A UNIFYING APPROACH TO MINIMAL PROBLEMS IN COLLINEAR AND PLANAR TDOA SENSOR NETWORK SELF-CALIBRATION

Erik Ask, Yubin Kuang, Kalle Åström

Centre for Mathematical Sciences, Lund University
{erikask,yubin,kalle}@maths.lth.se

ABSTRACT

This work presents a study of sensor network calibration from time-difference-of-arrival (TDOA) measurements for cases when the dimensions spanned by the receivers and the transmitters differ. This could for example be if receivers are restricted to a line or plane or if the transmitting objects are moving linearly in space. Such calibration arises in several applications such as calibration of (acoustic or ultrasound) microphone arrays, and radio antenna networks. We propose a non-iterative algorithm based on recent stratified approaches: (i) rank constraints on modified measurement matrix, (ii) factorization techniques that determine transmitters and receivers up to unknown affine transformation and (iii) determining the affine stratification using remaining nonlinear constraints. This results in a unified approach to solve almost all minimal problems. Such algorithms are important components for systems for self-localization. Experiments are shown both for simulated and real data with promising results.

Index Terms— Time-difference-of-arrival, anchor-free calibration, sensor networks.

1. INTRODUCTION

Sound ranging or sound localization are used to determine the sound source using a number of microphones at known locations and measuring the time-difference of arrival of sounds. Such techniques are used today with microphone arrays to enable beamforming and speaker tracking. Calibration of a sensor network using only TOA or TDOA measurements is a nonlinear optimization problem, for which proper initialization is essential. Several previous works rely on prior knowledge or extra assumptions of locations of the sensors to initialize the problem. In [1], the distances between pairs of microphones are manually measured and multi-dimensional

scaling is used to compute microphone positions. Other options include using GPS [2] to get approximate locations, or using transmitter-receiver pairs (radio or audio) that are close to each other [3, 4, 5]. In [6] it is shown how to estimate additional microphones, once an initial estimate of the positions of some microphones are known. Another line of work focus on solving the initialization without any additional assumptions. Initialization of TOA networks has been studied in [7], where solutions to the minimal case of 3 transmitters and 3 receivers in the plane is given and in [8], where solutions to the minimal cases of $(4, 6)$, $(5, 5)$ and $(6, 4)$ receiver-transmitter combinations are presented. Initialization of TDOA networks is studied in [9], where solutions were given to non-minimal cases in 3D (10 receivers, 5 transmitters) for TDOA and in [10] where four cases of $(9, 5)$, $(7, 6)$ and $(6, 8)$ receiver-transmitter combinations are presented. However solvers for the minimal cases $(10, 5)$, $(7, 5)$, $(6, 6)$ and $(5, 9)$ are still open research problems. A related work that is based on iterative solvers and similar rank constraints as we use is [11].

In this paper we study the initialization network calibration problem from only TDOA measurements for the case where there is a difference in dimension between the spaces spanned by the receivers and by the transmitters. We combine the techniques developed in [8] and [10]. This makes it possible to solve for many (almost all) of the relevant minimal cases. Solving these cases is of theoretical importance. The solvers can also be used in RANSAC [12] schemes to remove outliers in noisy data. The methods are validated both on synthetic and real data. The node localization is cross-validated against computer vision based approaches.

2. PROBLEM FORMULATION

Under the assumption that signals travel at constant speed measuring time of arrival (TOA) is equivalent to measuring distance. TOA requires synchronization between transmitters and receivers in the sense that both transmitting time and time of arrival is available for analysis. This is often not the case and only relative differences in time or distance is measur-

able, with either only synchronized transmitters or receivers. For clarity in the following discussions we will always assume that the *receivers* are synchronized.

Given a set $\{\mathbf{r}_i\}$ of receivers and a set of $\{\mathbf{s}_j\}$ of transmitters a TDOA measurement is

$$f_{ij} = ||\mathbf{r}_i - \mathbf{s}_j||_2 + o_j , \tag{1}$$

where o_j is an unknown offset, compensating for the lack of synchronization between transmitters and receivers.

