then,*

$$\lambda_{min}\left(\mathbf{X}\mathbf{X}^*/N\right) \to \sigma^2(1-\sqrt{c})^2 \text{ a.s.} \tag{13}$$

Moreover, for each pair of unit norm deterministic M dimensional vectors $\mathbf{u}$ and $\mathbf{v}$ it holds that, for $i = 1, 2$,

$$\mathbf{u}^*\left(\mathbf{X}\mathbf{X}^*/N\right)^{-i}\mathbf{v} = \frac{\mathbf{u}^*\mathbf{v}}{\sigma^{2i}(1-c_N)^{2i-1}} + \mathcal{O}_P(\frac{1}{\sqrt{N}}) \tag{14}$$

$$\frac{1}{N}\mathrm{Tr}\left(\mathbf{X}\mathbf{X}^*/N\right)^{-i} = \frac{c_N}{\sigma^{2i}(1-c_N)^{2i-1}} + \mathcal{O}_P(\frac{1}{N}) \tag{15}$$

We note that if $c_N \to 0$, the conventional results obtained when M remains fixed are recovered. We finally mention the following result.

Proposition 2 *We consider a $M \times L$ random matrix $\mathbf{Z} = (\mathbf{z}_1, \ldots, \mathbf{z}_L)$ with variance σ^2 i.i.d. complex Gaussian entries. Then, for each random $M \times M$ matrix $\mathbf{A}$ independent from $\mathbf{Z}$ for which $\sup_M \|\mathbf{A}\| < +\infty$, it holds that*

$$\frac{\mathbf{z}_k^*\mathbf{A}\mathbf{z}_l}{N} - \delta_{k-l}\,\sigma^2\frac{1}{N}\mathrm{Tr}(\mathbf{A}) \to 0 \text{ a.s} \tag{16}$$

$$\mathbb{E}_{\mathbf{Z}}\left|\frac{\mathbf{z}_k^*\mathbf{A}\mathbf{z}_l}{N} - \delta_{k-l}\,\sigma^2\frac{1}{N}\mathrm{Tr}(\mathbf{A})\right|^2 = \frac{\sigma^4}{N}\frac{1}{N}\mathrm{Tr}(\mathbf{A}\mathbf{A}^*) \tag{17}$$

where $\mathbb{E}_{\mathbf{Z}}$ represents the mathematical expectation w.r.t. the entries of $\mathbf{Z}$.

4.2. Hypothesis H_0.

The behaviour of $\eta_{w,N}$ under H_0 is given by the following theorem

Theorem 3 *$\eta_{w,N} - Lc_N$ converges almost surely towards 0, and the asymptotic distribution of $\sqrt{N}\left(\eta_{w,N} - Lc_N\right)$ is a zero mean normal distribution with variance Lc_N.*

We take (8) as a starting point. $\mathrm{Tr}(\mathbf{V}_2\mathbf{V}_2^*/N)$ converges a.s. to σ^2ML/N whereas $\mathrm{Tr}(\mathbf{V}\mathbf{V}^*/NM)$ converges to σ^2, which gives us our expected value. When $M, N \to \infty$, $\mathrm{Tr}(\mathbf{V}\mathbf{V}^*/NM) - \sigma^2 = \mathcal{O}_P(1/\sqrt{MN}) = \mathcal{O}_P(1/N)$. Therefore,

$$\eta_{w,N} = \frac{\mathrm{Tr}(\mathbf{V}_2\mathbf{V}_2^*/N)}{\sigma^2} + \mathcal{O}_P(1/N)$$

which justifies the Theorem. We now specify the behaviour of $\eta_{c,N}$ in our asymptotic regime.

