NORMALIZED RECURSIVE LEAST MODULI ALGORITHM WITH *p*-MODULUS OF ERROR AND *q*-NORM OF FILTER INPUT

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ABSTRACT

This paper proposes a new adaptation algorithm named Normalized Recursive Least Moduli (NRLM) algorithm which employs "p-modulus" of error and "q-norm" of filter input. p-modulus and q-norm are generalization of the modulus and norm used in complex-domain adaptive filters. The NRLM algorithm with p-modulus and q-norm makes adaptive filters fast convergent and robust against two types of impulse noise: one is found in observation noise and another at filter input. We develop theoretical analysis of the algorithm for calculating filter convergence. Through experiment with simulations and theoretical calculations, effectiveness of the proposed algorithm is demonstrated. We also find that the filter convergence does not critically depend on the value of p or q, allowing use of p = q = infinity that makes it easiest to calculate the p-modulus and q-norm. The theoretical convergence is in good agreement with the simulation results which validates the analysis.

***Index Terms*—**Adaptive filter, recursive least estimation, impulse noise, modulus, norm

1. INTRODUCTION

It is widely believed that adaptive filters consist of an essential part of the latest communication systems in practical applications. The LMS and NLMS algorithms are most popular adaptation algorithms used in many adaptive filters. These algorithms are intensively studied and attract many implementers for their excellent performance [1], [2]. However, their serious drawback is vulnerability to impulse noise. Impulse noise is found in practical adaptive filtering systems [3], [4]. Two types of impulse noise are identified: one is present in observation noise and another at filter input. It is known that the latter is often found in such applications as "active noise cancellation."

The author has long been interested in such adaptation algorithms which make adaptive filters robust against both types of impulse noise stated above and also fast convergent in the presence of highly correlated filter inputs. He has sought and proposed robust adaptation algorithms and methods for improving the filter convergence speed [5]-[10].

One of the effective algorithms for "de-correlation" is the Recursive Least Moduli (RLM) algorithm [6], from which we derive Normalized RLM algorithm. The NRLM algorithm is expected to be robust against both types of impulse noise and also realize fast convergence.

In the NRLM algorithm, we incorporate a modulus of error and a norm of filter input. In the paper, we derive a generalized modulus and a norm named *p-modulus* and *q-norm*. For a complex number $z = x + \mathrm{j}\, y$, we define *p*-modulus by $|z|_p = (|x|^p + |y|^p)^{1/p}$ $(p \geq 1)$. $|z|_2$ is the modulus usually denoted by $|z|$. For a complex vector $\mathbf{x} = [x_0 \cdots x_k \cdots x_{N-1}]^T$ of length N, q-norm of the vector $\mathbf{x}$ is defined by $\|\mathbf{x}\|_q = [\sum_{k=0}^{N-1} |x_k|^q]^{1/q}$ $(q \geq 1)$. $\|\mathbf{x}\|_2$ is the Euclidean norm usually written as $\|\mathbf{x}\|$. Using the *p*-modulus of error and *q*-norm of filter input, we can derive a modified NRLM algorithm which is denoted by "*q*NRL*p*M" for short.

In this paper, through theoretical analysis and experiment, we examine performance of the *q*NRL*p*M algorithm in the presence of both types of impulse noise and correlated filter inputs.

2. IMPULSE NOISE MODELS

In this section, we describe stochastic models used for two types of impulse noise found in adaptive filtering systems.

2.1. Impulsive Observation Noise

Impulse noise found in the additive observation noise is often modeled as Contaminated Gaussian Noise (CGN) that is mathematically a combination of two independent Gaussian noise sources [11], i.e., Gaussian noise $\nu^{(0)}(n)$ with variance $\sigma^2_\nu{}^{(0)}$ and probability of occurrence $p_\nu^{(0)}$, and $\nu^{(1)}(n)$ with $\sigma^2_\nu{}^{(1)}$ and $p_\nu^{(1)}$, where n is the time instant. Note that $p_\nu^{(0)} + p_\nu^{(1)} = 1$ holds. Usually, $\sigma^2_\nu{}^{(1)} \gg \sigma^2_\nu{}^{(0)}$ and $p_\nu^{(1)} < p_\nu^{(0)}$. The variance of CGN is given by $\sigma^2_\nu = p_\nu^{(0)}\sigma^2_\nu{}^{(0)} + p_\nu^{(1)}\sigma^2_\nu{}^{(1)}$. For "pure" Gaussian noise, $\sigma^2_\nu = \sigma^2_\nu{}^{(0)}$ and $p_\nu^{(1)} = 0$.

