FOURTH-ORDER CONFAC DECOMPOSITION APPROACH FOR BLIND IDENTIFICATION OF UNDERDETERMINED MIXTURES

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ABSTRACT

We have recently proposed a second-order method for the blind identification of underdetermined mixtures that relies on the constrained factor (CONFAC) decomposition. It consists in storing successive second-order derivatives of the cumulant generating function (CGF) of the observations computed at different points of the observation space in a third-order tensor following a CONFAC model. In this work, we extend this approach to the case of third-order derivatives by resorting to a fourth-order CONFAC decomposition. We show how different third-order derivative types can be combined into a single fourth-order CONFAC tensor model with the goal of increasing the diversity of the observations, so that higher underdeterminacy levels can be handled. Computer simulation results illustrate the performance of a CONFAC-based blind identification algorithm compared to some competing methods.

***Index Terms*—** Blind identification, cumulant generating function, complex sources, CONFAC decomposition.

1. INTRODUCTION

The problem of blind identification of underdetermined mixtures can be solved by resorting to second or higher even order statistics of the observations (see, e.g. [1–7] and references therein). When the diversity of the observations is not sufficient, one can resort to a second class of tensor-based methods that rely on the multilinearity properties of higher-order statistics (HOS) [3, 8]. A large majority of these methods solves the blind identification problem for underdetermined mixtures by means of the canonical polyadic (CP) decomposition[1] [12, 13] of a tensor storing the cumulants of the observations [1–3,9–11]. A particular class of blind identification methods exploiting the second characteristic function (CAF) of the observations, has been proposed in a few works [5, 14, 15]. In [5], the alternating least squares (ALS) algorithm is applied to blindly estimate the mixing matrix from a data tensor constructed from third-order derivatives of the characteristic function of the observations. In [16], we have considered a more general scenario where the sources are assumed to be complex (e.g. 4-PSK or 4-QAM). More recently [17], we have shown that the CGF-based blind underdetermined mixture identification problem can be addressed by means of the constrained factor (CONFAC) decomposition [18]. Therein, the authors propose to store a collection of successive second-order derivatives of the CGF of the observations in a third-order tensor. Exploiting three different derivative forms, an "extended" CONFAC tensor model is built and exploited for estimating the mixing matrix using the alternating least squares (ALS) algorithm.

In this work, we extend the approach of [17] to the case of third-order derivatives by resorting to a fourth-order CONFAC decomposition with known constrained structure. We combine the four derivative types of the generating functions of the observations into a single fourth-order CONFAC tensor model, which is exploited for blind identification on the mixing matrix. Compared to [17], the approach proposed in this work allows to further increase the diversity of the observations so that stronger underdeterminacy levels can be handled. Our numerical results show that the proposed solution offers improved performance over some competing blind identification algorithms.

Notations: In the following, vectors, matrices and tensors are denoted by lower case boldface ($\mathbf{a}$), upper case boldface ($\mathbf{A}$) and upper case calligraphic ($\mathcal{A}$) letters respectively. a_i is the i-th coordinate of vector $\mathbf{a}$ and $\mathbf{a}_i$ is the i-th column of matrix $\mathbf{A}$. The (i,j) entry of matrix $\mathbf{A}$ is denoted A_{ij} and the (i,j,k) entry of the third order tensor $\mathcal{A}$ is denoted $\mathcal{A}_{ijk}$. Complex objects are underlined, their real and imaginary parts are denoted $\Re\{\cdot\}$ and $\Im\{\cdot\}$ respectively. E[.] denotes the expected value of a random variable. $\mathbf{A}^T$ and $\mathbf{A}^\dagger$ stand, respectively, for the transpose and Moore-Penrose pseudo-inverse of $\mathbf{A}$. The Kronecker and Khatri-Rao products are denoted by $\otimes$ and $\odot$, respectively.

