OVERSAMPLED GRAPH LAPLACIAN MATRIX FOR GRAPH SIGNALS

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ABSTRACT

In this paper, we propose oversampling of graph signals by using oversampled graph Laplacian matrix. The conventional critically sampled graph filter banks have to decompose an original graph into bipartite subgraphs, and the transform has to be performed on each subgraph due to the spectral folding phenomenon caused by downsampling of graph signals. Therefore, they cannot always utilize all edges of the original graph for the one-stage transformation. Our proposed method is based on oversampling of the underlying graph itself, and it can append nodes and edges to the graph somewhat arbitrarily. We use this approach to make one oversampled bipartite graph that includes all edges of the original non-bipartite graph. We apply the oversampled graph with the critically sampled filter bank for decomposing graph signals, and show the performance of graph signal denoising.

***Index Terms*—** Graph signal processing, graph oversampling, multiresolution, spectral graph theory, graph wavelets

1. INTRODUCTION

Graphs are data structures that can represent complex relationships among data and can be used in many fields of engineering and science. A graph consists of nodes and edges, and each edge is usually assigned a weight determined by the similarity and connectivity of the nodes. A recent development is graph signal processing, in which a sample is placed on each node of a graph and the processing takes into account the structure of the samples [1–9]. Whereas signals of regular signal processing have very simple structures, those of graph signal processing are allowed to have complex irregular structures.

Multiresolution analysis is efficient for analyzing, processing or compressing signals [10]. Wavelet transforms for graph signals can be used to make multiresolution analysis [2–4]. An important topic in graph signal processing is downsampling and upsampling. Similar to the aliasing of regular signal processing, the *spectral folding phenomenon* is occurred by downsampling in graph signal processing. In order to deal with this challenge, studies on critically sampled filter banks have focused on bipartite graphs and determined the perfect reconstruction conditions [2, 3].

The graph-based transforms with the downsampling and upsampling operations, such as the critically sampled graph filter banks and the oversampled ones [11, 12], can only be applicable to bipartite graphs. For arbitrary non-bipartite graphs, we have to decompose an original graph into an edge-disjoint collection of bipartite subgraphs whose union is the original graph, and the transform is performed on each of these subgraphs. Since a subgraph has only a part of edges of the original graph, many edges are usually not utilized in one-stage transform.

In this paper, we propose graph oversampling, that yields oversampled graph Laplacian matrices. Furthermore, graph signals are oversampled taking into account the graph structures. The graph oversampling enables us to make one bipartite graph that includes all edges of the original graph, which is completely different from the graph used in the conventional critically sampled filter banks. The redundant multiresolution transform can be implemented by applying the critically sampled graph filter banks on the oversampled graph. We conduct graph signal denoising and demonstrate the effectiveness of the oversampled graph.

The rest of this paper is organized as follows. In Section 2, we describe notations used in this paper and the two-channel critically sampled wavelet filter bank on graphs [2,3]. Section 3 introduces methods of oversampling graph Laplacian matrices and input signals and shows the way of making one oversampled bipartite graph from a three-colorable graph. Section 4 describes signal spread and denoising experiments. Section 5 concludes the paper.

2. PRELIMINARIES

2.1. Graph Signals

A graph is represented as $\mathcal{G} = \{\mathcal{V}, \mathcal{E}\}$, where $\mathcal{V}$ and $\mathcal{E}$ denote sets of nodes and edges, respectively. The graph signal is defined as $\boldsymbol{f} \in \mathbb{R}^N$. We will only consider a finite undirected graph with no loops or multiple edges. The number of nodes is $N = |\mathcal{V}|$, unless otherwise specified. The

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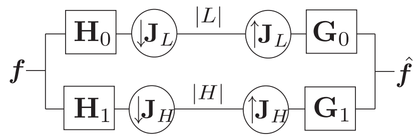


Fig. 1. Critically sampled two-channel graph filter bank.