2.2. Impulse Noise at Filter Input [8]

A "noisy" filter input $b(n)$ with impulse noise added to the reference input $a(n)$ is given by $b(n) = a(n) + \tau(n)\, \nu_a(n)$, where $\tau(n)$ is an independent Bernoulli random variable taking 1 with probability $p_{\nu a}$ and 0 with $1 - p_{\nu a}$. The impulse noise $\nu_a(n)$ itself is assumed to be an independent White & Gaussian process with variance $\sigma^2_{\nu a}$.

3. NORMALIZED LEAST MEAN MODULUS ALGORITHM

Define a cost function of the error as

$$L_e(n) = | e(n) | / \| \mathbf{a}(n) \|,$$

where $| e(n) |$ is the modulus of the error and $\| \mathbf{a}(n) \|$ is the Euclidean norm of the filter reference input vector $\mathbf{a}(n) = [a(n) \cdots a(n-k) \cdots a(n-N+1)]^T$. Taking the gradient of $L_e(n)$ with respect to the tap weight vector $\mathbf{c}(n)$, we derive a tap weight update equation for *Normalized Least Mean Modulus* (NLMM) algorithm [7] as given by

$$\mathbf{c}(n+1) = \mathbf{c}(n) + \alpha_c\, e^*(n) / | e(n) |\, \mathbf{a}(n) / \| \mathbf{a}(n) \|, \quad (1)$$

in which α_c is the step size and $(\,\cdot\,)^*$ denotes complex conjugate. The error $e(n) = \in(n) + \nu(n)$, where $\in(n) = \boldsymbol{\theta}^H(n)\, \mathbf{a}(n)$ is the excess error, $\boldsymbol{\theta}(n) = \mathbf{h} - \mathbf{c}(n)$ is the tap weight misalignment vector, $\mathbf{h}$ is the response vector of an unknown system to be identified, and $\nu(n)$ is the observation noise.

4. *p*-MODULUS AND *q*-NORM

4.1. *p*-Modulus

In this subsection, we define a generalized modulus of a complex number $z = x + \mathrm{j}\, y$ [9]. Define and denote

$$| z |_p = (| x |_p + | y |_p)^{1/p} \quad (p \geq 1)$$

which may be named "*p*-modulus." Typical values of p for practical use are: $p = 1$, 2 and ∞. $| z |_2$ is the *modulus* usually denoted by $| z |$. Note that $| z |_\infty = \max\{| x |, | y |\}$ holds.

Next, assume that x and y are uncorrelated zero-mean Gaussian random variables with unit variance. Let us calculate expectations $E(1/| z |_p)$ and $E(| z |^2/| z |_p^2)$. On the polar coordinates, using the PDFs of x and y, we find

$$E(1/| z |_p) = (\pi/2)^{1/2}\, \zeta(p) \text{ and}$$
$$E(| z |^2/| z |_p^2) = \zeta_2(p)$$

with

$$\zeta(p) = (4/\pi) \int_0^{\pi/4} (\cos^p\varphi + \sin^p\varphi)^{-1/p}\, d\varphi \text{ and}$$
$$\zeta_2(p) = (4/\pi) \int_0^{\pi/4} (\cos^p\varphi + \sin^p\varphi)^{-2/p}\, d\varphi.$$

Numerically calculated values of $\zeta(p)$ and $\zeta_2(p)$ for some values of p are listed in Table 1 below. Ratios $\zeta_2(p)/\zeta(p)$ and $\zeta_2(p)/\zeta^2(p)$ are also given in the table.