2. THE FOURTH-ORDER CONFAC DECOMPOSITION

The constrained factor (CONFAC) decomposition of a fourth-order tensor $\mathcal{X} \in \mathbb{C}^{P\times Q\times R\times S}$ is given by [18]:

$$X_{i_1 i_2 i_3 i_4} = \sum_{f_1=1}^{F_1}\sum_{f_2=1}^{F_2}\sum_{f_3=1}^{F_3}\sum_{f_4=1}^{F_4} A^{(1)}_{i_1 f_1} A^{(2)}_{i_2 f_2} A^{(3)}_{i_3 f_3} A^{(4)}_{i_4 f_4} W_{f_1 f_2 f_3 f_4}, \tag{1}$$

where

$$W_{f_1 f_2 f_3 f_4} = \sum_{f=1}^{F} \Theta_{f_1 f}\Psi_{f_2 f}\Gamma_{f_3 f}\Omega_{f_4 f}, \tag{2}$$

$\{\mathbf{A}^{(n)}\} \in \mathbb{C}^{I_n\times F_n}$, $n = 1,2,3,4$, are *factor matrices*, $\mathbf{\Theta} \in \mathbb{R}^{F_1\times F}$, $\mathbf{\Psi} \in \mathbb{R}^{F_2\times F}$, $\mathbf{\Gamma} \in \mathbb{R}^{F_3\times F}$, $\mathbf{\Omega} \in \mathbb{R}^{F_4\times F}$ are *constraint matrices*, and $F \geq \max(F_i)$, $i = 1,2,3,4$. The constraint

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[1]This decomposition is sometimes also called Candecomp/Parafac, which can be referred to with the same acronynm.

matrices $\mathbf{\Theta}, \mathbf{\Psi}, \mathbf{\Gamma}$, and $\mathbf{\Omega}$ have full row-rank and their columns are canonical vectors possibly multiplied by -1. These canonical vectors belong to the bases $\mathbb{R}^{F_1}$, $\mathbb{R}^{F_2}$, $\mathbb{R}^{F_3}$ and $\mathbb{R}^{F_4}$, respectively. From (1)-(2), we can note that the fourth-order CONFAC decomposition can be viewed as a "constrained" Tucker decomposition [19] with the particular feature of having a core tensor that follows a rank-F sparse polyadic decomposition structure. At the same time, it can be interpreted as an "augmented" CP decomposition whose factor matrices may have repeated columns. The third-order CONFAC decomposition was introduced in [18] its uniqueness properties have been investigated in [20].

As will be shown later, in this work we resort to the fourth-order CONFAC decomposition (1)-(2) to solve the blind identification problem by exploiting third-order derivatives of the second generating of the observations.

3. PROBLEM FORMULATION

We consider a noisy linear mixture of K narrowband sources received by an array of N sensors. The mixing matrix is defined by $\mathbf{H} = [\mathbf{h}_1, \ldots, \mathbf{h}_K] \in \mathbb{R}^{N\times K}$. Define $\mathbf{z}(m) = [z_1(m), \ldots, z_N(m)]^T \in \mathbb{R}^N$, $\mathbf{s}(m) = [s_1(m), \ldots, s_K(m)]^T \in \mathbb{R}^K$ and $\mathbf{n}(m) \in \mathbb{R}^N$ as the m^{th} discrete-time realizations of the observations, source and additive white Gaussian noise vectors, respectively, $m = 1, \ldots, M$. According to this model we have:

$$\mathbf{z}(m) = \mathbf{H}\mathbf{s}(m) + \mathbf{n}(m). \tag{3}$$

The problem consists in finding $\widehat{\mathbf{H}}$ such that $\widehat{\mathbf{H}} = \mathbf{H}\mathbf{\Pi}\mathbf{\Lambda}$, where $\mathbf{\Pi}$ is a permutation matrix and $\mathbf{\Lambda}$ is a diagonal matrix. This means that $\mathbf{H}$ can be identified up to permutation and scaling of its columns. Column permutation and scaling are referred to as *trivial ambiguities*. The identification of the mixing matrix $\mathbf{H}$ relies on the following assumptions:

- H1. The mixing matrix $\mathbf{H}$ does not contain pairwise collinear columns;
- H2. The sources $s_1, \ldots, s_K$ are non-Gaussian and mutually statistically independent;
- H3. The number of sources K is known.

In the following, we recall from [16] the core equations that formulate the problem in the case of complex sources.

