FILTER DESIGN WITH HARD SPECTRAL CONSTRAINTS

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ABSTRACT

Filter design is a fundamental problem in signal processing and important in many applications. In this paper we consider a communication application with spectral constraints, using filter designs that can be solved globally via convex optimization. Tradeoffs are discussed in order to determine which design is the most appropriate, and for these applications, finite impulse response filters appear to be more suitable than infinite impulse response filters since they allow for more flexible objective functions, shorter transients, and faster filter implementations.

***Index Terms*—** OFDM, filter design, convex optimization.

1. INTRODUCTION

Filter design is a fundamental problem that is omnipresent in applications such as communications and radar. The frequency spectrum is becoming excessively congested [1], and the use of filter design is a systematic method for creating signals with constrained spectra to minimize interference [2]. Here we will consider a case of filter design using convex optimization for a power line communication (PLC) application [3, 4].

We consider orthogonal frequency-division multiplexing (OFDM) [5], where the symbol $(x_1, x_2, \ldots, x_K)$ is encoded into the OFDM-symbol

$$s(t) = \frac{1}{T}\sum_{k=1}^{K} x_k e^{j\omega_k t}, \qquad \text{for } t = 0, 1, \ldots, T-1, \tag{1}$$

using a set of carrier frequencies $(\omega_1, \omega_2, \ldots, \omega_K)$. Here T is the length of the OFDM-symbol, and for practical reasons we assume that $|x_k| \le 1$, for $k = 1, \ldots, K$. Regulations put constraints on the transmit power in specific frequency bands and the carriers are therefore selected in the frequency bands where transmission is allowed (see Figure 1). Despite such a carrier selection, the spectral valleys in the resulting spectrum, between the sets of carriers, are often not deep enough to satisfy the constraints. To deal with this, the signal is passed through a filter $H(\omega)$ before transmission.

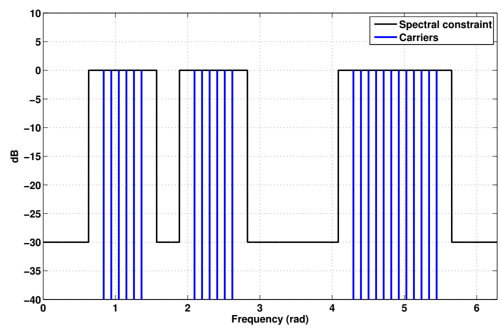

Fig. 1. Carrier placement and spectral constraints.

Denote the energy spectral density of $s(t)$ by $\Phi(\omega)$ and the output constraints by $\Psi_{\text{output}}(\omega)$. The spectral energy over all possible symbols $(x_1, x_2, \ldots, x_K)$ is bounded by

$$\begin{aligned} \Phi(\omega) &= \left| \sum_{k=1}^{K} x_k \frac{\sin((\omega-\omega_k)T/2)}{T\sin((\omega-\omega_k)/2)} e^{j(\omega-\omega_k)(T-1)/2} \right|^2 \\ &\le \left(\sum_{k=1}^{K} \left| \frac{\sin((\omega-\omega_k)T/2)}{T\sin((\omega-\omega_k)/2)} \right| \right)^2 =: \Phi_{\max}(\omega), \end{aligned}$$

which is depicted in the left plot of Figure 2, and it can be seen that the valleys are not deep enough.[1] In order to guarantee that the filtered output satisfies the spectral constraints, the filter magnitude needs to be bounded by

$$|H(\omega)|^2 \le \Psi_{\text{filter}}(\omega) := \Psi_{\text{output}}(\omega)/\Phi_{\max}(\omega), \tag{2}$$

which is depicted in the right plot of Figure 2.

In addition to the constraints (2), the magnitude of the frequency response in the frequency bands of the carriers $\Omega = \{\omega : |\omega - \omega_k| \le \Delta\omega/2, k = 1, \ldots, K\}$, where $\Delta\omega = 2\pi/T$, should be as large as possible to minimize the attenuation of the OFDM-symbol.

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[1] In this example $T = 60$ and the distance between two adjacent carriers is $\Delta\omega = 2\pi/T$.