Table 1. $\zeta(p)$ and $\zeta_2(p)$ versus p and ratios on ω.

p	$\zeta(p)$	$\zeta_2(p)$	$\zeta_2(p)/\zeta(p)$	$\zeta_2(p)/\zeta^2(p)$
1	0.793515	0.636620	0.8023	1.0110
2	1.000000	1.000000	1.0000	1.0000
8	1.111705	1.245815	1.1206	1.0080
32	1.121493	1.271285	1.1336	1.0108
128	1.122155	1.273113	1.1345	1.0110
∞	1.122200	1.273240	1.1346	1.0110

In the table above, we find that the ratio $\zeta_2(p)/\zeta(p)$ increases as p increases to converge to an upper bound. However, the ratio $\zeta_2(p)/\zeta^2(p)$ takes a value 1 to 1.01 and does not critically depend on the value of p.

4.2. *q*-Norm

Let $\mathbf{x} = [x_0 \cdots x_k \cdots x_{N-1}]^T$ be a complex-valued vector. Then, "*q*-norm" of the vector $\mathbf{x}$ is defined by

$$\| \mathbf{x} \|_q = [\textstyle\sum_{k=0}^{N-1} | x_k |^q]^{1/q} \quad (q \geq 1) \text{ [10]},$$

where $| x_k |$ is the modulus of x_k. Typical values of q are: $q = 1$, 2 and ∞. For $q = \infty$, "infinity-norm (or ∞-norm)" is calculated as $\| \mathbf{x} \|_\infty = \max\{| x_0 |, \cdots | x_k |, \cdots | x_{N-1}|\}$.

Next, let $\mathbf{x} = \mathbf{x}(n)$ be a stationary zero-mean Gaussian process with uncorrelated real and imaginary parts, covariance matrix $\mathbf{R_x} = E[\mathbf{x}(n)\mathbf{x}^H(n)]/2$ and unit variance. We define the following expectations.

$$\omega(q) = E(1/\| \mathbf{x} \|_q) \text{ and}$$
$$\omega_2(q) = E(1/\| \mathbf{x} \|_q^2).$$

Generally, it is found difficult to calculate these expectations analytically.

In order to obtain $\omega(q)$ and $\omega_2(q)$ approximately, we run simulations to calculate ensemble averages $\langle 1/\| \mathbf{x} \|_q \rangle$ and $\langle 1/\| \mathbf{x} \|_q^2 \rangle$, respectively, as listed in Table 2 below, where $N = 32$ and $\mathbf{x}(n)$ is an AR1 process with regression coefficient $\eta = 0.9$. In the table, ratios $\omega_2(q)/\omega(q)$ and $\omega_2(q)/\omega^2(q)$ are also given. The number of samples of $\mathbf{x}$ is 10 million.

Table 2. Simulation results and ratios on ω.

q	$\langle 1/\| \mathbf{x} \|_q \rangle$	$\langle 1/\| \mathbf{x} \|_q^2 \rangle$	$\omega_2(q)/\omega(q)$	$\omega_2(q)/\omega^2(q)$
1	0.02658	0.000753	0.0283	1.064
2	0.1363	0.01968	0.144	1.059
8	0.3813	0.1529	0.401	1.052
32	0.4311	0.1955	0.453	1.052
128	0.4348	0.1988	0.457	1.052
∞	0.4350	0.1990	0.458	1.052

From the results above, we learn that the ratio $\omega_2(q)/\omega(q)$ increases as q increases to converge to an upper bound, whereas the ratio $\omega_2(q)/\omega^2(q)$ takes a value 1.05 to 1.06 and thus it does very weakly depend on q.

5. NORMALIZED RECURSIVE LEAST MODULI ALGORITHM WITH *p*-MODULUS OF ERROR AND *q*-NORM OF FILTER INPUT

In this section, we propose a new adaptation algorithm related to the NLMM algorithm described in Section 3.