Theorem 4 *$\eta_{c,N}$ converges almost surely towards $\overline{\eta}_{c,N} = L\,\log\left(\frac{1}{1-c_N}\right)$, and the asymptotic distribution of $\sqrt{N}\left(\eta_{c,N} - \overline{\eta}_{c,N}\right)$ is a zero mean normal distribution with variance $\frac{Lc_N}{1-c_N}$.*

We use (7), (10) and the matrix inversion lemma, and obtain that

$$\eta_{c,N} = \log\det\left(\mathbf{I}_L + \mathbf{V}_2^*/\sqrt{N}\,\left(\mathbf{V}_1\mathbf{V}_1^*/N\right)^{-1}\,\mathbf{V}_2/\sqrt{N}\right)$$

In the following, we denote by $\mathbf{F}_N$ the $L \times L$ matrix defined by

$$\mathbf{F}_N = \mathbf{V}_2^*/\sqrt{N}\,\left(\mathbf{V}_1\mathbf{V}_1^*/N\right)^{-1}\,\mathbf{V}_2/\sqrt{N} \tag{18}$$

We provide only a sketch of proof. We first evaluate the almost sure behaviour of $\eta_{c,N}$. We use Proposition (1) and Proposition (2) for $\mathbf{Z} = \mathbf{V}_2/\sqrt{N}$ and $\mathbf{A} = (\mathbf{V}_1\mathbf{V}_1^*/N)^{-1}$. By (13), matrix $\mathbf{A}$ satisfies $\sup_M \|\mathbf{A}\| < +\infty$ almost surely for M, N large enough, and we deduce immediately that $L \times L$ matrix $\mathbf{F}_N$ converges almost surely towards the matrix $\frac{c_N}{1-c_N}\mathbf{I}_L$. We have used that, for a fixed L, the ratio $\frac{M}{N-L}$ behaves as c_N. This justifies the first statement of the Theorem.

We do not prove the asymptotic Gaussianity of the $\eta_{c,N}$, but provide only an informal justification of the expression for the asymptotic variance. Using a standard first order expansion of $\eta_{c,N}$,

$$\eta_{c,N} = L\log\frac{1}{1-c_N} + \mathrm{Tr}\left((1-c_N)\mathbf{\Delta}_N\right) + \mathcal{O}_P(1/N) \tag{19}$$

where $\mathbf{\Delta}_N$ represents the difference between matrix $\mathbf{F}_N$ defined by (18) and matrix $\frac{c_N}{1-c_N}\mathbf{I}_L$. $\mathrm{Tr}\left((1-c_N)\mathbf{\Delta}_N\right)$ can be written as $\mathrm{Tr}\left((1-c_N)\mathbf{\Delta}_N\right) = (1-c_N)\sum_{l=1}^{L}\epsilon_l$ where

$$\epsilon_l = \mathbf{V}_{2,l}^*/\sqrt{N}\,\left(\mathbf{V}_1\mathbf{V}_1^*/N\right)^{-1}\,\mathbf{V}_{2,l}/\sqrt{N} - \frac{c_N}{1-c_N}$$

and $\mathbf{V}_{2,l}$ represents column l of matrix $\mathbf{V}_2$.

By Proposition 1, ϵ_l can be written as

$$\epsilon_l = \mathbf{V}_{2,l}^*/\sqrt{N}\,\left(\mathbf{V}_1\mathbf{V}_1^*/N\right)^{-1}\,\mathbf{V}_{2,l}/\sqrt{N} - \frac{\sigma^2}{N}\mathrm{Tr}\left(\mathbf{V}_1\mathbf{V}_1^*/N\right)^{-1} + \mathcal{O}_P(1/N)$$

By (17), it holds that the expectation of $|\epsilon_l|^2$ w.r.t. $\mathbf{V}_{2,l}$ is given by

$$\mathbb{E}_{\mathbf{V}_{2,l}}|\epsilon_l|^2 = \frac{\sigma^4}{N}\frac{1}{N}\mathrm{Tr}\left(\mathbf{V}_1\mathbf{V}_1^*/N\right)^{-2} + o_P(1/N)$$

Taking the expectation w.r.t. $\mathbf{V}_1$ and using (15), we obtain that the variance of ϵ_l is equal to $\frac{c_N}{(1-c_N)^3}$. It is clear that the random variables $(\epsilon_l)_{l=1,\ldots,L}$ are decorrelated. Therefore, the asymptotic variance of $(1-c_N)\sum_{l=1}^{L}\epsilon_l$ coincides with $\frac{Lc_N}{(1-c_N)}$ as expected.