3.1. Second generating function of the observations

We recall from [16] the main steps that formulate the second-order derivatives of the cumulant generating function (CGF) of the observations in the case of complex sources. The CGF of the observations, Φ_z, can be decomposed in a sum of marginal second generating functions of the sources, φ_k, $k = 1\cdots K$. We start by defining Φ_z and φ_k in the complex field. The second generating function φ_k of the k-th source taken at the point x of $\mathbb{C}$ defined $\mathbb{R}^2$ is given by

$$\varphi_k(\Re\{x\}, \Im\{x\}) = \log \mathrm{E}[\exp(\Re\{x^* s_k\})]. \tag{4}$$

Similarly, the second generating function Φ_z of the observations taken at the point $\mathbf{w} = (\mathbf{u}, \mathbf{v})$ defined in $\mathbb{R}^{2N}$ can be written as

$$\Phi_z(\mathbf{u}, \mathbf{v}) \stackrel{\text{def}}{=} \log \mathrm{E}[\exp(\mathbf{x}^{\mathrm{T}}\mathbf{u} + \mathbf{y}^{\mathrm{T}}\mathbf{v})],$$

where $\mathbf{x} = \Re\{\mathbf{z}\}$ and $\mathbf{y} = \Im\{\mathbf{z}\}$. Define $\mathbf{A}$ and $\bar{\mathbf{A}}$ as the real and imaginary parts of the mixing matrix so that $\mathbf{H} = \mathbf{A} + i\bar{\mathbf{A}}$. Next, denote $\mathbf{a}_k$ (resp. $\bar{\mathbf{a}}_k$) the k-th column of $\mathbf{A}$ (resp. $\bar{\mathbf{A}}$). Replacing $\mathbf{z}$ by its model and using sources' mutual statistical independence hypothesis yields:

$$\Phi_z(\mathbf{u}, \mathbf{v}) = \sum_k \varphi_k\left(\mathbf{u}^{\mathrm{T}}\mathbf{a}_k + \mathbf{v}^{\mathrm{T}}\bar{\mathbf{a}}_k, \mathbf{v}^{\mathrm{T}}\mathbf{a}_k - \mathbf{u}^{\mathrm{T}}\bar{\mathbf{a}}_k\right) + \Phi_e(\mathbf{u}, \mathbf{v}), \tag{5}$$

where $\Phi_e(\mathbf{u}, \mathbf{v})$ is the corresponding second generating function of the Gaussian noise. From these definitions, we can rewrite (5) as

$$\Phi_z(\mathbf{w}) = \sum_k \varphi_k\left(g_1(\mathbf{w})\,,\, g_2(\mathbf{w})\right) + \Phi_e(\mathbf{w}), \tag{6}$$

where $g_1(\mathbf{w}) = \sum_n A_{nk}u_n + \bar{A}_{nk}v_n$ and $g_2(\mathbf{w}) = \sum_n A_{nk}v_n - \bar{A}_{nk}u_n$. Defining

$$\begin{aligned} g:\quad & \mathbb{R}^{2N} && \longrightarrow \mathbb{R}^2 \\ & \mathbf{w} && \longmapsto g(\mathbf{w}) = (g_1(\mathbf{w}), g_2(\mathbf{w})), \end{aligned}$$

yields a compact representation for (6) as

$$\Phi_z(\mathbf{w}) = \sum_k \varphi_k\left(g(\mathbf{w})\right) + \Phi_e(\mathbf{w}). \tag{7}$$

The differentiation of Φ_z and φ_k has been done in [16], and is summarized in the following section.

3.2. Decomposing third-order derivatives of $\Phi_z(\Re\{\mathbf{w}\}, \Im\{\mathbf{w}\})$

We focus on third-order derivatives of the CGF and compute partial derivatives of Φ_z in S different points of $\mathbb{R}^{2N}$, denoted here as $\mathbf{w}^{(s)} = (u^{(s)}, v^{(s)})$, $s = 1\cdots S$. Let $\{\Phi_z(\mathbf{w}^{(1)}), \Phi_z(\mathbf{w}^{(2)}), \ldots, \Phi_z(\mathbf{w}^{(S)})\}$ be the set containing the CGFs evaluated at S different points of the observation space, with $\mathbf{w}^{(s)} = (\mathbf{u}^{(s)}, \mathbf{v}^{(s)})$. Let us define

