ON THE STEADY-STATE AND TRACKING ANALYSIS OF THE COMPLEX SRLMS ALGORITHM

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ABSTRACT

In this paper, the steady-state and tracking behavior of the complex signed regressor least mean square (SRLMS) algorithm are analyzed in stationary and nonstationary environments, respectively. Here, the SRLMS algorithm is analyzed in the presence of complex-valued white and correlated Gaussian input data. Moreover, a comparison between the convergence performance of the complex SRLMS algorithm and the complex least mean square (LMS) algorithm is also presented. Finally, simulation results are presented to support our analytical findings.

***Index Terms*—** LMS, SRLMS, Steady-state, Tracking.

1. INTRODUCTION

Computational complexity reduction of the adaptive noise cancelation system, particularly, in applications such as wireless biotelemetry system is very important [1]. The signed regressor least mean square (SRLMS) algorithm is known to have a reduced computational complexity compared to that of the traditional least mean square (LMS) algorithm [2]. Therefore, adaptive filters equipped with the signed versions of the LMS algorithm (such as the SRLMS algorithm) are extensively used for the processing and analysis of electrocardiogram (ECG) signals [1].

The SRLMS algorithm is obtained from the conventional LMS algorithm by replacing the regressor vector by its sign. The SRLMS algorithm is also referred to as simply the signed regressor algorithm (SRA) [2]–[3]. Theoretical studies of the SRLMS algorithm can be found in [2]–[5]. To the best of the authors knowledge, the steady-state and tracking analysis of the SRLMS algorithm for the case of complex-valued data are not available in the literature of adaptive filtering. Therefore, this work reports the findings of the steady-state and tracking analysis of the SRLMS algorithm for the case of complex-valued data.

The organization of the paper is as follows. The complex SRLMS algorithm is described briefly in Section 2. The steady-state and tracking analysis of the complex SRLMS algorithm are derived in Sections 3 and 4, respectively. Finally, simulation results and some concluding remarks are presented in Sections 5 and 6, respectively.

2. THE COMPLEX SRLMS ALGORITHM

The weight update recursion for the complex SRLMS algorithm is governed by

$$\mathbf{w}_i = \mathbf{w}_{i-1} + \mu \,\mathrm{csgn}[\mathbf{u}_i]^* e_i, \qquad i \geq 0, \tag{1}$$

where $\mathbf{w}_i$ is the updated weight vector, μ is the step-size, $\mathbf{u}_i$ is the regressor vector, and e_i denotes the estimation error given by

$$e_i = d_i - \mathbf{u}_i \mathbf{w}_{i-1}, \tag{2}$$

where d_i is the desired value.

3. STEADY-STATE ANALYSIS

We shall assume that the data $\{d_i, \mathbf{u}_i\}$ satisfy the following assumptions of the stationary data model [6]:

A.1 There exists an optimal weight vector $\mathbf{w}^o$ such that

$$d_i = \mathbf{u}_i \mathbf{w}^o + v_i. \tag{3}$$

A.2 The additive noise sequence v_i is independent and identically distributed (i.i.d.) circular with variance $\sigma_v^2 = \mathrm{E}[|v_i|^2]$ and is independent of $\mathbf{u}_j$ for all i, j.

A.3 The initial condition $\mathbf{w}_{-1}$ is independent of the zero mean random variables $\{d_i, \mathbf{u}_i, v_i\}$.

A.4 The regressor covariance matrix is $\mathbf{R} = \mathrm{E}\left[\mathbf{u}_i^* \mathbf{u}_i\right] > \mathbf{0}$.

For the adaptive filter of the form in (1), and for any data $\{d_i, \mathbf{u}_i\}$, assuming filter operation in steady-state, the following variance relation holds [6]:

$$\mu \mathrm{E}\left[||\mathbf{u}_i||_{\mathrm{H}}^2 |\mathrm{g}[e_i]|^2\right] = 2\mathrm{Re}\left[\mathrm{E}\left[e_{a_i}^* \mathrm{g}[e_i]\right]\right], \text{ as } i \to \infty, \tag{4}$$

where

$$\begin{aligned} \mathrm{E}[||\mathbf{u}_i||_{\mathrm{H}}^2] &= \mathrm{E}[\mathrm{Re}[\mathbf{u}_i \mathrm{H}[\mathbf{u}_i]\mathbf{u}_i^*]], & (5)\\ e_i &= e_{a_i} + v_i, & (6) \end{aligned}$$