If the size of the set $\{\mathbf{r}_i\}$ is k and the size of $\{\mathbf{s}_j\}$ is n, we have kn measurements $\{f_{ij}\}$. Assuming all positions are unknown, the basic TDOA problem is

Problem 1. *Given all pairwise measurements $\{f_{ij}\}$ find all positions $\mathbf{r}_i$ and all positions $\mathbf{s}_j$*

Note that solving problem 1 implicitly includes solving the unknown offsets o_j. The topic of this paper is to determine for what choices of k and n problem 1 is solvable when either transmitters or receivers can be seen as belonging to a lower or higher dimension than its counterpart and gives closed formed solutions for these cases. This leads us to the subproblems

Problem 2. $\mathcal{D}_s - \mathcal{D}_r = 1$, *and structure as in Problem 1.*
Problem 3. $\mathcal{D}_r - \mathcal{D}_s = 1$, *and structure as in Problem 1.*

Here $\mathcal{D}_r$ is the dimension of measurements $\mathbf{r}$ and $\mathcal{D}_s$ the dimension of $\mathbf{s}$.

Since all obtained measurement in both the TOA and TDOA setting depend only on relative distances between points, subjecting all points in any given constellation to a common Euclidean transformation will not affect the measurements. This observation has two important implications, summarized in the following lemma.

Lemma 1. *Problem 2 and problem 3 cover all difference in dimension configurations and can only be solved up to a Euclidean transformation of the coordinate system.*

Proof. The second part should be clear as there is no fixed global coordinate system, and distances are preserved under Euclidean transformations.

Since one set of points span a lower dimensional space the transformation that allows us to express these points as $(\mathbf{x}^T, \mathbf{0}^T)$ exists. Assuming the higher dimesion is m and the lower is k de distance d_{ij} between points $\mathbf{x}_i$ and $\mathbf{y}_i$ fulfill

$$||(\mathbf{x}_i^T, \mathbf{0}^T)^T - \mathbf{y}_j)||^2 = \sum_{h=1}^{k}(x_h^{(i)} - y_h^{(j)})^2 + \sum_{h=k+1}^{m} y_h^{(j)2} = d_{ij}^2 .$$

Assume now that for any fixed j and arbitrary number of points $\mathbf{x}_i$ all true coordinates for $h = 1, \cdots, k$ are known, implying that the first sum is known in each equation, and we want to determine the remaining coordinates for $\mathbf{y_j}$, the above is then

$$\underbrace{\sum_{h=1}^{k}(x_h^{(i)} - y_h^{(j)})^2}_{e_{ij}} + \sum_{h=k+1}^{m} y_h^{(j)2} = d_{ij}^2 \Rightarrow$$

$$\sum_{h=k+1}^{m} y_h^{(j)2} = d_{ij}^2 - e_{ij} \ \forall \ i.$$

But since for any two choices of i deducted from each other the left hand side is 0, we only have 1 independent equation irregardless of how many points $\mathbf{x}_i$ we have. Since we don't measure distances between points $\mathbf{y}_j$, adding more such points gives $(m - k)$ more unknowns and 1 more independent equation. Thus The last coordinates can only be solved up to distance from the lower dimensional subspace, i.e. we can replace the last coordinates with one that represents this distance. $\square$

Naturally this give an unsolvable ambiguity if the difference is larger than 1. For instance if the lower dimension is one and the higher is 3, we can only solve for the higher dimension up to a rotation around the the line.

3. MINIMAL CASES IN DIFFERENCE IN DIMENSION

Each pair $(\mathbf{r}_i, \mathbf{s}_j)$ give a measurement f_{ij} and there are $\mathcal{D}_r$ unknowns for each $\mathbf{r}$ and $\mathcal{D}_s + 1$ unknowns for every $\mathbf{s}$. The number of unknowns per sensor is at most $\mathcal{D}_\vee = \max(\mathcal{D}_r, \mathcal{D}_s)$. If we set $\mathcal{D}_\wedge = \min(\mathcal{D}_r, \mathcal{D}_s)$ the following must hold for problems 2 and 3 to be solvable

$$kn \geq \mathcal{D}_r k + (\mathcal{D}_s + 1)n - \frac{(\mathcal{D}_\wedge + 1)\mathcal{D}_\wedge}{2} . \tag{2}$$

The kn, $\mathcal{D}_r k$ and $\mathcal{D}_s$ terms are straightforward. The final term comes from the ambiguity in coordinate system and is as follows: Place the first lower dimensional coordinate at the origin, place the second along the first axis, the third in the plane spanned by the first and second axis, continue until the entire subspace is defined and place all remaining lower dimensional points in the subspace.