We recall that if M is fixed, $N\eta_{w,N}$ and $N\eta_{c,N}$ both behave like a χ^2 distribution with $2ML$ degrees of freedom. Therefore, the behaviour of $\eta_{c,N}$ in the two asymptotic regimes deeply differ. However, if $c_N \to 0$, $-\log(1-c_N) \simeq c_N$, and the asymptotic means and variances of $\eta_{c,N}$ tend to coincide. For $\eta_{w,N}$, the asymptotic expected value and variance in the two regimes are the same, but when $M \to \infty$, the asymptotic distribution is a normal distribution instead of a χ^2 distribution.

4.3. Asymptotic behaviour under hypothesis H_1.

The limit distribution of the statistics $\eta_{w,N}$ under hypothesis H_1, defined by (11), is given by the following result.

Theorem 5 *$\eta_{w,N}$ converges almost surely towards $\overline{\eta}_{w,N}$ defined by*

$$\overline{\eta}_{w,N} = \frac{\mathrm{Tr}(\mathbf{H}\mathbf{H}^*)}{\frac{1}{M}\mathrm{Tr}(\mathbf{H}\mathbf{H}^* + \sigma^2\mathbf{I})} + Lc_N$$

Moreover, the asymptotic distribution of $\sqrt{N}(\eta_{w,N} - \overline{\eta}_{w,N})$ is a zero-mean Gaussian distribution with variance κ_3, where

$$\begin{aligned}\kappa_3 &= \frac{2\sigma^2\,\mathrm{Tr}\,\mathbf{H}\mathbf{H}^* + \sigma^4 Lc_N}{\frac{1}{M^2}[\mathrm{Tr}(\mathbf{H}\mathbf{H}^* + \sigma^2\mathbf{I})]^2} \\ &= \kappa_1 + Lc_N + \mathcal{O}_P(1/N) \end{aligned} \quad (20)$$

Since M is of the same order of magnitude as N, the term $\mathrm{Tr}(\mathbf{V}_2\mathbf{V}_2^*/N)$ in the numerator of (11) is no longer negligeable. In the denominator, $\mathrm{Tr}(\mathbf{V}_1\mathbf{V}_1^* + \mathbf{V}_2\mathbf{V}_2^*)/MN$ still converges towards σ^2, but $\mathrm{Tr}(\mathbf{H}\mathbf{H}^*/M)$ is now an $\mathcal{O}_P(1/N)$ term. For sake of comparison, however, we will keep the $\mathrm{Tr}(\mathbf{H}\mathbf{H}^*/M)$ term in the expression of the expected value and variance. This justifies the first statement of the Theorem.

To calculate the variance, we note that the fluctuations of the denominator of (11) are faster than those of the numerator. In the numerator, $\mathrm{Var}(\mathrm{Tr}(\mathbf{V}_2\mathbf{H}^*/\sqrt{N})) = \mathrm{Var}(\mathrm{Tr}(\mathbf{H}\mathbf{V}_2^*/\sqrt{N})), = \frac{\sigma^2}{N}\mathrm{Tr}(\mathbf{H}\mathbf{H}^*)$, and $\mathrm{Var}(\mathrm{Tr}(\mathbf{V}_2\mathbf{V}_2^*/N)) = \frac{\sigma^2 ML}{N^2}$. These are all $\mathcal{O}_P(1/N)$ terms, whereas the corresponding terms, scaled by M, in the denominator, are $\mathcal{O}_P(1/N^2)$ terms. The asymptotic variance can thus be expressed as

$$\frac{\mathrm{Var}(\mathrm{Tr}(\mathbf{H}\mathbf{V}_2^*/\sqrt{N} + \mathbf{V}_2\mathbf{H}^*/\sqrt{N} + \mathbf{V}_2\mathbf{V}_2^*/N))}{\frac{1}{M^2}[\mathrm{Tr}(\mathbf{H}\mathbf{H}^* + \sigma^2\mathbf{I})]^2}$$

Since the three terms in the above expression are decorrelated, the variance of $\eta_{w,N}$ equals the sum of the variances of the three terms. Elementary calculations lead to (20). The behaviour of $\eta_{c,N}$ under hypothesis H_1 is given by the following result.

Theorem 6 *$\eta_{c,N}$ converges almost surely towards $\overline{\eta}_{N,1}$ defined by*

$$\overline{\eta}_{c,N} = L\log\frac{1}{1-c_N} + \log\det\left(\mathbf{I} + \mathbf{H}^*\mathbf{H}/\sigma^2\right)$$

Moreover, the asymptotic distribution of $\sqrt{N}(\eta_{c,N} - \overline{\eta}_{c,N})$ is a zero-mean Gaussian distribution with variance $\frac{Lc_N}{1-c_N} + \kappa_2$, where κ_2 is defined by (12).