(m, n)-th element of the adjacency matrix $\mathbf{A}$ is the weight of the edge between m and n if m and n are connected, and 0 otherwise. The degree matrix $\mathbf{D}$ is a diagonal matrix and its m-th diagonal element is $d_{mm} = \sum_n a_{mn}$. The unnormalized graph Laplacian matrix (GLM) is defined as $\mathbf{L} := \mathbf{D} - \mathbf{A}$ and the symmetric normalized GLM is $\boldsymbol{\mathcal{L}} := \mathbf{D}^{-1/2}\mathbf{L}\mathbf{D}^{-1/2}$. The symmetric normalized GLM has the property that its eigenvalues are within the interval $[0, 2]$, and we will use $\boldsymbol{\mathcal{L}}$ in this paper. The eigenvalues of $\boldsymbol{\mathcal{L}}$ are λ_i and ordered as: $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \ldots \leq \lambda_{N-1} \leq 2$ without loss of generality. The eigenvector $\boldsymbol{u}_{\lambda_i}$ corresponds to λ_i and satisfies $\boldsymbol{\mathcal{L}}\boldsymbol{u}_{\lambda_i} = \lambda_i \boldsymbol{u}_{\lambda_i}$. The entire spectrum of $\mathcal{G}$ is defined by $\sigma(\mathcal{G}) := \{\lambda_0, \ldots, \lambda_{N-1}\}$. The projection matrix for the eigenspace V_{λ_i} is

$$\mathbf{P}_{\lambda_i} = \sum_{\lambda=\lambda_i} \boldsymbol{u}_\lambda \boldsymbol{u}_\lambda^T, \tag{1}$$

where $\boldsymbol{u}_\lambda^T$ is the transpose of $\boldsymbol{u}_\lambda$. Let $h(\lambda_i)$ be the spectral kernel of filter $\mathbf{H}$. The spectral domain filter can be written as

$$\mathbf{H} = h(\boldsymbol{\mathcal{L}}) = \sum_{\lambda_i \in \sigma(\mathcal{G})} h(\lambda_i)\mathbf{P}_{\lambda_i}. \tag{2}$$

The spectral domain filtering of graph signals can be simply denoted as $\mathbf{H}\boldsymbol{f}$.

2.2. Two-Channel Graph Wavelet Filter Banks

A bipartite graph whose nodes can be decomposed into two disjoint sets L and H such that every edge connects a node in L to one in H can be represented as $\mathcal{G} = \{L, H, \mathcal{E}\}$. The downsampling function β_H of a bipartite graph is defined as

$$\beta_H(m) = \begin{cases} +1 & \text{if } m \in H, \\ -1 & \text{if } m \in L. \end{cases} \tag{3}$$

The diagonal downsampling matrix is $\mathbf{J}_H = \mathrm{diag}\{\beta_H(n)\}$ and satisfies $\mathbf{J} = \mathbf{J}_H = -\mathbf{J}_L$. The downsampling-then-upsampling operation can be defined as follows:

$$\mathbf{D}_{du,L} = \frac{1}{2}(\mathbf{I}_N + \mathbf{J}_L),\ \mathbf{D}_{du,H} = \frac{1}{2}(\mathbf{I}_N + \mathbf{J}_H). \tag{4}$$

where $\mathbf{I}_N$ is an $N \times N$ identity matrix.

$\mathbf{J}$ and $\mathbf{P}_{\lambda_i}$ are related as follows [3] (spectral folding phenomenon):

$$\mathbf{J}\mathbf{P}_{\lambda_i} = \mathbf{P}_{2-\lambda_i}\mathbf{J}. \tag{5}$$

The critically sampled filter banks decompose N input signals into $|L|$ lowpass coefficients and $|H|$ highpass coefficients, where $|L| + |H| = N$, as illustrated in Fig. 1. The overall transfer function of graph-QMF [3] and graphBior [2] can be written as

$$\begin{aligned} \mathbf{T} &= \frac{1}{2}\mathbf{G}_0(\mathbf{I} - \mathbf{J})\mathbf{H}_0 + \frac{1}{2}\mathbf{G}_1(\mathbf{I} + \mathbf{J})\mathbf{H}_1 \\ &= \frac{1}{2}(\mathbf{G}_0\mathbf{H}_0 + \mathbf{G}_1\mathbf{H}_1) + \frac{1}{2}(\mathbf{G}_1\mathbf{J}\mathbf{H}_1 - \mathbf{G}_0\mathbf{J}\mathbf{H}_0). \end{aligned} \tag{6}$$