with $\mathrm{H}[\mathbf{u}_i]$ denoting some positive-definite Hermitian matrix-valued function of $\mathbf{u}_i$, $e_{a_i} = \mathbf{u}_i(\mathbf{w}^o - \mathbf{w}_{i-1})$ is the a priori estimation error, $\mathrm{g}[e_i]$ denotes some function of e_i and for the complex SRLMS algorithm $\mathrm{g}[e_i] = e_i$. Then, by using the fact that e_{a_i} is independent of v_i, we reach at the following expression for the term $\mathrm{E}\left[e_{a_i}^* \mathrm{g}[e_i]\right]$:

$$\mathrm{E}\left[e_{a_i}^* \mathrm{g}[e_i]\right] = \mathrm{E}[|e_{a_i}|^2]. \tag{7}$$

To evaluate the term $\mathrm{E}\left[||\mathbf{u}_i||_{\mathrm{H}}^2 |\mathrm{g}[e_i]|^2\right]$, we start by noting that

$$|\mathrm{g}[e_i]|^2 = |e_{a_i}|^2 + |v_i|^2 + e_{a_i}^* v_i + e_{a_i} v_i^*. \tag{8}$$

Now, if we multiply (8) by $||\mathbf{u}_i||_{\mathrm{H}}^2$ from the left, then take the expected value of the resulting equation and use the fact that v_i is independent of both $\mathbf{u}_i$ and e_{a_i}, we obtain

$$\mathrm{E}\left[||\mathbf{u}_i||_{\mathrm{H}}^2 |\mathrm{g}[e_i]|^2\right] = \mathrm{E}[||\mathbf{u}_i||_{\mathrm{H}}^2 |e_{a_i}|^2] + \sigma_v^2 \mathrm{E}[||\mathbf{u}_i||_{\mathrm{H}}^2]. \tag{9}$$

In [7], we have shown that

$$\mathrm{E}[||\mathbf{u}_i||_{\mathrm{H}}^2] = \frac{4\mathrm{Tr}(\mathbf{R})}{\sqrt{\pi\sigma_u^2}}. \tag{10}$$

Substituting (10) into (9) we get

$$\mathrm{E}\left[||\mathbf{u}_i||_{\mathrm{H}}^2 |\mathrm{g}[e_i]|^2\right] = \mathrm{E}[||\mathbf{u}_i||_{\mathrm{H}}^2 |e_{a_i}|^2] + \frac{4\sigma_v^2 \mathrm{Tr}(\mathbf{R})}{\sqrt{\pi\sigma_u^2}}. \tag{11}$$

Substituting (7) and (11) into (4) we get

$$\mu \mathrm{E}[||\mathbf{u}_i||_{\mathrm{H}}^2 |e_{a_i}|^2] + \frac{4\mu\sigma_v^2 \mathrm{Tr}(\mathbf{R})}{\sqrt{\pi\sigma_u^2}} = 2\mathrm{E}[|e_{a_i}|^2]. \tag{12}$$

In order to simplify (12), we use the separation principle, namely, that at steady-state, $||\mathbf{u}_i||_{\mathrm{H}}^2$ is independent of $e_{a_i}^2$. Therefore, we obtain

$$\frac{4\mu \mathrm{Tr}(\mathbf{R}) \mathrm{E}[|e_{a_i}|^2]}{\sqrt{\pi\sigma_u^2}} + \frac{4\mu\sigma_v^2 \mathrm{Tr}(\mathbf{R})}{\sqrt{\pi\sigma_u^2}} = 2\mathrm{E}[|e_{a_i}|^2]. \tag{13}$$

This leads to the expression for the steady-state excess-mean-square error (EMSE), $\zeta = \mathrm{E}[|e_{a_i}|^2]$, of the complex SRLMS algorithm which is given by

$$\zeta = \frac{4\mu\sigma_v^2 \mathrm{Tr}(\mathbf{R})}{2\sqrt{\pi\sigma_u^2} - 4\mu \mathrm{Tr}(\mathbf{R})}. \tag{14}$$