We again just give a sketch of proof. Defining the $L \times L$ matrix

$$\mathbf{G}_N = \left(\mathbf{H} + \mathbf{V}_2/\sqrt{N}\right)^* (\mathbf{V}_1\mathbf{V}_1^*/N)^{-1}\left(\mathbf{H} + \mathbf{V}_2/\sqrt{N}\right) \quad (21)$$

we obtain after some standard algebra that, under hypothesis H_1, $\eta_{c,N}$ can be written as $\eta_{c,N} = \log\det(\mathbf{I}_L + \mathbf{G}_N)$ In order to evaluate its almost sure behaviour, we expand $\mathbf{G}_N$. Using (14), it is easy to check that the matrix $\mathbf{H}^*(\mathbf{V}_1\mathbf{V}_1^*/N)^{-1}\mathbf{H}$ converges towards $\frac{1}{\sigma^2(1-c_N)}\mathbf{H}^*\mathbf{H}$ and that the matrix $(\mathbf{V}_2/\sqrt{N})^*(\mathbf{V}_1\mathbf{V}_1^*/N)^{-1}\mathbf{H}$ converges towards the zero matrix. As we already showed that matrix $\mathbf{F}_N$ converges towards $\frac{c_N}{1-c_N}\mathbf{I}_L$, it holds that $\mathbf{I}_L + \mathbf{G}_N$ behaves as $\frac{1}{1-c_N}(\mathbf{I}_L + \mathbf{H}^*\mathbf{H}/\sigma^2)$ as expected.

In order to prove the asymptotic Gaussianity of $\eta_{c,N}$, we again use the linearization trick, and express $\eta_{c,N} - \overline{\eta}_{c,N}$ as

$$\eta_{c,N} - \overline{\eta}_{c,N} = \mathrm{Tr}\left((1-c_N)(\mathbf{I}_L + \mathbf{H}^*\mathbf{H}/\sigma^2)^{-1}\mathbf{\Delta}_N\right) + \mathcal{O}_P(1/N)$$

where matrix $\mathbf{\Delta}_N$ is defined by

$$\mathbf{\Delta}_N = \mathbf{G}_N - \left(\frac{c_N}{1-c_N}\mathbf{I} + \frac{\mathbf{H}^*\mathbf{H}}{\sigma^2(1-c_N)}\right)$$

$\eta_{c_N} - \overline{\eta}_{c,N}$ can be expressed as the sum of four terms that are asymptotically decorrelated.

The asymptotic Gaussianity and the evaluation of the variance of three of these terms can be addressed as before using Propositions 1 and 2. The proof of the asymptotic Gaussianity of the contribution of the term $\mathbf{H}^*(\mathbf{V}_1\mathbf{V}_1^*/N)^{-1}\mathbf{H}$ to $\eta_{c_N} - \overline{\eta}_{c,N}$ needs to establish a central limit theorem for $\sum_{l=1}^{L}\mathbf{a}_l^*(\mathbf{V}_1\mathbf{V}_1^*/N)^{-1}\mathbf{b}_l$ where $(\mathbf{a}_l, \mathbf{b}_l)_{l=1,\ldots,L}$ are bounded deterministic vectors. For this, we use an approach developed by Pastur and his colleagues ([12] and [9], see also [13] where the approach is used to study the mutual information of large MIMO systems).

Interestingly, it is seen that the asymptotic variances of both $\eta_{w,N}$ and $\eta_{c,N}$ coincide with the sum of their asymptotic variance in the standard regime $N \to +\infty$ and M fixed, and an extra term which coincides with their asymptotic variance under H_0.

5. NUMERICAL RESULTS

In this section, we validate the relevance of the Gaussian approximation of section 4 and compare it with the asymptotic distributions obtained by the standard asymptotic analysis of section 3.

In our numerical experiments, we have calculated the theoretical expected values and variances as well as their empirical values, evaluated by Monte Carlo simulations with 30.000 trials. In the experiments, L has been fixed to 5, M to 40, while N is increased, starting from $N = 80$. Moreover, to avoid variance problems due to channel variations, the channel matrix $\mathbf{H}$ is fixed through all trials.