The spectral folding term $\mathbf{G}_1\mathbf{J}\mathbf{H}_1 - \mathbf{G}_0\mathbf{J}\mathbf{H}_0$, arising from downsampling and upsampling, must be zero. In addition, $\mathbf{T} = \mathbf{I}_N$ should be satisfied for perfect reconstruction. Hence, the perfect reconstruction condition can be expressed as

$$\begin{aligned} g_0(\lambda)h_0(\lambda) + g_1(\lambda)h_1(\lambda) &= 2, \\ -g_0(\lambda)h_0(2-\lambda) + g_1(\lambda)h_1(2-\lambda) &= 0. \end{aligned} \tag{7}$$

The orthogonal transform, graph-QMF, has the orthogonality condition $h_0^2(\lambda) + h_0^2(2-\lambda) = c^2$. Therefore, the filters are chosen in a way that satisfies $h_1(\lambda) = h_0(2-\lambda)$, $h_0(\lambda) = g_0(\lambda)$ and $h_1(\lambda) = g_1(\lambda)$. Unfortunately, filters that satisfy these conditions are not compact support. That is, if graph-QMF were forced to be compact support, it would suffer from a loss of orthogonality and a reconstruction error. On the other hand, graphBior relaxes the orthogonal condition of graph-QMF and has a perfect reconstruction condition and compact support because it uses a design method similar to Cohen-Daubechies-Feauveau's construction for regular signals [13].

The critically sampled filter bank is designed for bipartite graphs. For any arbitrary graph, the original graph should be decomposed into an edge-disjoint collection of K bipartite subgraphs [3, 14] and the transform is performed on each subgraph. Each subgraph has the same node set as the original graph and their union is the original graph. This decomposition leads to a *multi-dimensional* graph wavelet filter bank.

3. GRAPH OVERSAMPLING

In this section, we propose the way to make oversampled GLMs and oversampled graph signals, and show the example of the oversampled graph.

3.1. Oversampled Graph Laplacian Matrix

Fig. 2 shows an example of the transform using graph oversampling. By appending the nodes and the edges, the original graph $\mathcal{G} = \{L, H, \mathcal{E}\}$ is expanded to the oversampled graph $\widetilde{\mathcal{G}} = \{\widetilde{L}, \widetilde{H}, \widetilde{\mathcal{E}}\}$ that $\widetilde{L}$ and $\widetilde{H}$ includes L and H, respectively. The downsampling matrices $\mathbf{J}_{\widetilde{L}}$ and $\mathbf{J}_{\widetilde{H}}$ of the oversampled graph are defined by $\widetilde{L}$ and $\widetilde{H}$. The oversampled signal $\widetilde{\boldsymbol{f}}$ is written as

$$\widetilde{\boldsymbol{f}} = \begin{bmatrix} \boldsymbol{f} \\ \boldsymbol{f}' \end{bmatrix}, \tag{8}$$

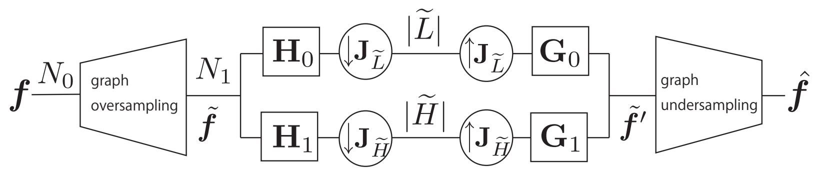


Fig. 2. Graph oversampling followed by the critically sampled graph filter bank.