It is shown in [10] that the underlying TOA difference in dimension case requires $1 + \mathcal{D}_\wedge + \mathcal{D}_\wedge(\mathcal{D}_\wedge + 1)/2$ of sensors in the lower dimension to be solvable. It is straightforward to confirm that the generalization with offsets does not alleviate this requirement. This together with equation 2 give us the necessary requirements on k and n and all solvable cases for problem 2 and 3 are summarized in figure 1a and 1b respectively.

4. SOLUTION

To derive solvers for feasible k and n we will employ rank constraint strategies introduced in [8] and [14] and modify them for the dimension difference setting. In the cases where the offsets can be completely solved separately from the remaining unknowns we will use methods from [10] to solve for the remaining unknowns. For cases where the offset cannot be computed separately we will show how the rank constraints can be used in conjunction with other constraints to obtain the full solution. Implementation is based on techniques presented in [15].

4.1. Rank Constraints

The rank constraint strategy requires a reformulation of the measurement equations, as well as some observations on their

Case	(k, n)	Solutions	Solved
i	$(9, 3)$	75	This paper
ii	$(7, 4)$	1	This paper
iii	$(6, 5)$	10	This paper
iv	$(5, 2)$	10	In [13]
v	$(4, 3)$	6	In [13]
vi	$(5, 6)$	5	This paper
vii	$(4, 9)$	NA	unsolved
viii	$(5, 3)$	1	This paper
ix	$(4, 4)$	5	This paper
x	$(3, 5)$	16	This paper
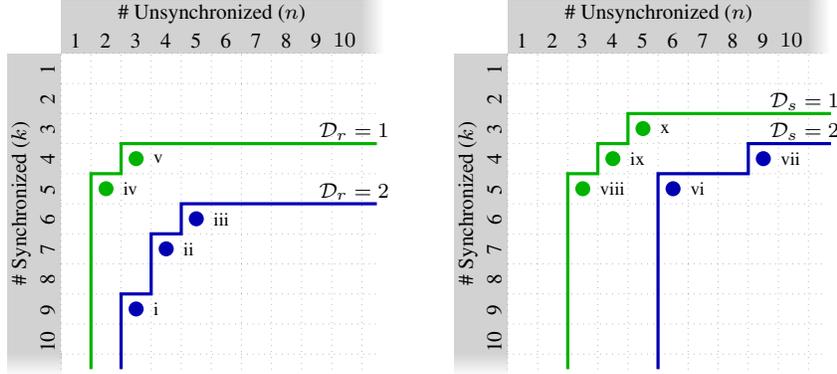
(a) Solvable configurations for problem 2. (b) Solvable configurations for problem 3. (c) Properties of solvers for problem 2 and 3.

Fig. 1: Summary of all solvable cases for difference in dimensions TDOA, in dimensions 1-3. All configurations (k, n) below the green curves mark solvable cases when the lower dimension is 1, all configurations below the blue lines mark solvable configurations when the lower dimension is 2. The table gives the properties of the configurations.