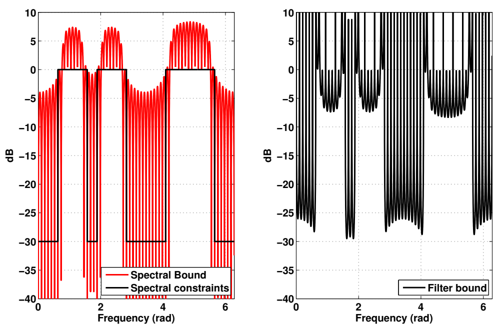

Fig. 2. Left subplot: Maximum signal spectrum, and spectral constraints. Right subplot: Filter upper bound $\Psi_{\text{filter}}(\omega)$.

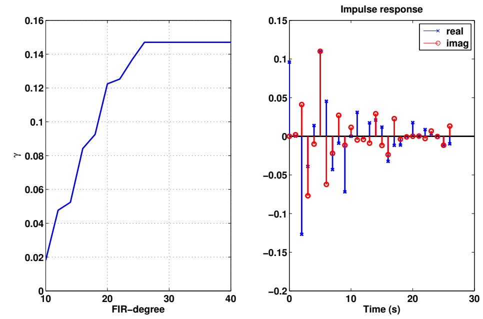

Fig. 3. Left subplot: γ as function of filter degree. Right subplot: The impulse response (minimum phase) corresponding to the optimal max/min filter of degree 26.

In this article we will describe how such a filter can be constructed in a systematic way for spectral shaping of an OFDM signal.[2] We will consider designs for finite impulse response (FIR) filters as well as infinite impulse response (IIR) filters and examine the relevant trade-offs for the current application. Our goal is to highlight possible pitfalls and to determine which methods are appropriate for PLC.

2. FIR FILTER DESIGN

First consider using an FIR filter of order N, i.e., a filter H with frequency response function

$$H(\omega) = b_0 + b_1 e^{j\omega} + \cdots + b_N e^{j\omega N}, \tag{3}$$

where $b_k \in \mathbb{C}, k = 0, 1, \ldots, N$. The design of the filter may be formulated as the max/min problem

$$\begin{array}{lll} \max\limits_{H,\gamma} & \gamma & \\ \text{subject to} & \gamma \leq |H(\omega)|^2, & \text{for } \omega \in \Omega, \\ & |H(\omega)|^2 \leq \Psi_{\text{filter}}(\omega), & \text{for } \omega \in [0, 2\pi] \\ & H \in \mathcal{P}(N) & \end{array} \tag{4}$$

where $\mathcal{P}(N)$ denotes the set of FIR transfer functions of the form (3). The magnitude $|H(\omega)|^2$ is nonlinear in the filter coefficients b_k, making the maximization of γ non-trivial. To overcome this problem, we use the lead of [8–11], and reparameterize the filter design as a convex optimization problem. Note that

$$|H(\omega)|^2 = \sum_{k=-N}^{N} p_k e^{jk\omega} =: p(\omega) \tag{5}$$

where

$$p_k = \sum_{\ell=0}^{N-k} b_{\ell+k} \bar{b}_\ell \tag{6}$$

and where $\bar{b}$ denotes the complex conjugate of b. Conversely, if the trigonometric polynomial $p(\omega)$ of degree N is non-negative on $[0, 2\pi]$ then there exists $H(\omega)$ such that (6) holds and the coefficients $b_0, b_1, \ldots, b_N$ can be recovered from $p(\omega)$ by spectral factorization. This allows for a linear parametrization of $|H(\omega)|^2 = p(\omega)$ via $p_0, p_1, \ldots, p_N$ subject to the convex constraints $p(\omega) \geq 0$, $\omega \in [0, 2\pi]$. Hence (4) may be formulated as a convex optimization problem

$$\begin{array}{lll} \max\limits_{p,\gamma} & \gamma & \\ \text{subject to} & \gamma \leq p(\omega), & \text{for } \omega \in \Omega, \\ & 0 \leq p(\omega) \leq \Psi_{\text{filter}}(\omega), & \text{for } \omega \in [0, 2\pi], \\ & p \in \mathcal{P}_{\text{trig}}(N), & \end{array} \tag{7}$$

where $\mathcal{P}_{\text{trig}}(N)$ denotes the set of real trigonometrical polynomials[3] (5).

We continue the example associated with Figures 1 and 2, where $T = 60$ and the distance between two adjacent carriers is $\Delta\omega = 2\pi/T$. The objective value, corresponding to $\gamma = \min\{|H(\omega)| : \omega \in \Omega\}$ is depicted in Figure 3 as a function of the filter length. In this example the objective value essentially reaches its maximum for $N = 26$. The second subplot in Figure 3 shows the impulse response of the optimal filter with length 26. Figure 4 depicts the magnitude of the FIR filter (left plot) and the resulting bound on the spectrum of the transmitted signal, i.e., $|H(\omega)|^2 \Phi_{\max}(\omega)$ (right plot). Figure 5 shows $s(t)$ and its spectrum $\Phi(\omega)$ for a random binary sequence x_k, whereas Figure 6 corresponds to the filtered signal.