$$G_{sk}^{ijl} = \frac{\partial^3 \varphi_k(g(\mathbf{w}^{(s)}))}{\partial g_i(\mathbf{w}^{(s)})\partial g_j(\mathbf{w}^{(s)})\partial g_l(\mathbf{w}^{(s)})},$$

and note that $G_{sk}^{211} = G_{sk}^{121} = G_{sk}^{112}$ and $G_{sk}^{221} = G_{sk}^{122} = G_{sk}^{212}$. By successively differentiating (6), we can obtain the four different third-order derivative equations:

$$\begin{aligned} \frac{\partial^3 \Phi_z(\mathbf{w}^{(s)})}{\partial u_p \partial u_q \partial u_r} = \quad & \sum_{k=1}^K A_{pk}A_{qk}A_{rk}G_{sk}^{111} - \sum_{k=1}^K A_{pk}A_{qk}\bar{A}_{rk}G_{sk}^{211} - \\ & \sum_{k=1}^K A_{pk}\bar{A}_{qk}A_{rk}G_{sk}^{211} - \sum_{k=1}^K \bar{A}_{pk}A_{qk}A_{rk}G_{sk}^{211} + \\ & \sum_{k=1}^K A_{pk}\bar{A}_{qk}\bar{A}_{rk}G_{sk}^{221} + \sum_{k=1}^K \bar{A}_{pk}A_{qk}\bar{A}_{rk}G_{sk}^{221} + \\ & \sum_{k=1}^K \bar{A}_{pk}\bar{A}_{qk}A_{rk}G_{sk}^{221} - \sum_{k=1}^K \bar{A}_{pk}Bqk\bar{A}_{rk}G_{sk}^{222}, \end{aligned} \tag{8}$$

$$\frac{\partial^3\Phi_z(\mathbf{w}^{(s)})}{\partial u_p \partial u_q \partial v_r} = \sum_{k=1}^{K} A_{pk}A_{qk}\bar{A}_{rk}G_{sk}^{111} + \sum_{k=1}^{K} A_{pk}A_{qk}A_{rk}G_{sk}^{211} - \sum_{k=1}^{K} A_{pk}\bar{A}_{qk}\bar{A}_{rk}G_{sk}^{211} - \sum_{k=1}^{K} \bar{A}_{pk}A_{qk}\bar{A}_{rk}G_{sk}^{211} - \sum_{k=1}^{K} A_{pk}\bar{A}_{qk}A_{rk}G_{sk}^{221} - \sum_{k=1}^{K} \bar{A}_{pk}A_{qk}A_{rk}G_{sk}^{221} + \sum_{k=1}^{K} \bar{A}_{pk}\bar{A}_{qk}\bar{A}_{rk}G_{sk}^{221} + \sum_{k=1}^{K} \bar{A}_{pk}BqkA_{rk}G_{sk}^{222}, \tag{9}$$

$$\frac{\partial^3\Phi_z(\mathbf{w}^{(s)})}{\partial v_p \partial v_q \partial v_r} = \sum_{k=1}^{K} \bar{A}_{pk}\bar{A}_{qk}\bar{A}_{rk}G_{sk}^{111} + \sum_{k=1}^{K} \bar{A}_{pk}\bar{A}_{qk}A_{rk}G_{sk}^{211} + \sum_{k=1}^{K} \bar{A}_{pk}A_{qk}\bar{A}_{rk}G_{sk}^{211} + \sum_{k=1}^{K} A_{pk}\bar{A}_{qk}\bar{A}_{rk}G_{sk}^{211} + \sum_{k=1}^{K} \bar{A}_{pk}A_{qk}A_{rk}G_{sk}^{221} + \sum_{k=1}^{K} A_{pk}\bar{A}_{qk}A_{rk}G_{sk}^{221} + \sum_{k=1}^{K} A_{pk}A_{qk}\bar{A}_{rk}G_{sk}^{221} + \sum_{k=1}^{K} A_{pk}AqkA_{rk}G_{sk}^{222}, \tag{10}$$