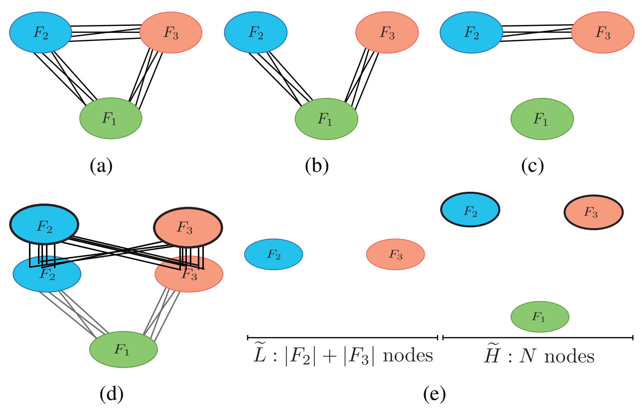


Fig. 3. Bipartite oversampled graph construction for a three colorable graph. (a) Three-colorable graph whose node sets are F_1, F_2 and F_3. (b) Bipartite subgraph G_1. (c) Bipartite subgraph G_2. (d) Oversampled bipartite graph. The gray lines are edges contained in G_1 and the black lines are additional edges. (e) The sets $\widetilde{L}$ and $\widetilde{H}$ of the oversampled bipartite graph.

where $\boldsymbol{f}'$ is the signal for additional nodes and its length is $N_1 - N_0$. The spectral domain filtering is performed based on the oversampled GLM.

Let $\mathbf{A}_0$ be an adjacency matrix of the original bipartite graph whose size is $N_0 \times N_0$. The normalized oversampled GLM $\widetilde{\boldsymbol{\mathcal{L}}}$ is $N_1 \times N_1$ ($N_1 > N_0$), and $N_1 - N_0$ is the number of the additional nodes. It is represented as

$$\widetilde{\boldsymbol{\mathcal{L}}} = \widetilde{\mathbf{D}}^{-1/2}\widetilde{\mathbf{L}}\widetilde{\mathbf{D}}^{-1/2} \tag{9}$$

where

$$\widetilde{\mathbf{L}} = \widetilde{\mathbf{D}} - \widetilde{\mathbf{A}} \tag{10}$$

$$\widetilde{\mathbf{A}} = \begin{bmatrix} \mathbf{A}_0 & \mathbf{A}_{01} \\ \mathbf{A}_{01}^T & \mathbf{0}_{N_1 - N_0} \end{bmatrix}, \tag{11}$$

in which $\widetilde{\mathbf{A}}$ is the oversampled adjacency matrix whose size is $N_1 \times N_1$ and $\widetilde{\mathbf{D}}$ is a degree matrix that normalizes the new GLM. Additionally, $\mathbf{A}_{01}$ contains information on the connection between the original nodes and appended ones. Note that nodes are appended so that $\widetilde{\boldsymbol{\mathcal{L}}}$ is still a bipartite graph. The filters in Fig. 2 can be represented as $\mathbf{H}_i = h_i(\widetilde{\boldsymbol{\mathcal{L}}})$ and $\mathbf{G}_i = g_i(\widetilde{\boldsymbol{\mathcal{L}}})$ for $i = 0, 1$.

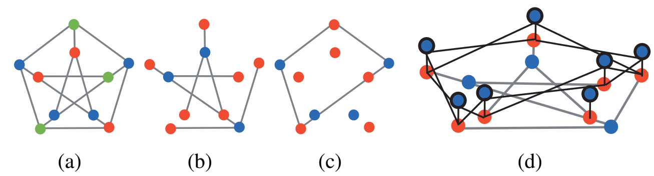


Fig. 4. (a) Petersen graph. (b) Bipartite subgraph #1. (c) Bipartite subgraph #2. (d) Proposed bipartite graph. The gray lines are edges contained bipartite subgraph #1 and the black lines are additional edges.