relations to the locations of the sensors. Assuming the coordinates of the lower dimension is "zero-padded" as per the previous discussions we have that $(f_{ij} - o_j)^2 = (\mathbf{r}_i - \mathbf{s}_j)^T(\mathbf{r}_i - \mathbf{s}_j)$. If we introduce the vectors $\mathbf{R}_i = [1 \ \mathbf{r}_i^T \ \mathbf{r}_i^T\mathbf{r}_i]^T$ and $\mathbf{S}_j = [\mathbf{s}_j^T\mathbf{s}_j - o_j^2 \ \mathbf{s}_j^T \ 1]^T$ we get by collecting $\mathbf{R}_i$ into the $(\mathcal{D}_\vee + 2) \times k$ matrix $\mathbf{R}$ and $\mathbf{S}_j$ into the $n \times (\mathcal{D}_\vee + 2)$ matrix $\mathbf{S}$ the relation $\mathbf{F} = \mathbf{R}^T\mathbf{S}$, where $\mathbf{F}$ is a matrix containing $\{f_{ij}^2 - 2f_{ij}o_j\}$. The rank of this matrix is bounded by $(\mathcal{D}_\vee + 2)$ as k and n increases, and the only unknowns are the offsets o_j. It is possible to further exploit the structure of $\mathbf{R}$ and $\mathbf{S}$ to obtain tighter rank constraints, the details are shown in [14] but it is based on exploiting the row of 1s in $\mathbf{R}$ and the row of 1s in $\mathbf{S}$. Effectively we will introduce two matrices $\mathbf{C}_k$ and $\mathbf{C}_n$ both on the form $[-\mathbf{1} \ \mathbf{I}]^T$ that by the operations $\hat{\mathbf{R}}^T = \mathbf{C}_k^T\mathbf{R}^T$ and $\hat{\mathbf{S}} = \mathbf{S}\mathbf{C}_n$ turns the rows of ones into zeros. This results in the final system

$$\hat{\mathbf{F}} = \mathbf{C}_k^T\mathbf{F}\mathbf{C}_n = \hat{\mathbf{R}}^T\hat{\mathbf{S}} , \quad (3)$$

that due to the introduction of zero rows holds after removing the last row of $\mathbf{R}$ and the first row of $\mathbf{S}$. Note that these are not the zero rows. The resulting matrix $\hat{\mathbf{F}}$ is of rank at most $\mathcal{D}_\vee$. However as $\mathbf{S}$ and consequently $\hat{\mathbf{S}}$ is rank deficient prior to the above operations due to the last coordinates of $\mathbf{s}_j$ being zero, the rank of $\hat{\mathbf{F}}$ is in fact at most $\mathcal{D}_\wedge$. It has entries

$$\hat{f}_{ij} = g_{ij} - g_{0j} - g_{i0} + g_{00}, \quad (4)$$

where $g_{ij} = f_{i+1,j+1}^2 - 2f_{i+1,j+1}o_{j+1}$. Therefore, given that each entry of $\hat{\mathbf{F}}$ is a (first order) function of the unknown offsets $\{o_1, \ldots, o_n\}$, we can enforce these rank constraints on the sub-matrices of $\hat{\mathbf{F}}$. Specifically, any matrix has the entries as in (4), all its $(\mathcal{D}_\wedge + 1) \times (\mathcal{D}_\wedge + 1)$ sub-matrices will be rank-deficient and have rank $\mathcal{D}_\wedge$. The existence of such sub-matrices is not guaranteed. For instance case $(5,2)$, the resulting compacted matrix will be of size 4 by 1, and $\mathcal{D}_\wedge + 1 = 2$. This gives equivalently constraints on the determinants of the set of $(\mathcal{D}_\wedge + 1) \times (\mathcal{D}_\wedge + 1)$ sub-matrices $\Lambda_{\mathcal{D}_\wedge+1}$:

$$\det \mathbf{Q} = 0, \ \forall \mathbf{Q} \in \Lambda_{\mathcal{D}_\wedge+1}. \quad (5)$$

For a $(k - 1) \times (n - 1)$ matrix $\hat{\mathbf{F}}$, the number of constraints N_c is

$$N_c = |\Lambda_{\mathcal{D}_\wedge+1}| = \binom{k - 1}{\mathcal{D}_\wedge + 1}\binom{n - 1}{\mathcal{D}_\wedge + 1}.$$

Each constraint is a polynomial equation of degree $\mathcal{D}_\wedge + 1$.

In general, for a case with k receivers and n transmitters, with the minimal affine span of the two as $\mathcal{D}_\wedge$, there exist $N_o = (k - 1 - \mathcal{D}_\wedge)(n - 1 - \mathcal{D}_\wedge)$ linearly independent constraints on the offsets in (5). For cases where $n = N_o$, determining the offsets using only the rank constraints is minimal and well-defined. For linear cases, these correspond to $(4, 4), (5, 3)$. And for the planar cases, $(7,4)$ and $(5,6)$ are the two minimal problems for determining the offsets. Note that such properties are independent of $\mathcal{D}_s$ and $\mathcal{D}_r$.