$$\frac{\partial^3\Phi_z(\mathbf{w}^{(s)})}{\partial v_p \partial v_q \partial u_r} = \sum_{k=1}^{K} \bar{A}_{pk}\bar{A}_{qk}A_{rk}G_{sk}^{111} - \sum_{k=1}^{K} \bar{A}_{pk}\bar{A}_{qk}\bar{A}_{rk}G_{sk}^{211} + \sum_{k=1}^{K} \bar{A}_{pk}A_{qk}A_{rk}G_{sk}^{211} + \sum_{k=1}^{K} A_{pk}\bar{A}_{qk}A_{rk}G_{sk}^{211} - \sum_{k=1}^{K} \bar{A}_{pk}A_{qk}\bar{A}_{rk}G_{sk}^{221} - \sum_{k=1}^{K} A_{pk}\bar{A}_{qk}\bar{A}_{rk}G_{sk}^{221} + \sum_{k=1}^{K} A_{pk}A_{qk}A_{rk}G_{sk}^{221} - \sum_{k=1}^{K} A_{pk}Aqk\bar{A}_{rk}G_{sk}^{222}. \tag{11}$$

As can be seen from (8)-(11), each third-order derivative equation corresponds to a sum of eight fourth-order CP decompositions [12, 13]. Moreover, notice that for each derivative equation, the first-, second-, third- and fourth-mode factor matrices are appear repeated times across the different CP decompositions involved in the sum. Otherwise stated, different CP decompositions associated with a given derivative equation involve repeated real and imaginary parts of the mixing matrix, and the repetition pattern depends on the derivative equation.

4. CONFAC FORMULATION

Define $\mathcal{X}^{\Phi_t} \in \mathbb{R}^{N\times N\times N\times S}$, $t = 1,2,3,4$, as fourth-order tensors storing the third-order derivatives of $\Phi_z(\mathbf{w}^{(s)})$ taken at a specific point s of the observation space, $s = 1,\dots,S$, as follows:

$$\mathcal{X}^{\Phi_1}_{pqrs} \overset{\text{def}}{=} \frac{\partial^3\Phi_z(\mathbf{w}^{(s)})}{\partial u_p \partial u_q \partial u_r}, \quad \mathcal{X}^{\Phi_2}_{pqrs} \overset{\text{def}}{=} \frac{\partial^3\Phi_z(\mathbf{w}^{(s)})}{\partial u_p \partial u_q \partial v_r},$$
$$\mathcal{X}^{\Phi_3}_{pqrs} \overset{\text{def}}{=} \frac{\partial^3\Phi_z(\mathbf{w}^{(s)})}{\partial v_p \partial v_q \partial v_r}, \quad \mathcal{X}^{\Phi_4}_{pqrs} \overset{\text{def}}{=} \frac{\partial^3\Phi_z(\mathbf{w}^{(s)})}{\partial v_p \partial v_q \partial u_r} \tag{12}$$

where $\mathcal{X}^{\Phi_t}_{pqrs}$ is the (p,q,r,s)-th entry of tensor $\mathcal{X}^{\Phi_t}$, $t = 1,2,3,4$. We call $\mathcal{X}^{\Phi_1}$, $\mathcal{X}^{\Phi_2}$, $\mathcal{X}^{\Phi_3}$ and $\mathcal{X}^{\Phi_4}$ simply as "derivative tensors". Let $\mathbf{A}^{(k)} \in \mathbb{R}^{N\times 2}$ and $\mathbf{G}^{(k)} \in \mathbb{R}^{S\times 4}$, $k = 1,\dots,K$, be defined as:

$$\mathbf{A}^{(k)} \overset{\text{def}}{=} \begin{bmatrix} A_{1k} & \bar{A}_{1k} \\ \vdots & \vdots \\ A_{Nk} & \bar{A}_{Nk} \end{bmatrix} = [\mathbf{a}_k, \bar{\mathbf{a}}_k] \tag{13}$$

$$\mathbf{G}^{(k)} \overset{\text{def}}{=} \begin{bmatrix} G_{1k}^{111} & G_{1k}^{211} & G_{1k}^{221} & G_{1k}^{222} \\ \vdots & \vdots & \vdots & \\ G_{Sk}^{111} & G_{Sk}^{211} & G_{Sk}^{221} & G_{Sk}^{211} \end{bmatrix} = [\mathbf{g}_{1,k}, \mathbf{g}_{2,k}, \mathbf{g}_{3,k}, \mathbf{g}_{4,k}] \tag{14}$$