In the tap weight update equation (1), we first replace the modulus $| e(n) |$ by *p*-modulus $| e(n) |_p$ and the norm $\| \mathbf{a}(n) \|$ by *q*-norm $\| \mathbf{a}(n) \|_q$, yielding "*q*NLM*p*M" algorithm. Next, we introduce a recursive least estimate of the *inverse* covariance matrix of the filter input as used in the well-known RLS algorithm.

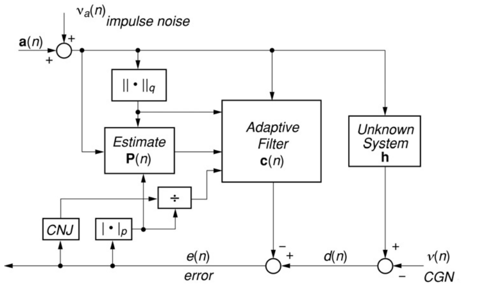

Fig. 1. Schematic diagram for *q*NRL*p*M algorithm.

The resulting tap weight update equation is given by

$$\mathbf{c}(n+1) = \mathbf{c}(n) + \alpha_c\, \mathbf{P}(n)\, e^*(n) \,/\, |\, e(n)\, |_p \cdot \mathbf{a}(n) \,/\, \|\, \mathbf{a}(n)\, \|_q, \quad (2)$$

where the estimate $\mathbf{P}(n)$ is iteratively calculated by either of the two methods below.

Method <A> (*Indirect Method*)

$$\mathbf{P}(n+1) = \mathbf{Q}^{-1}(n+1) \quad (3a)$$

with

$$\mathbf{Q}(n+1) = \lambda\, \mathbf{Q}(n) + 1 \,/\, |\, e(n)\, |_p \cdot \mathbf{a}(n)\, \mathbf{a}^H(n) \,/\, \|\, \mathbf{a}(n)\, \|_q, \quad (3b)$$

or

Method <B> (*Direct Method*)

$$\mathbf{P}(n+1) = \lambda^{-1}\{\mathbf{P}(n) - \mathbf{P}(n)\, \mathbf{a}(n)\, \mathbf{a}^H(n)\, \mathbf{P}(n) \,/\, [\lambda\, |\, e(n)\, |_p \|\, \mathbf{a}(n)\, \|_q + \mathbf{a}^H(n)\, \mathbf{P}(n)\, \mathbf{a}(n)]\}, \quad (4)$$

where λ is the forgetting factor.

The adaptation algorithm proposed above is *Normalized Recursive Least Moduli Algorithm With p-Modulus of Error and q-Norm of Filter Input*, denoted as "*q*NRL*p*M." Fig. 1 is a schematic diagram for the *q*NRL*p*M algorithm.

6. PERFORMANCE ANALYSIS

In this section, for ease of analysis, we assume *absence* of impulse noise. For ease of reading, only main results of the analysis are summarized, leaving the details out.

6.1. Assumptions

A1: The number of tap weights N is sufficiently large.

A2: The filter reference input $\mathbf{a}(n)$ is a Gaussian process with covariance matrix $\mathbf{R_a} = E[\mathbf{a}(n)\mathbf{a}^H(n)]/2$ and variance $\sigma^2_a = E[|\, a(n)\, |^2]/2$.

A3: The filter input $\mathbf{a}(n)$ and the tap weights $\mathbf{c}(n)$ are mutually independent (*Independence Assumption*).

A4: The estimate $\mathbf{P}(n)$ is independent of $e(n)$ and $\mathbf{a}(n)$.

A5: The error $e(n)$ given $\mathbf{a}(n)$ is Gaussian distributed.