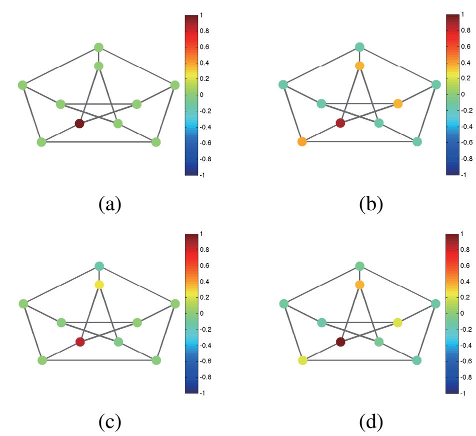
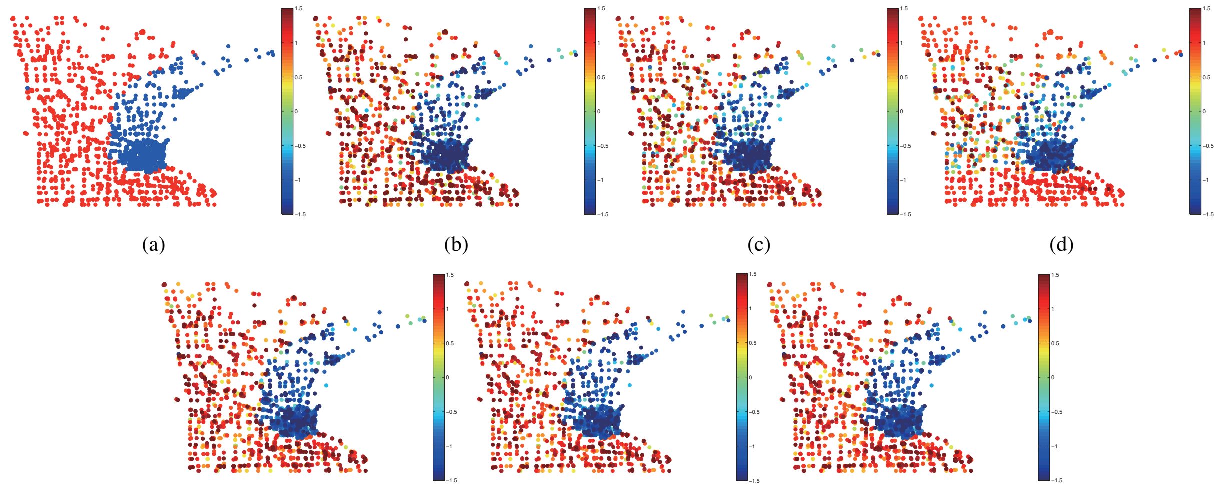


Fig. 5. Signal spread. (a) Input signal. (b) Lowpass filtered signal using the (non-bipartite) original graph. (c) Lowpass filtered signal using bipartite subgraph #1 (see Fig. 4(b)). (d) Lowpass filtered signal using oversampled bipartite graph (see Fig. 4(d)).

3.2. Graph Expansion Methods

As described in Section 3.1, the appended nodes of the oversampled GLM can be arbitrarily connected to the nodes, as long as the oversampled graph is bipartite. We describe an efficient way to construct such oversampled graphs. Since the oversampled graph has to be a bipartite graph, we first decompose the original graph into bipartite subgraphs. On the basis of one bipartite subgraph, we append nodes and edges in another bipartite subgraph to it. In this way, we can make one oversampled bipartite graph that has all edges of several bipartite subgraphs. For instance, the way to convert a three-colorable graph into one bipartite graph containing all edges



(a) (b) (c) (d)

(e) (f) (g)

Fig. 6. Denoising results. (a) Original signal. (b) Noisy signal ($\sigma = 1/2$). (c) Signal denoised by sym8 (1 level). (d) Signal denoised by sym8 (5 levels). (e) Signal denoised by graphBior(6,6). (f) Signal denoised by the graph Laplacian pyramid. (g) Signal denoised by the proposed method.

Table 1. Denoised Results: SNR (dB)

σ	1	1/2	1/4	1/8	1/16	1/32	redundancy
sym8 (1 level)	1.80	6.14	11.88	18.91	24.15	29.99	1.00
sym8 (5 levels)	3.08	5.61	11.07	18.27	24.14	30.09	1.00
graphBior [2]	2.81	8.38	14.49	20.65	25.52	31.42	1.00
graph Laplacian pyramid [15]	2.79	8.37	14.48	20.51	25.57	31.56	2.05
oversampled GLM + graphBior	**3.50**	**9.13**	**15.13**	**21.64**	**26.85**	**32.73**	2.05
noisy	0.15	5.81	12.06	18.41	23.94	30.07	–

of the original graph is described below.