Using these definitions, as subsequently shown, we can decompose the t-th derivative tensor $\mathcal{X}^{\Phi_t}$, $t = 1,2,3,4$, as follows:

$$\mathcal{X}^{\Phi_t}_{pqrs} = \sum_{k=1}^{K} \Big(\underbrace{\sum_{f_1=1}^{2}\sum_{f_2=1}^{2}\sum_{f_3=1}^{2}\sum_{f_4=1}^{4} A^{(k)}_{pf_1}A^{(k)}_{qf_2}A^{(k)}_{rf_3}G^{(k)}_{sf_4}W^{(t)}_{f_1f_2f_3f_4}}_{\mathcal{X}^{\Phi_t(k)}_{pqrs}} \Big) \tag{15}$$

where

$$W^{(t)}_{f_1f_2f_3f_4} = \sum_{f=1}^{8} \Theta^{(t)}_{f_1f}\Psi^{(t)}_{f_2f}\Gamma^{(t)}_{f_3f}\Omega^{(t)}_{f_4f}. \tag{16}$$

which corresponds to a sum of K CONFAC decomposition blocks that yield the t-th derivative tensor $\mathcal{X}^{\Phi_t} \in \mathbb{R}^{N\times N\times N\times S}$ of the observations, $t = 1,2,3,4$. Its k-th block is given by a sum of 8 outer products involving repeated columns of matrices $\mathbf{A}^{(k)}$ and $\mathbf{G}^{(k)}$. The repetition pattern involving the columns of $\mathbf{A}^{(k)}$ and $\mathbf{G}^{(k)}$ are determined by the joint structure of $\mathbf{\Theta}^{(t)}$, $\mathbf{\Psi}^{(t)}$, $\mathbf{\Gamma}^{(t)}$, $\mathbf{\Omega}^{(t)}$, which in turn, depends on the pair of differentiation variables with respect to which the second generating function $\Phi_z(\mathbf{w}^{(s)})$ is successively derived. Comparing individually each derivative form (8), (9), (10), and (11) with the decomposition in (15), a possible structural choice for the constraint matrices satisfying the decomposition can be identified as follows:

$$\mathbf{\Theta}^{(t)} = \mathbf{\Theta} = \begin{bmatrix} 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{bmatrix}, \tag{17}$$

$$\mathbf{\Psi}^{(t)} = \mathbf{\Psi} = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \end{bmatrix}, \tag{18}$$

$$\mathbf{\Gamma}^{(t)} = \mathbf{\Gamma} = \begin{bmatrix} 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 \end{bmatrix}, \tag{19}$$

$$\mathbf{\Omega}^{(1)} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \tag{20}$$

$$\mathbf{\Omega}^{(2)} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & -1 & -1 & 0 & 0 \\ 0 & 0 & -1 & -1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}, \tag{21}$$

$$\mathbf{\Omega}^{(3)} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \tag{22}$$

$$\mathbf{\Omega}^{(4)} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & -1 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \tag{23}$$

$t = 1, 2, 3$. Other structures for $\mathbf{\Theta}^{(t)}$, $\mathbf{\Psi}^{(t)}$, $\mathbf{\Gamma}^{(t)}$, $\mathbf{\Omega}^{(t)}$ satisfying the decomposition are possible. Note that the convenience of the adopted choice comes from the fact that only the fourth-mode constraint matrix changes as the differentiation variables are changed, while the three first ones are fixed, i.e. they do not depend on the differentiation variables.

Define the block matrices

$$\mathbf{A} = [\mathbf{A}^{(1)}, \ldots, \mathbf{A}^{(K)}] \in \mathbb{R}^{N \times 2K} \tag{24}$$

$$\mathbf{G} = [\mathbf{G}^{(1)}, \ldots, \mathbf{G}^{(K)}] \in \mathbb{R}^{S \times 4K}, \tag{25}$$

which concatenate the contributions of the K sources, where $\mathbf{A}^{(k)}$ and $\mathbf{G}^{(k)}$ are defined in (13) and (14), respectively. Define also the block-diagonal constraint matrices

$$\bar{\mathbf{\Theta}} = \mathbf{I}_K \otimes \mathbf{\Theta} \in \mathbb{R}^{2K \times 8K}, \tag{26}$$