6.2. Difference Equations for Tap Weight Misalignment

From (2), we obtain an update equation for $\boldsymbol{\theta}(n)$ given by

$$\boldsymbol{\theta}(n+1) = \boldsymbol{\theta}(n) - \alpha_c\, \mathbf{P}(n)\, e^*(n) \,/\, |\, e(n)\, |_p \cdot \mathbf{a}(n) \,/\, \|\, \mathbf{a}(n)\, \|_q$$

whence we derive a set of difference equations for the mean vector $\mathbf{m}(n) = E[\boldsymbol{\theta}(n)]$ and the second-order moment matrix $\mathbf{K}(n) = E[\boldsymbol{\theta}(n)\boldsymbol{\theta}^H(n)]$:

$$\mathbf{m}(n+1) = \mathbf{m}(n) - \alpha_c\, \mathbf{p}(n) \quad \text{and}$$

$$\mathbf{K}(n+1) = \mathbf{K}(n) - \alpha_c\, [\mathbf{V}(n) + \mathbf{V}^H(n)] + \alpha^2_c\, \mathbf{T}(n), \quad (5)$$

where $\mathbf{p}(n) = E[\mathbf{P}(n)]\, E[e^*(n)/|\, e(n)\, |_p \cdot \mathbf{a}(n)/\|\, \mathbf{a}(n)\, \|_q]$, $\mathbf{V}(n) = E[\mathbf{P}(n)]\, E[e^*(n)/|\, e(n)\, |_p \cdot \mathbf{a}(n)/\|\, \mathbf{a}(n)\, \|_q\, \boldsymbol{\theta}^H(n)]$, $\mathbf{T}(n) \cong E[\mathbf{P}(n)]\, \mathbf{T_a}\, E[\mathbf{P}(n)]$, and $\mathbf{T_a} = E[|\, e(n)\, |^2/|\, e(n)\, |_p^2 \cdot \mathbf{a}(n)\mathbf{a}^H(n)/\|\, \mathbf{a}(n)\, \|_q^2]$. For N>>1, we calculate $\mathbf{p}(n) = E[\mathbf{P}(n)]\, \mathbf{W}(n)\, \mathbf{m}(n)$ and $\mathbf{V}(n) = E[\mathbf{P}(n)]\, \mathbf{W}(n)\, \mathbf{K}(n)$, where

$$\mathbf{W}(n) \cong E[1/|\, e(n)\, |_p]\, (1/2)E[\mathbf{a}(n)\mathbf{a}^H(n)]E[1/\|\, \mathbf{a}(n)\, \|_q]$$

$$\cong (\pi/2)^{1/2}\, \zeta(p)\, \omega(q) \,/\, [\sigma_e(n)\, \sigma_a] \cdot \mathbf{R_a},$$

$$\mathbf{T_a} \cong \zeta_2(p)\, \omega_2(q) \,/\, \sigma^2_a \cdot 2\mathbf{R_a},$$

$\sigma^2_e(n) = \varepsilon(n) + \sigma^2_v$ is the error variance, and we define Excess Mean Square Error (EMSE) by $\varepsilon(n) = E[|\epsilon(n)|^2]/2 = \mathrm{tr}[\mathbf{R_a}\mathbf{K}(n)]$.

6.3. Calculation of E[P(n+1)] – Analysis of Method <A>

From (3b), we derive a difference equation

$$E[\mathbf{Q}(n+1)] = \lambda\, E[\mathbf{Q}(n)] + \boldsymbol{\Xi}_\mathbf{Q}(n),$$

where, for N>>1,

$$\boldsymbol{\Xi}_\mathbf{Q}(n) = 2\mathbf{W}(n)$$

$$\cong (2\pi)^{1/2}\, \zeta(p)\, \omega(q) \,/\, [\sigma_e(n)\, \sigma_a] \cdot \mathbf{R_a}.$$

Having updated $E[\mathbf{Q}(n+1)]$, we approximately calculate in (3a) $E[\mathbf{P}(n+1)] = E[\mathbf{Q}^{-1}(n+1)] \cong E[\mathbf{Q}(n+1)]^{-1}$. It is known that this $E[\mathbf{P}(n+1)]$ is less accurate than that calculated by the analysis of Method <B> in the next subsection.