Ultimately, the steady-state mean-square error (MSE), $\varphi = \mathrm{E}\left[|e_i|^2\right]$, of the complex SRLMS algorithm is given by

$$\varphi = \frac{4\mu\sigma_v^2 \mathrm{Tr}(\mathbf{R})}{2\sqrt{\pi\sigma_u^2} - 4\mu \mathrm{Tr}(\mathbf{R})} + \sigma_v^2. \tag{15}$$

4. TRACKING ANALYSIS

Here, let us assume that the data $\{d_i, \mathbf{u}_i\}$ satisfy the following assumptions of the nonstationary data model [6]:

A.5 There exists an optimal weight vector $\mathbf{w}_i^o$ such that

$$d_i = \mathbf{u}_i \mathbf{w}_i^o + v_i. \tag{16}$$

A.6 The weight vector varies according to the random-walk model

$$\mathbf{w}_i^o = \mathbf{w}_{i-1}^o + \mathbf{q}_i, \tag{17}$$

where the Gaussian noise sequence $\mathbf{q}_i$ is i.i.d. with variance σ_q^2 and covariance matrix $\mathbf{Q}$. Moreover, $\mathbf{q}_i$ is independent of $\{v_j, \mathbf{u}_j\}$ for all i, j.

A.7 The initial conditions $\{\mathbf{w}_{-1}, \mathbf{w}_{-1}^o\}$ are independent of the zero mean random variables $\{d_i, \mathbf{u}_i, v_i, \mathbf{q}_i\}$.

In this case, the following variance relation holds [6]:

$$\mu \mathrm{E}\left[||\mathbf{u}_i||_{\mathrm{H}}^2 |\mathrm{g}[e_i]|^2\right] + \mu^{-1}\mathrm{Tr}(\mathbf{Q}) = 2\mathrm{Re}\left[\mathrm{E}\left[e_{a_i}^* \mathrm{g}[e_i]\right]\right], \quad \text{as } i \to \infty. \tag{18}$$

Therefore, by substituting (7) and (11) into (18), the tracking EMSE of the complex SRLMS algorithm can be shown to be

$$\zeta = \frac{4\mu\sigma_v^2 \mathrm{Tr}(\mathbf{R}) + \sqrt{\pi\sigma_u^2}\, \mu^{-1}\mathrm{Tr}(\mathbf{Q})}{2\sqrt{\pi\sigma_u^2} - 4\mu \mathrm{Tr}(\mathbf{R})}. \tag{19}$$

The optimum value of the step-size of the complex SRLMS algorithm can be obtained by minimizing (19) with respect to μ and is given by

$$\mu_{\mathrm{opt}} = \frac{1}{2}\sqrt{\frac{\mathrm{Tr}(\mathbf{Q})}{\sigma_v^2}\left[\frac{\mathrm{Tr}(\mathbf{Q})}{\sigma_v^2} + \frac{\sqrt{\pi\sigma_u^2}}{\mathrm{Tr}(\mathbf{R})}\right]} - \frac{\mathrm{Tr}(\mathbf{Q})}{2\sigma_v^2}. \tag{20}$$

Ultimately, the corresponding minimum value of the tracking MSE of the complex SRLMS algorithm is given by

$$\varphi_{\min} = \frac{4\mu_{\mathrm{opt}}\sigma_v^2 \mathrm{Tr}(\mathbf{R}) + \sqrt{\pi\sigma_u^2}\, \mu_{\mathrm{opt}}^{-1}\mathrm{Tr}(\mathbf{Q})}{2\sqrt{\pi\sigma_u^2} - 4\mu_{\mathrm{opt}} \mathrm{Tr}(\mathbf{R})} + \sigma_v^2. \tag{21}$$

Finally, Table 1 and Table 2 report the expressions for the steady-state EMSE and the tracking EMSE of the complex SRLMS and LMS algorithms, respectively.

Table 1. Performance comparison of the steady-state EMSE for the LMS and the complex SRLMS algorithms.