A few remarks are in order. Firstly, the use of an FIR filter increases the duration of the signal from T to $T + N$, and hence a high filter length may cause inter-symbol interference. Secondly, the bound used in (7) is uniform on Ω, hence the objective value saturates as the filter magnitude reaches the bound in one area and does not seek to improve in the other areas. This is a problem when the passbands and

[2] In this paper we do not consider certain practical aspects, such as cyclic prefix and trade-offs between guard intervals, standard windowed OFDM, and filters (c.f., [6, 7]).

[3] That is, with $p_{-k} = \bar{p}_k$ for $k = 0, 1, \ldots, N$.

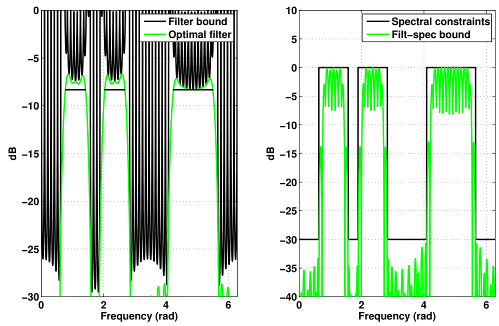

Fig. 4. Magnitude of the optimal max/min FIR filter of length 26 and the corresponding output spectral bound.

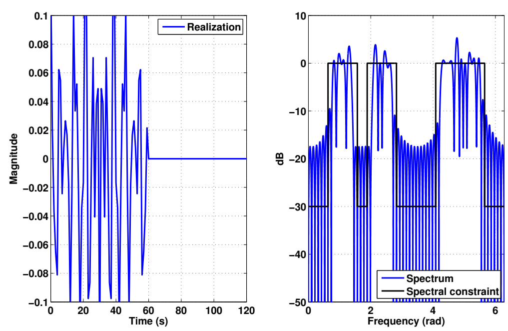

Fig. 5. Unfiltered signal (real part) and its spectrum.

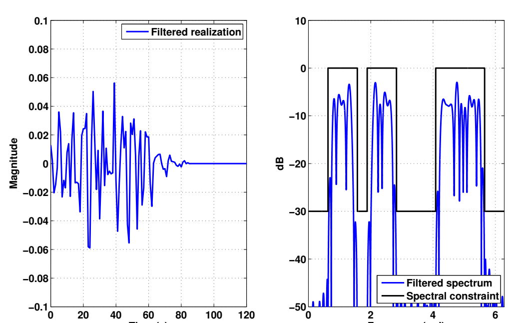

Fig. 6. Filtered signal (real part) and its spectrum using the optimal max/min FIR filter of length 26.

notches have varying depths and widths, in which case all the passbands suffer when one constraint is difficult to achieve. In this case $\Psi(\omega)$ is smaller for the third set of carriers (in the interval $[1.3\pi, 1.8\pi]$) than for the first two sets, which means that the optimization problem does not increase the lower value of $|H(\omega)|$ for the first two sets of carriers even though there is room for that. This suggests that the objective function in the max/min problem is not the best choice for these cases.

One alternative to the max/min objective function is to use a least-squares (LS) objective. This choice also leads to a convex optimization problem for FIR models and may be posed as follows:

$$\begin{array}{ll} \min\limits_{p,\gamma} & \displaystyle\int_{\Omega} (\Psi_{\text{filter}}(\omega) - p(\omega))^2 d\omega \\ \text{subject to} & 0 \leq p(\omega) \leq \Psi_{\text{filter}}(\omega), \text{ for } \omega \in [0, 2\pi], \\ & p \in \mathcal{P}_{\text{trig}}(N). \end{array} \tag{8}$$

This cost function does not run the risk of being saturated by one or few frequency points. Instead it decreases as the filter magnitude increases in the passbands. The design result for the optimal LS FIR filter of length 26 is shown in Figure 7. The filter frequency response tightly follows the bound in the bands where the carriers are. The corresponding impulse response is depicted in Figure 8, and the filtered signal is shown in Figure 9.