6.4. Calculation of E[P(n+1)] – Analysis of Method <B>

From (4), we derive a difference equation

$$E[\mathbf{P}(n+1)] = \lambda^{-1}E[\mathbf{P}(n)]\{\mathbf{I} - \boldsymbol{\Phi}_\mathbf{P}(n)\, E[\mathbf{P}(n)]\},$$

where $\mathbf{I}$ is identity matrix and

$$\boldsymbol{\Phi}_\mathbf{P}(n) = E\{\mathbf{a}(n)\mathbf{a}^H(n)/[\lambda\, |\, e(n)\, |_p \|\, \mathbf{a}(n)\, \|_q + \mathbf{a}^H(n)\mathbf{P}(n)\mathbf{a}(n)]\}.$$

For N>>1, though details are not given, we finally obtain

$$\boldsymbol{\Phi}_\mathbf{P}(n) \cong S_c[y(n)] \,/\, \lambda \cdot \boldsymbol{\Xi}_\mathbf{Q}(n) \text{ [12]},$$

where we define a function

$$S_c(x) = (2/\pi)^{1/2} \int_0^\infty t \,/\, (t + x) \cdot \exp(-t^2/2)\, dt \quad (x \geq 0) \quad \text{and}$$

$$y(n) = 2\, \zeta(p)\, \omega(q)\, \mathrm{tr}\{\mathbf{R_a}E[\mathbf{P}(n)]\} \,/\, [\lambda\, \sigma_e(n)\, \sigma_a].$$

6.5. Initial Conditions

For $\mathbf{c}(0) = \mathbf{0}$, we have $\mathbf{m}(0) = \mathbf{h}$ and $\mathbf{K}(0) = \mathbf{h}\, \mathbf{h}^H$. The initial value of the estimate matrix is usually assumed to be $\mathbf{P}(0) = P_0\, \mathbf{I}$, where $P_0 \cong (\pi/2)^{1/2}\, \zeta(p)/\zeta_2(p) \cdot \omega(q)/\omega_2(q) \,/\, (2\alpha_c N)$ that minimizes $\varepsilon(1)$.

6.6. Steady-State Solution

As $n \to \infty$, we derive, from (5), an equation

$$\mathbf{K}(\infty) = (\alpha_c/2)\mathbf{W}^{-1}(\infty)\mathbf{T_a}\, E[\mathbf{P}(\infty)],$$

where for Method <A> $E[\mathbf{P}(\infty)] \cong \lambda_c\, \boldsymbol{\Xi}_\mathbf{Q}^{-1}(\infty)$, and for Method <B> $E[\mathbf{P}(\infty)] \cong \lambda_c\, \rho_P\, \boldsymbol{\Xi}_\mathbf{Q}^{-1}(\infty)$. Here, $\lambda_c = 1 - \lambda$ and ρ_P (>1) is a root of an equation $\rho_P = \lambda \,/\, S_c[(2/\pi)^{1/2}\, \lambda_c N \rho_P \,/\, \lambda]$. Then, we can solve the steady-state EMSE

$$\varepsilon(\infty) = \delta \,/\, (1 - \delta) \cdot \sigma^2_v$$

with

$$\delta = \alpha_c\, \lambda_c\, \rho_P\, \pi^{-1}[\zeta_2(p)/\zeta^2(p)]\, [\omega_2(q)/\omega^2(q)]\, N. \quad (6)$$

Note here that on the RHS of (6) the ratio $\zeta_2(p)/\zeta^2(p)$ in the first square bracket has a value 1 to 1.01 (see Table 1), and the ratio $\omega_2(q)/\omega^2(q)$ in the second square bracket takes a value 1.05 to 1.06 (see Table 2). Thus, the steady-state EMSE for the proposed *q*NRL*p*M algorithm does not critically depend on the value of either p or q. We can even select $p = q = \infty$, for which it is easiest to calculate the p-modulus of the error and the q-norm of the filter input.

For $N = 32$, an AR1 input with regression coefficient $\eta = 0.9$, $\sigma^2_v = 0.01$, $\alpha_c = 1$ and $\lambda_c = 2^{-10}$, we calculate $\varepsilon(\infty) \cong -39.5$ dB for $p = q = 1$, and $\varepsilon(\infty) \cong -39.6$ dB for $p = q = \infty$.