$$\bar{\mathbf{\Psi}} = \mathbf{I}_K \otimes \mathbf{\Psi} \in \mathbb{R}^{2K \times 8K}, \tag{27}$$

$$\bar{\mathbf{\Gamma}} = \mathbf{I}_K \otimes \mathbf{\Gamma} \in \mathbb{R}^{2K \times 8K}, \tag{28}$$

$$\bar{\mathbf{\Omega}}^{(t)} = \mathbf{I}_K \otimes \mathbf{\Omega}^{(t)} \in \mathbb{R}^{4K \times 8K}. \tag{29}$$

With these definitions, we can treat (15) simply as an "augmented" CONFAC decomposition composed of K blocks, the k-th block being associated with the k-th source. In this case, the following correspondences can be obtained by analogy with (1):

$$\begin{aligned}
&(\mathbf{A}^{(1)}, \mathbf{A}^{(2)}, \mathbf{A}^{(3)}, \mathbf{A}^{(4)}) \leftrightarrow (\mathbf{A}, \mathbf{A}, \mathbf{A}, \mathbf{G}),\\
&(\mathbf{\Theta}, \mathbf{\Psi}, \mathbf{\Gamma}, \mathbf{\Omega}) \leftrightarrow (\bar{\mathbf{\Theta}}, \bar{\mathbf{\Psi}}, \bar{\mathbf{\Gamma}}, \bar{\mathbf{\Omega}}^{(t)}),\\
&(F_1, F_2, F_3, F_4, F) \leftrightarrow (2K, 2K, 2K, 4K, 8K),\\
&(P, Q, R, S) \leftrightarrow (N, N, N, S).
\end{aligned}$$

Let us define $\mathbf{X}_1^{(t)} \in \mathbb{C}^{N^3 \times S}$ as a matrix unfolding of the t-th derivative tensor $\mathcal{X}^{\Phi_t}$, $t = 1, \ldots, T$. It can be shown [18] that this matrix admits the following quadrilinear CONFAC factorization:

$$\mathbf{X}_1^{(t)} = \big((\mathbf{A}\bar{\mathbf{\Theta}}) \odot (\mathbf{A}\bar{\mathbf{\Psi}}) \odot (\mathbf{A}\bar{\mathbf{\Gamma}})\big)(\mathbf{G}\bar{\mathbf{\Omega}}^{(t)})^T. \tag{30}$$

Following the approach of [17], we take all the four derivative types into account by defining a $\bar{\mathbf{X}}_1 = [\mathbf{X}_1^{(1)T}, \ldots, \mathbf{X}_1^{(4)T}]^T \in \mathbb{R}^{4N^3 \times S}$, yielding a bigger CONFAC model as follows

$$\bar{\mathbf{X}}_1 = \begin{bmatrix} \big((\mathbf{A}\bar{\mathbf{\Theta}}) \odot (\mathbf{A}\bar{\mathbf{\Psi}}) \odot (\mathbf{A}\bar{\mathbf{\Gamma}})\big)(\mathbf{G}\bar{\mathbf{\Omega}}^{(1)})^T \\ \vdots \\ \big((\mathbf{A}\bar{\mathbf{\Theta}}) \odot (\mathbf{A}\bar{\mathbf{\Psi}}) \odot (\mathbf{A}\bar{\mathbf{\Gamma}})\big)(\mathbf{G}\bar{\mathbf{\Omega}}^{(4)})^T \end{bmatrix}, \tag{31}$$

which can be compactly written as

$$\bar{\mathbf{X}}_1 = \big(\mathbf{I}_4 \otimes \big((\mathbf{A}\bar{\mathbf{\Theta}}) \odot (\mathbf{A}\bar{\mathbf{\Psi}}) \odot (\mathbf{A}\bar{\mathbf{\Gamma}})\big)\big)(\mathbf{G}\tilde{\mathbf{\Omega}})^T, \tag{32}$$

where

$$\tilde{\mathbf{\Omega}} = \left[\bar{\mathbf{\Omega}}^{(1)}, \bar{\mathbf{\Omega}}^{(2)}, \bar{\mathbf{\Omega}}^{(3)}, \bar{\mathbf{\Omega}}^{(4)}\right] \in \mathbb{R}^{4K \times 32K}.$$