3. IIR FILTER DESIGN

Another filter design option is to use an IIR filter, which has a transfer function of the form

$$H(\omega) = \frac{b_0 + b_1 e^{j\omega} + \cdots + b_N e^{j\omega N}}{a_0 + a_1 e^{j\omega} + \cdots + a_M e^{j\omega M}}.$$

By using the idea in (5), once again, and letting

$$p_k = \sum_{\ell=0}^{N-k} b_{\ell+k}\bar{b}_\ell, \quad q_k = \sum_{\ell=0}^{M-k} a_{\ell+k}\bar{a}_\ell,$$

the numerator and the denominator of $|H(\omega)|^2$ may be parameterized linearly as follows:

$$|H(\omega)|^2 = \frac{\sum_{k=-N}^{N} p_k e^{jk\omega}}{\sum_{k=-M}^{M} q_k e^{jk\omega}} =: \frac{p(\omega)}{q(\omega)}$$

where $p(\omega) \geq 0, q(\omega) \geq 0$ for $\omega \in [0, 2\pi]$. The max/min problem (4) for IIR filters may now be posed as

$$\begin{array}{lll} \max\limits_{p,q,\gamma} & \gamma & \\ \text{subject to} & \gamma q(\omega) \leq p(\omega), & \text{for } \omega \in \Omega, \\ & 0 \leq p(\omega) \leq \Psi(\omega) q(\omega), & \text{for } \omega \in [0, 2\pi], \\ & p \in \mathcal{P}_{\text{trig}}(N), \quad q \in \mathcal{P}_{\text{trig}}(M). & \end{array}$$

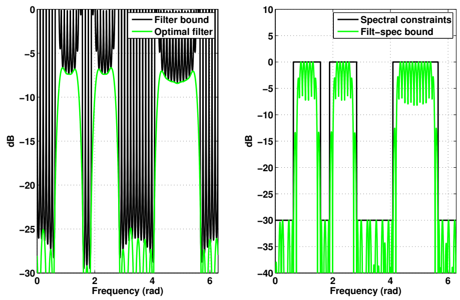

Fig. 7. Magnitude of the LS-optimal FIR filter of length 26 and the output spectral bound.

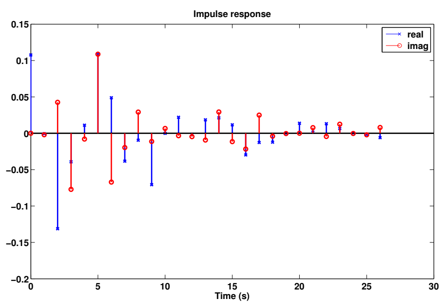

Fig. 8. Impulse response (minimum phase) of the LS-optimal FIR filter of length 26.

This is a non-convex problem, but its feasibility sub-problem is convex for a fixed γ. Hence the problem is quasi-convex and can be solved globally, e.g., by using the bisection algorithm on γ.

The objective value, corresponding to $\gamma = \min\{|H(\omega)| : \omega \in \Omega\}$ as a function of the filter length (with $N = M$) is depicted in Figure 10. The optimal IIR filter of length $(8, 8)$ and the resulting spectral bound of its output are shown in Figure 11. The corresponding filtered realization and its spectrum are depicted in Figure 12.

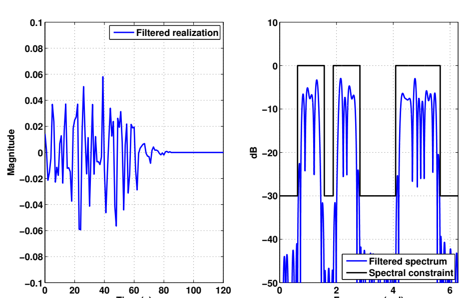

Fig. 9. Filtered signal (real part) and its spectrum using the LS-optimal FIR filter of length 26 (minimum phase).

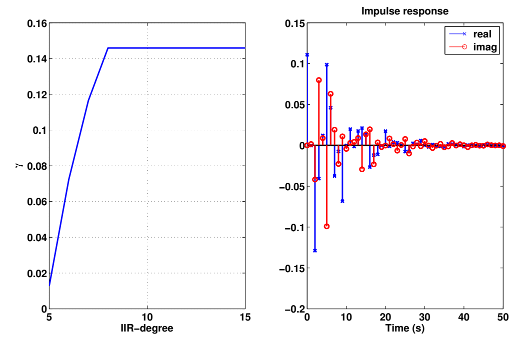

Fig. 10. Left subplot: γ as a function of the IIR filter length (with $N = M$). Right subplot: Impulse response (minimum phase) of the optimal max/min IIR filter with $N = M = 8$.