7. EXPERIMENT

In this section, experiment is carried out with simulations and theoretical calculations of adaptive filter convergence using the *q*NRL*p*M algorithm. The following examples are prepared for the experiment. The simulation result is an ensemble average of squared excess error $\langle|\epsilon(n)|^2\rangle/2$ over 1000 independent runs of filter convergence.

Example #1 $N = 32$

filter input: AR1 Gaussian process with
$\sigma^2_a = 1$ (0 dB) and $\eta = 0.9$
"pure" Gaussian noise: $\sigma^2_v = 0.01$ (–20 dB)
(a) $p = q = 1$, (b) $p = q = 2$, (c) $p = q = 8$,
(d) $p = q = 32$, (e) $p = q = 128$, (f) $p = q = \infty$
for *q*NRL*p*M: $\alpha_c = 1$, $\lambda_c = 2^{-10}$
simulated convergence

Example #2 $N = 32$

filter input: same as in *Example #1*
"pure" Gaussian noise: $\sigma^2_v = 0.01$ (–20 dB)
(a) $p = q = 1$, (b) $p = q = \infty$
for *q*NRL*p*M: $\alpha_c = 1$, $\lambda_c = 2^{-10}$
for *q*NLM*p*M: (a) $\alpha_c = 2^{-9}$, (b) $\alpha_c = 2^{-13}$
analysis of Methods <A> and <B>

Example #3 $N = 32$

filter input: same as in *Example #1*
$p = q = \infty$
for *q*NRL*p*M: $\alpha_c = 1$, $\lambda_c = 2^{-10}$
Case 1: "pure" Gaussian noise $\sigma^2_v = 0.01$
no impulse noise at filter input
Case 2: CGN $\sigma^{2\,(0)}_v = 0.01$; $p_v^{(0)} = 0.9$
$\sigma^{2\,(1)}_v = 10$; $p_v^{(0)} = 0.1$
no impulse noise at filter input
Case 3: "pure" Gaussian noise $\sigma^2_v = 0.01$
impulse noise at filter input
$\sigma^2_{va} = 1000$; $p_{va} = 0.1$
Case 4: CGN as in Case 2
impulse noise at filter input
as in Case 3
analysis of Method <B>

For *Example #1*, simulated filter convergence for some values of p and q is given in Table 3.

Table 3. Ensemble average $\langle|\epsilon(n)|^2\rangle/2$ (dB) versus n.

n	p=q=1	p=q=2	p=q=8	p=q=32	p=q=128	p=q=∞
0	0.5	0.4	0.3	0.4	0.5	0.3
200	–18.9	–18.8	–18.6	–18.6	–18.7	–18.6
400	–25.0	–24.6	–24.7	–24.5	–24.7	–24.6
800	–29.7	–29.6	–29.9	–29.7	–29.7	–29.9
1200	–32.0	–32.1	–32.2	–32.2	–32.4	–32.4
2000	–35.3	–35.3	–35.4	–35.3	–35.4	–35.4
4000	–38.7	–38.5	–38.5	–38.4	–38.4	–38.6
6000	–39.3	–39.2	–39.2	–39.1	–39.4	–39.3
8000	–39.4	–39.3	–39.4	–39.2	–39.4	–39.5

In Table 3, we observe that the filter convergence curves are almost the same for the above p and q. In fact, the filter convergence curves are indistinguishable if plotted on a graph. Generally, the filter convergence does not critically depend on any combination of p and q.

In *Example #2*, results of experiment are given in Fig. 2 for $p = q = 1$ and in Fig. 3 for $p = q = \infty$. In the figures, theoretical convergence curves are calculated and drawn according to the analysis of Methods <A> and <B>, in comparison with simulation results. We observe that the theoretical convergence for Method <B> is more accurate than that for Method <A>. Also in the figures, theoretical convergence for the *q*NLM*p*M algorithm is plotted. We find that the convergence for the proposed *q*NRL*p*M algorithm is much faster than that for *q*NLM*p*M algorithm, showing the effectiveness of the recursive least estimation.