For cases where $N_o > n$, the rank constraints are overdetermined for the offsets. There are two ways to estimate the offsets using these overdetermined set of equations. The first one is to utilize the fact that there exist a unique solution to the overdetermined system, using techniques from [9], the offsets can be solved linearly. The second scheme is to ignore a subset of constraints such that the remaining constraints render the problem minimal and well-defined. One possible drawback of this scheme is the possible existence of multiple solutions.

If the minimal TDOA cases that are minimal in determined offsets using only the rank constraints, i.e. $(5,3)$ and $(5,6)$, the full problem can be solved by combining the corresponding linear difference in dimension TOA solver from [10]. Again accounting for the inherent ambiguity of the last coordinate in the high dimensional space, the linear solver is unique and the number of solutions is entirely dependent on the number of solutions of the offset equation. These are summarized in figure 1c. In a few cases there are multiple valid

solutions to a given set of measurements, but in general the excess solutions are complex and can be directly discarded.

As for the cases where the rank constraints give underdetermined systems, one needs to exploit other, often nonlinear constraints.

4.2. Distance Equations

We here derive additional non-linear equations on the offsets. To make the presentation clear, in the following discussion, it is assumed that the receivers are in the lower dimension. It is straightforward to convert the formulation for cases where the transmitters are in the lower dimension.

According to [8], each factorization of $\hat{\mathbf{F}} = \hat{\mathbf{R}}^T \hat{\mathbf{S}}$ provides the receiver and transmitter coordinates up to an coordinate change described by a full rank matrix $\mathbf{L}$ and translation $\mathbf{b}$. Let $\tilde{\mathbf{R}}$ be the first $\mathcal{D}_\wedge$ columns of the rank-$\mathcal{D}_\wedge$ matrix $\hat{\mathbf{F}}$ which is parameterized by the offsets $\mathbf{o}$. This then corresponds to a choice of factorization that has the identity matrix on the corresponding places in $\hat{\mathbf{S}}$. Based on this and the formulation in (3), we can write the positions of the receivers $\mathbf{r}_i = \mathbf{L}\tilde{\mathbf{r}}_i(\mathbf{o})$. Following the derivation in [8] this gives the following constraints on the unknown transformation $\mathbf{H}$ and translation $\mathbf{b}$ for $i = \{1, \ldots, m-1\}$,

$$d_{i+1,1}^2 - d_{11}^2 = \tilde{\mathbf{r}}_i^T \mathbf{H} \tilde{\mathbf{r}}_i - 2\mathbf{b}^T \tilde{\mathbf{r}}_i, \qquad (6)$$

where $d_{ij} = f_{ij} - o_j$, $\mathbf{H} = (\mathbf{L}^T\mathbf{L})^{-1} \in \mathbb{R}^{\mathcal{D}_\wedge \times \mathcal{D}_\wedge}$ and $\mathbf{b} \in \mathbb{R}^{\mathcal{D}_\wedge}$. Since the equations are linear in the entries in $\mathbf{H}$ and $\mathbf{b}$, the system can be rewritten as

$$\mathbf{W} \begin{bmatrix} \mathbf{h} \\ \mathbf{b} \\ 1 \end{bmatrix} = 0, \qquad (7)$$

where $\mathbf{W}$ is a $(m-1) \times k$ matrix parameterized by the offsets and $\mathbf{h}$ is the vector representation of the unknowns in $\mathbf{H}$. Here $k = \mathcal{D}_\wedge(\mathcal{D}_\wedge + 1)/2 + \mathcal{D}_\wedge + 1$. From (7), we know that all $k \times k$ sub-determinants of $\mathbf{W}$ are equal to 0. By forming these equations, we remove the unknowns $\mathbf{h}$ and $\mathbf{b}$ and reduce (6) to a polynomial system of only n unknowns. Combining these equations with the rank constraints, one arrives at a set of well-defined equations for the offsets. In principle, both the (3,5) (9,3) as well as the (4,9) cases can be solved using this formulation. Fast and stable solvers have been implemented based on Gröbner basis methods for (3,5) and (9,3) cases. Efficient solvers for the (4,9) case is still difficult to derive due to the large number of unknowns (9 offsets) and high degree (degree 9). Such idea can also be extended to cases where $\mathcal{D}_\wedge = 3$. The number of solutions for these cases using the above solving strategy are presented in figure 1c.