Algorithm	**Steady-state EMSE**
LMS	$\frac{\mu\sigma_v^2 \mathrm{Tr}(\mathbf{R})}{2-\mu \mathrm{Tr}(\mathbf{R})}$
Complex SRLMS	$\frac{4\mu\sigma_v^2 \mathrm{Tr}(\mathbf{R})}{2\sqrt{\pi\sigma_u^2}-4\mu \mathrm{Tr}(\mathbf{R})}$

5. SIMULATION RESULTS

Several simulations are carried out in order to corroborate our theoretical findings. In all the simulations, the filter length is fixed at $M = 5$. In Fig. 2 and Fig. 4, the correlated data is obtained by passing a unit-variance i.i.d. Gaussian data through a first-order auto-regressive model with transfer function $\frac{\sqrt{1-a^2}}{(1-az^{-1})}$ and $a = 0.8$. In Figures 3–4, we have chosen $\sigma_q^2 = 10^{-8}$. Additive white Gaussian noise (AWGN)

Table 2. Performance comparison of the tracking EMSE for the LMS and the complex SRLMS algorithms.

Algorithm	**Tracking EMSE**
LMS	$\frac{\mu\sigma_v^2\mathrm{Tr}(\mathbf{R})+\mu^{-1}\mathrm{Tr}(\mathbf{Q})}{2-\mu\mathrm{Tr}(\mathbf{R})}$
Complex SRLMS	$\frac{4\mu\sigma_v^2\mathrm{Tr}(\mathbf{R})+\sqrt{\pi\sigma_u^2}\,\mu^{-1}\mathrm{Tr}(\mathbf{Q})}{2\sqrt{\pi\sigma_u^2}-4\mu\mathrm{Tr}(\mathbf{R})}$

environment is considered in Figures 1–6, while in Figures 7–8 uniform noise environment is considered. The signal-to-noise ratio (SNR) is fixed at 30 dB in Figures 1–4 and 10 dB in Figures 5–8.

First, the steady-state MSE of the complex SRLMS algorithm using white and correlated Gaussian regressors is shown in Figures 1–2, respectively. As can be seen from these figures, the simulation results are in a very good match with the theoretical result in equation (15) for values of μ ranging from 0.0001 to 0.01.

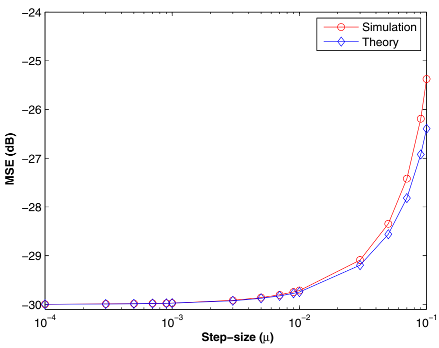

Fig. 1. Theoretical and simulated steady-state MSE of the SRLMS algorithm using white Gaussian regressors.

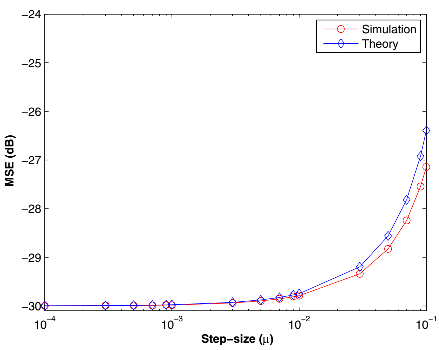

Fig. 2. Theoretical and simulated steady-state MSE of the SRLMS algorithm using correlated Gaussian regressors.

Second, Figures 3–4 demonstrate the tracking performance of the complex SRLMS algorithm using white and correlated Gaussian regressors, respectively. A zoom into the region around $\mu = 0.002$ shows that the tracking MSE possesses a minimum value of -29.828394 in Fig. 3 and -29.871226 in Fig. 4 at $\mu = 0.003$, which are in very good agreement with the corresponding theoretical values of $\varphi_{\min} = -29.896845$ and $\mu_{\text{opt}} = 0.00208$ obtained from expressions (21) and (20), respectively.

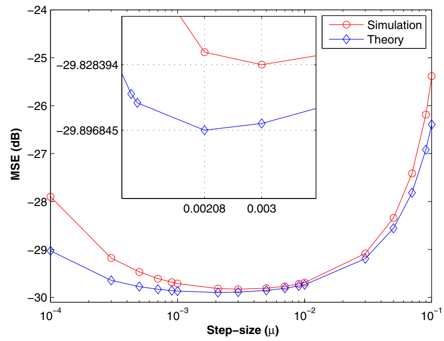

Fig. 3. Theoretical and simulated tracking MSE of the SRLMS algorithm using white Gaussian regressors.