In Fig. 4, results for Cases 1 to 4 of *Example #3* are shown for $p = q = \infty$. Case 1 is for "pure" Gaussian noise (no impulse noise). In Case 2, filter convergence in the presence of CGN is depicted, where we observe that the EMSE increase from Case 1 is only 1 dB. For Case 3 (impulse noise at filter input) and Case 4 (both types of impulse noise), simulation results are plotted which demonstrate sufficient robustness of the *q*NRL*p*M algorithm against *both* types of impulse noise.

In general, the theoretically calculated convergence (analysis of Method <B>) is in good agreement with the simulated convergence that validates the analysis developed.

8. CONCLUSION

This paper has proposed a new adaptation algorithm named *q*NRL*p*M algorithm. Stochastic models for two types of impulse noise have been used.

Through performance analysis and experiment, we have found that the algorithm realizes fast convergent and robust adaptive filters in the presence of both types of impulse noise. We have also learned that the filter convergence behavior does not critically depend on the value of either p or q, allowing us to use ∞-modulus of the error and ∞-norm of the filter input. Note that ∞-modulus and ∞-norm are easiest to calculate. Theoretical convergence has been found sufficiently accurate that validates the analysis.

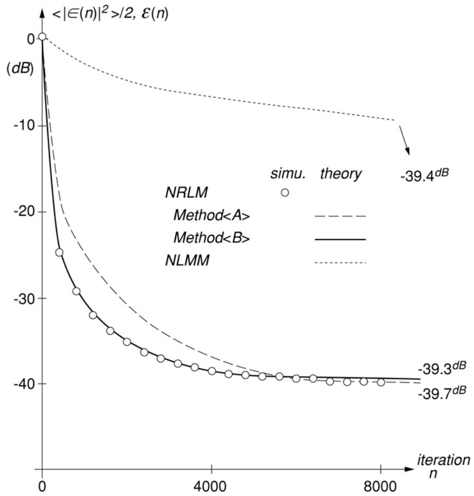

Fig. 2. Adaptive filter convergence (*Example #2*, $p = q = 1$).

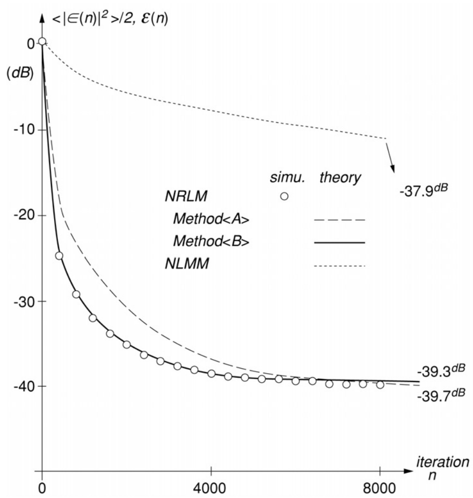

Fig. 3. Adaptive filter convergence (*Example #2*, $p = q = \infty$).

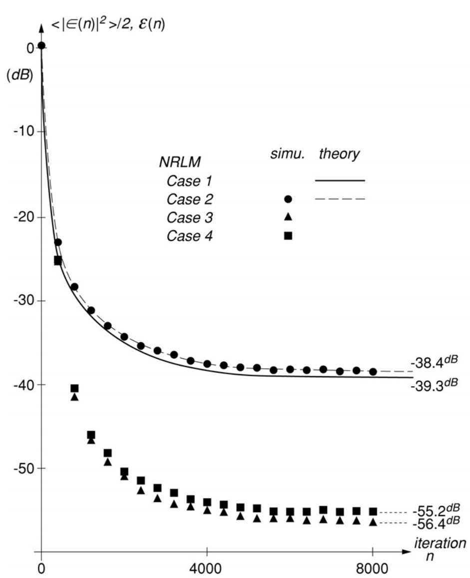

Fig. 4. Adaptive filter convergence (*Example #3*, Cases 1 to 4).

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