5. SUMMARY

The classification for the found cases is shown in figure 1. Two cases were solved using direct manipulation of the distance equations in [13]. Some cases are overdetermined by
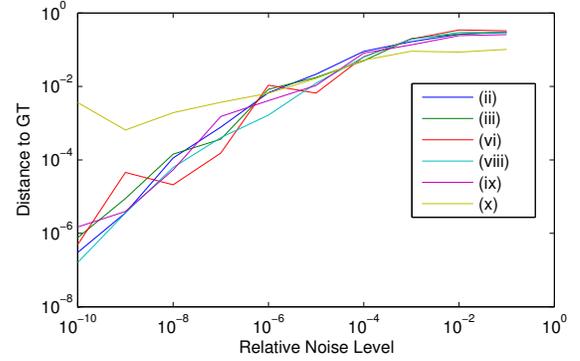
Fig. 2: Stability of the derived solvers, except (i).

one equation, however further reducing either k or n would make the system underdetermined and thus unsolvable. As described above this sometimes allows for linear solvers to be employed. In general the resulting systems have relatively low total degree and few solutions, with the exception of case (i). Case (vii) is even more complex and using the presented strategy we were unsuccessful in constructing a solver that displayed good numerics.

6. EXPERIMENTS

We will present the numerical stability for all implemented solvers using generated examples. We further will present results on real data using a microphone setup in 2D, with sounds in 3D. The accuracy of the solution will be measured by comparing it to a 3D reconstruction from images. The visual reconstruction is obtained using standard techniques from computer vision.

6.1. Numerical Stability

Synthetic data is generated by randomly placing sensors in a $[0,1]$ cube, meeting the requirements of dimensionality assumed by the solvers. The solver for case (i) requires that the original equations are expanded to a 1400 by 500 coefficient matrix, and it has very poor stability even if no noise is added. Typical accuracy without noise is RMS on the order of 10^{-4}. All other solvers had consistent accuracy of the order 10^{-10} to 10^{-13} with the exception of (x) that on rare occasions had values of 10^{-2}, skewing its mean quite severely. We believe this is caused by close to degenerate configurations. This behavior is also visible in the presence of noise, as illustrated in figure 2. The figure shows the mean over 200 cases for different levels of relative added gaussian noise, applied to the measurements. The RMS is calculated against the generated ground truth (GT). Again the poor performance of (x) is due to single events with substantially less accurate result.

6.2. Reconstruction of Microphone Array

A total of 8 microphones are placed on a floor (2D), see figure 3, and sequences of distinct sounds generated from several locations in the room (3D). The sounds are far enough apart to be distinct in the matching, but due to echoes, disturbances
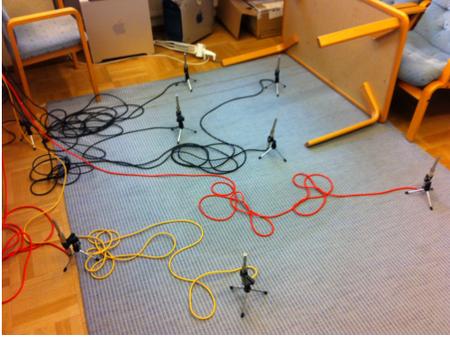
Fig. 3: Microphones placed on office floor.
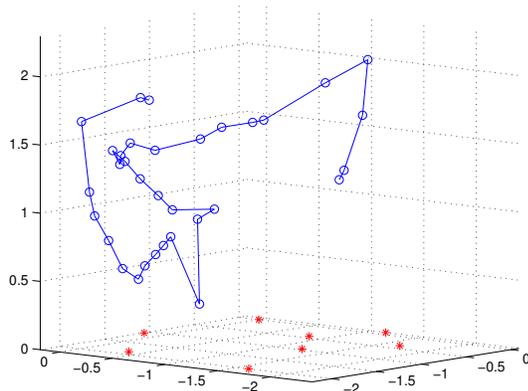
Fig. 4: Reconstructed layout of microphone positions (red stars) and motion trajectory of sound sources (blue circles and line), all units in meter.