Finally, the convergence behavior of the complex SRLMS algorithm is compared to that of the complex LMS algorithm in an unknown system identification setup. Figures 5 and 7 show the convergence performance of both the algorithms using white Gaussian regressors, while Fig. 6 and Fig. 8 show the convergence comparison using correlated Gaussian regressors. As observed from these figures, the complex LMS algorithm results in superior performance over the complex SRLMS algorithm for the same misadjustment.

6. CONCLUSIONS

In this work, analytical expressions are derived for the steady-state MSE, optimal step-size, and the corresponding optimal tracking MSE of the SRLMS algorithm for complex-valued data case. We observed that the theoretical values of the op-

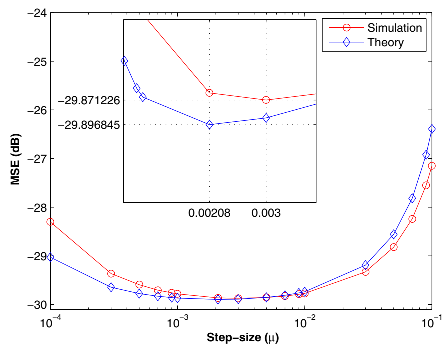

Fig. 4. Theoretical and simulated tracking MSE of the SRLMS algorithm using correlated Gaussian regressors.

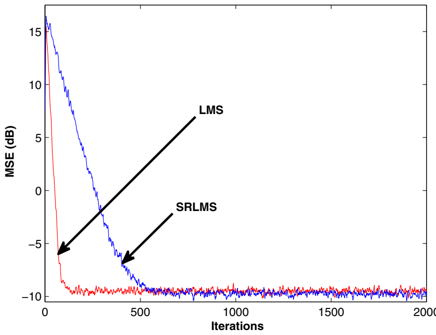

Fig. 5. Convergence comparison of the LMS and the SRLMS algorithms using white Gaussian regressors in an AWGN environment.

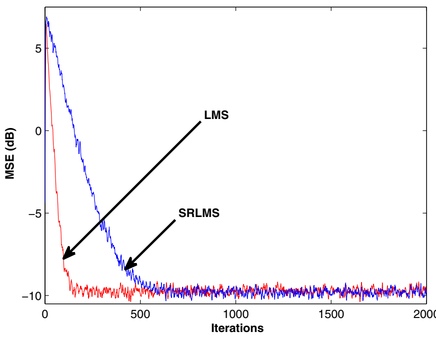

Fig. 6. Convergence comparison of the LMS and the SRLMS algorithms using correlated Gaussian regressors in an AWGN environment.

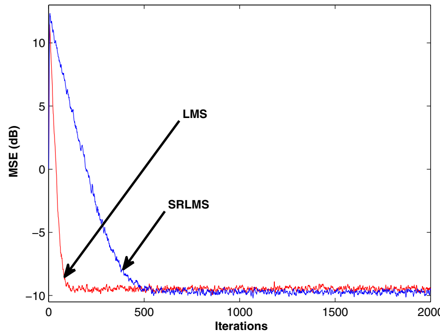

Fig. 7. Convergence comparison of the LMS and the SRLMS algorithms using white Gaussian regressors in a uniform noise environment.

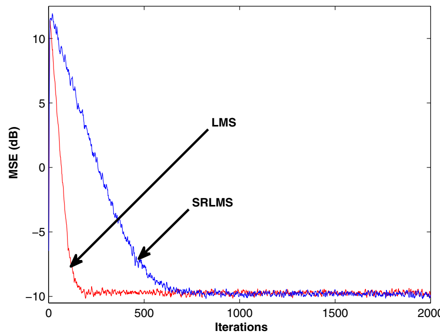

Fig. 8. Convergence comparison of the LMS and the SRLMS algorithms using correlated Gaussian regressors in a uniform noise environment.

timal step-size and the resulting minimum MSE of the complex SRLMS algorithm are similar for white and correlated Gaussian data. Furthermore, we also observed that the theoretical and simulation values of the optimal MSE of the complex SRLMS algorithm are in much closer agreement for correlated Gaussian data than white Gaussian data. Finally, as expected, the complex SRLMS algorithm has been shown to exhibit slower convergence rate than the complex LMS algorithm for the same misadjustment.

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