Figures 1 and 2 compare the theoretical variances with the empirical variance for $\eta_{w,N}$ and $\eta_{c,N}$ respectively. When c_N is small, the two theoretical variances coincide, as expected, and are very close to the true variance. When c_N is large, the assumption that M is small compared to N is no longer valid, and the classical asymptotic analysis fails. It is interesting to note that even for small values of M, N, the theoretical variance obtained by considering M, N the same order of magnitude is close to the empirical variance.

If N, M are increased while keeping c_N constant, the empirical results are even closer to the theoretical values, since the number of samples is larger.

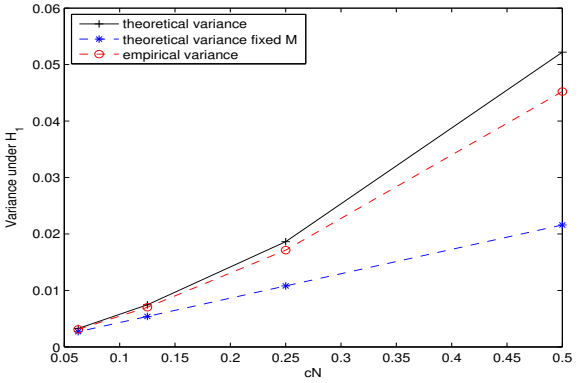

Fig. 1. Variance of $\eta_{w,N}$ under hypothesis H_1

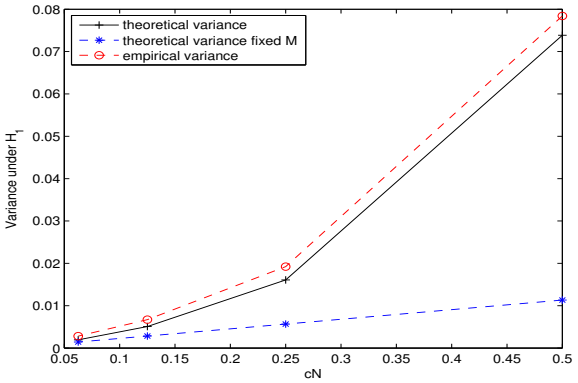

Fig. 2. Variance of $\eta_{c,N}$ under hypothesis H_1

Note that despite the white noise assumption for calculating the limit distribution, detailed in section 2, we are able to characterize the limit distribution of both statistics in a satisfactory way.

Due to lack of space, we omit the results for the expected values and the variance under H_0. We mention however that for both statistics, the theoretical expected values and variances calculated with the assumption that M, N are of the same order of magnitude, are very close to their empirical counterparts.

To validate the asymptotic Gaussianity of the synchronisation statistics, Figure 3 contains a histogram of the empirical values of $\eta_{c,N}$ under H_1, along with a normal distribution of the theoretical expected value and variance, for the assumption $N, M \to \infty$ at the same rate. Similar results also apply for the other statistics, with the same assumption, and for hypothesis H_0.

6. CONCLUSION

In this paper, the asymptotic distributions of two generalized likelihood ratios are studied. The corresponding asymptotic expected values and variances are characterized when the number of samples in the training sequence N increases at the same rate as the number of sensors M. It is seen that the asymptotic means and variances are close to their empirical counterparts for realistic values of M and N.

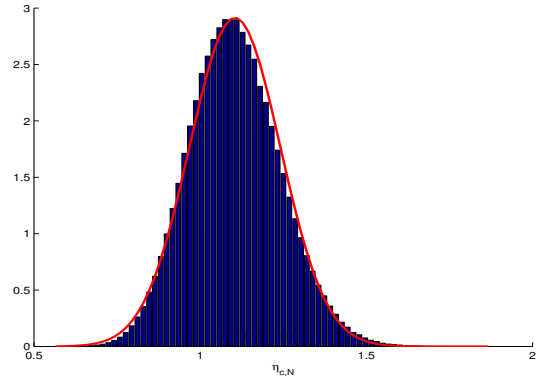

Fig. 3. Empirical $\eta_{c,N}$ superposed with a Gaussian distribution with its theoretical expected value and variance. N=80, M=10, L=5.

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