exact time differences are unavailable, and in some cases the matches are bad enough to be considered outliers. We then use the (6,5) minimal solver in a RANSAC-like algorithm. As a final step the solution is locally optimized using all found inliers. The result is very promising with an RMS of 6.7cm in microphone positions between the visual and audio based reconstructions. The reconstructed path for the sound source is consistent with the dimensions of the room, and form a smooth track. The reconstructed layout is illustrated in figure 4.

7. CONCLUSIONS

We have classified all solvable minimal cases in a difference in dimension TDOA setting. Further we have devised solution strategies and implemented solvers for most of these cases. With the exception of 2 solvers the overall performance is excellent, and one of the bad solvers still maintain a very high success rate.

REFERENCES

[1] S. T. Birchfield and A. Subramanya, "Microphone array position calibration by basis-point classical multidimensional scaling," *IEEE transactions on Speech and Audio Processing*, vol. 13, no. 5, 2005.

[2] D. Niculescu and B. Nath, "Ad hoc positioning system (aps)," in *GLOBECOM-01*, 2001.

[3] E. Elnahrawy, Xl. Li, and R. Martin, "The limits of localization using signal strength," in *SECON-04*, 2004.

[4] J. Sallai, G. Balogh, M. Maroti, and A. Ledeczi, "Acoustic ranging in resource-constrained sensor networks," in *eCOTS-04*, 2004.

[5] Marco Crocco, Alessio Del Bue, and Vittorio Murino, "A bilinear approach to the position self-calibration of multiple sensors," *Trans. Sig. Proc.*, vol. 60, no. 2, pp. 660–673, feb 2012.

[6] J. C. Chen, R. E. Hudson, and K. Yao, "Maximum likelihood source localization and unknown sensor location estimation for wideband signals in the near-field," *IEEE transactions on Signal Processing*, vol. 50, 2002.

[7] H. Stewénius, *Gröbner Basis Methods for Minimal Problems in Computer Vision*, Ph.D. thesis, Lund University, APR 2005.

[8] Yubin Kuang, Simon Burgess, Anna Torstensson, and Kalle Åström, "A complete characterization and solution to the microphone position self-calibration problem," in *The 38th International Conference on Acoustics, Speech, and Signal Processing*, 2013.

[9] M. Pollefeys and D. Nister, "Direct computation of sound and microphone locations from time-difference-of-arrival data," in *Proc. of International Conference on Acoustics, Speech and Signal Processing*, 2008.

[10] Simon Burgess, Yubin Kuang, and Kalle Åström, "Toa sensor network calibration for receiver and transmitter spaces with difference in dimension," in *Proceeding of the 21st European Signal Processing Conference 2013*, 2013.

[11] Nikolay D. Gaubitch, W. Bastiaan Kleijn, and Richard Heusdens, "Auto-localization in ad-hoc microphone arrays.," in *ICASSP*. 2013, pp. 106–110, IEEE.

[12] M. A. Fischler and R. C. Bolles, "Random sample consensus: a paradigm for model fitting with applications to image analysis and automated cartography," *Communications of the ACM*, vol. 24, no. 6, pp. 381–95, 1981.

[13] Erik Ask, Simon Burgess, and Kalle Åström, "Minimal structure and motion problems for toa and tdoa measurements with collinearity constraints," in *Proceedings of the 2nd International Conference on Pattern Recognition Applications*, 2013.

[14] Yubin Kuang and Kalle Åström, "Stratified sensor network self-calibration from tdoa measurements," in *21st European Signal Processing Conference 2013*, 2013.

[15] Martin Byröd, Klas Josephson, and Kalle Åström, "Fast and stable polynomial equation solving and its application to computer vision," *Int. Journal of Computer Vision*, vol. 84, no. 3, pp. 237–255, 2009.