The oversampled graph construction for a three-colorable graph is illustrated in Fig. 3. We assign three colors to nodes such that adjacent nodes have different colors and distinguish these nodes as F_1, F_2, and F_3, respectively. The original graph can be decomposed into two bipartite subgraphs, G_1 that contains edges linking F_1 and $F_2 \cup F_3$ and G_2 that contains edges between F_2 and F_3 (Figs. 3(b) and (c)). Hence, the edges in G_2 only have connections on one side of the subsets (F_2 and F_3) of G_1. By adding the nodes *just above* F_2 and F_3 and edges between F_2 and F_3 to G_1, we can convert the original graph into one bipartite graph that contains all edges and nodes in the original graph (Fig. 3(d)). Finally, the oversampled graph is decomposed by the critically sampled graph filter bank into sets $\widetilde{L}$ and $\widetilde{H}$, as shown in Fig. 3(e).

For example, the Petersen graph (Fig. 4(a)) is a well-known three-colorable graph, and it can be decomposed into two bipartite graphs (Figs. 4(b) and 4(c)). In order to make the oversampled bipartite graph shown in Fig. 4(d), we place blue nodes right above the red ones of the bipartite subgraph #1 (Fig. 4(b)) and add edges by referring to the information about the edges of the bipartite subgraph #2 (Fig. 4(c)). The additional blue nodes have the same values as the corresponding red nodes and are treated as $\boldsymbol{f}'$ in (8). Therefore, the oversampled graph can be regarded as the bipartite graph with all edges of the original graph.

4. EXPERIMENTAL RESULTS

This section describes experiments that assess the performances of the proposed method.

4.1. Signal Spread on Graphs

To demonstrate the advantage of the oversampled bipartite graph, we compared the signal spreads of the critically sampled graph and the oversampled one. The original graph in this case is the Petersen graph (Fig. 4(a)) and it is decomposed into the two bipartite subgraphs shown in Figs. 4(b) and 4(c). The input signal is shown in Fig. 5(a). The comparison is performed between the original bipartite graph (Fig. 4(b)) and the proposed bipartite graph (Fig. 4(d)). The low-pass filtered signals are shown in Figs. 5(b)–(d). As expected, the spread of the signal after using the oversampled bipartite graph is more similar to the original (non-bipartite) graph than that of the critically sampled bipartite graph.

4.2. Graph Signal Denoising

Here, graph signals corrupted by additive white Gaussian noise are denoised. We compared the proposed method with the regular one-dimensional wavelet *sym8* with one-level and five-level decompositions, the critically sampled filter bank (*graphBior(6, 6)*) [2], and the Laplacian pyramid for graph signals [15]. Since *sym8* treats the signal as a vector, it does not take into account the structure of the signals. All of graph-based methods perform one-level transforms and have the same number of coefficients in the lowpass channel. The lowest frequency subband was kept and the other high frequency subbands were hard-thresholded with the threshold $T = 3\sigma$, where σ is the standard deviation of the noise.

The original graph is the *Minnesota Traffic Graph*. It is three-colorable, therefore it is a good example of the oversampled bipartite graph introduced in Section 3.2. We make the oversampled graph and perform the critically sampled filter bank (*graphBior(6, 6)*) on that graph for the proposed method. Note that the number of the downsampled lowpass coefficients is the same as that of the critically sampled filter banks, whereas the number of highpass coefficients are the same as the input signal. The graph Laplacian pyramid uses the same bipartite graph and the downsampling operation as those of graphBior for the lowpass channel, in order to have the equal number of lowpass coefficients.

The original signal is shown in Fig. 6(a). Table 1 compares SNRs and the redundancy of the transforms. Figs. 6(c)–(g) show the denoised signals. The graph-based transforms outperform the regular wavelet transforms. The proposed method is much better than the graphBior for all noise levels. It also outperforms the graph Laplacian pyramid even though their redundancies are the same.

5. CONCLUSION

This paper presented the oversampling method for graph signals. It appends nodes and edges to the original graph to construct an oversampled GLM. We showed examples of the oversampled graphs for arbitrary graphs. The experiments were conducted on signal spreads and graph signal denoising, and the proposed method, that implements the critically sampled filter bank on the oversampled graph, outperformed the other transforms.

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