Clearly, IIR filters are more general than FIR filters and require a lower degree to satisfy a set of constraints. There are however some difficulties that make IIR filters less suitable for this particular application. The objective functions which give rise to convex optimization problems are more restrictive for IIR filters compared to FIR filters. For example, the least squares objective function in (8) for IIR filters gives rise to a non-convex optimization problem. Secondly, the transients of IIR filters are infinitely long, at least in principle, potentially causing more inter-symbol interference problems. Thirdly, even if an IIR filter of significantly lower degree may be found, there is little or no computational gain since an FIR filter may be implemented using the fast Fourier transform (FFT), which in general is more computationally efficient.

4. IMPLEMENTATION AND SPECTRAL FACTORIZATION

The convex optimization problems in this paper were solved using CVX, a Matlab package for specifying and solving convex programs [12]. The constraints were enforced on a fine grid (5000 grid points). A rule of thumb is that the grid should contain at least 15 times as many points as the order of the filter [13]. In practice this works well for the bounds $\gamma \leq p(\omega)$ and $p(\omega) \leq \Psi(\omega)$, but for the positivity constraint $0 \leq p(\omega)$ this is not enough. The reason is that if $p(\omega)$ has negative values for some ω, then there does not exist any filter such that $|H(\omega)|^2 = p(\omega)$. Fortunately the positivity constraint on $p(\omega)$ may be formulated as an linear matrix inequality (LMI) using the so-called trace parameterization (see [9–11]). The trigonometric polynomial

$$p(\omega) = \sum_{k=-N}^{N} p_k e^{jk\omega}$$

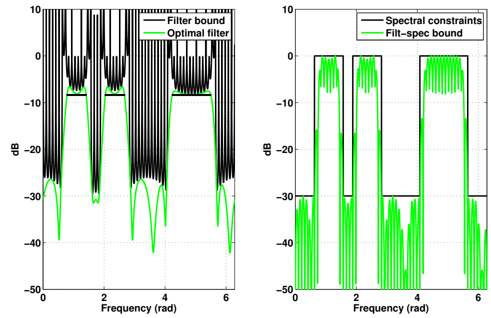

Fig. 11. Left subplot: Magnitude of the optimal max/min IIR filter with $N = M = 8$. Right subplot: The corresponding output spectral bound.

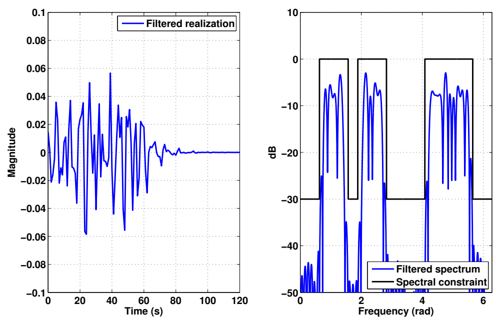

Fig. 12. Filtered signal (real part) and its spectrum using the max/min optimal IIR filter with $N = M = 8$ (min. phase).

is non-negative for $\omega \in [0, 2\pi]$ if and only if there exists a positive semidefinite matrix $X \in \mathbb{C}^{(N+1)\times(N+1)}$ such that

$$p_k = \mathrm{trace}(E_k^T X), \qquad \text{for } k = -N, \ldots, N,$$

where E_k is the kth shift matrix. That is, $E_k \in \mathbb{R}^{(N+1)\times(N+1)}$ with $[E_k]_{\ell,m} = 1$ if $\ell - m = k$ and $[E_k]_{\ell,m} = 0$ otherwise.

This fact allows us to state the positivity constraint as an LMI, which ensures that $p(\omega)$ is non-negative for the entire interval and not just for isolated grid points. Then we can use Wilson's algorithm [14] for spectral factorization of $p(\omega)$ (and $q(\omega)$) to determine the filter $H(\omega)$.

5. CONCLUSIONS

We have discussed designs of FIR and IIR filters for signals with constrained spectra such as those appearing in some communication applications. The designs were formulated as convex or quasiconvex optimization problems and globally optimal solutions were found in each case. For the applications of interest in this paper, FIR filters appear to be more suitable than IIR filters. Indeed FIR filters allow for more flexible objective functions, shorter transients, and faster filter implementations.

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