Using a similar reasoning, we can also define three additional matrix unfoldings $\bar{\mathbf{X}}_2 \in \mathbb{R}^{4N^2 S \times N}$, $\bar{\mathbf{X}}_3 \in \mathbb{R}^{4N^2 S \times N}$ and $\bar{\mathbf{X}}_4 \in \mathbb{R}^{4N^2 S \times N}$, which can be factored as[2]

$$\bar{\mathbf{X}}_2 = \big((\mathbf{A}\bar{\mathbf{\Psi}}) \odot (\mathbf{A}\bar{\mathbf{\Gamma}}) \odot ((\mathbf{I}_4 \otimes \mathbf{G})\bar{\mathbf{\Omega}})\big)(\mathbf{A}\bar{\mathbf{\Theta}})^T, \tag{33}$$

$$\bar{\mathbf{X}}_3 = \big((\mathbf{A}\bar{\mathbf{\Gamma}}) \odot ((\mathbf{I}_4 \otimes \mathbf{G})\bar{\mathbf{\Omega}}) \odot (\mathbf{A}\bar{\mathbf{\Theta}})\big)(\mathbf{A}\bar{\mathbf{\Psi}})^T, \tag{34}$$

$$\bar{\mathbf{X}}_4 = \big(((\mathbf{I}_4 \otimes \mathbf{G})\bar{\mathbf{\Omega}}) \odot (\mathbf{A}\bar{\mathbf{\Theta}}) \odot (\mathbf{A}\bar{\mathbf{\Psi}})\big)(\mathbf{A}\bar{\mathbf{\Gamma}})^T, \tag{35}$$

where $\bar{\mathbf{\Omega}} = [\bar{\mathbf{\Omega}}^{(1)T}, \bar{\mathbf{\Omega}}^{(2)T}, \bar{\mathbf{\Omega}}^{(3)T}, \bar{\mathbf{\Omega}}^{(4)T}]^T \in \mathbb{R}^{16K \times 8K}$.

The identifiability of the mixing matrix $\mathbf{A}$ in the least squares (LS) sense from the unfolded factorizations (32)-(35) requires that the inequalities $N^3 \geq 2K$ and $N^2 S \geq 2K$ are satisfied, implying that $min(N^3, N^2 S) \geq 2K$. This condition yields the maximum number of sources that can be handled by the proposed blind identification method, which is based on the alternating least squares algorithm.

5. NUMERICAL RESULTS

We propose to fit a fourth-order CONFAC model to the derivative tensor. The alternating least squares (ALS) algorithm [13] is used to estimate $\mathbf{A}$ and $\mathbf{G}$ by exploiting the unfolded matrix representations (32)-(35) of the proposed CONFAC model. Note that a CP model cannot be fitted to the derivative tensor, once $\mathbf{A}$ and $\mathbf{G}$ have identical columns. Thus, a CP-based ALS algorithm would fail in estimating the mixing matrix. The LS estimation steps of the CONFAC-based ALS algorithm follow directly from (32)-(35), and have been omitted due to lack of space (see [18] for further details). After convergence of the ALS algorithm, a final estimate of the complex-valued mixing matrix $\mathbf{H}$ is obtained by properly combining pairs of columns of the real-valued $\mathbf{A}$, as explained in [16]. We evaluate the normalized mean square error (NMSE)

$$f_H(\mathbf{H}, \hat{\mathbf{H}}) = \frac{\text{vec}(\mathbf{H} - \hat{\mathbf{H}})^T \text{vec}(\mathbf{H} - \hat{\mathbf{H}})}{\text{vec}(\mathbf{H})^T \text{vec}(\mathbf{H})}$$

as a function of the signal to noise ratio (SNR). Each NMSE curve represents the median value from 50 Monte Carlo runs. Sources are synthesized 4-PSK signals.

In Figures 1 and 2, we compare the proposed method, named "CONFAC-3", with the FOOBI method that also relies on fourth-order statistics [6], and the 6-BIOME method [4] that is based in sixth-order statistics. We consider two different situations regarding the number of sources, sensors and observations. It can be noted that in both situations the CONFAC-3 method offers the best performance in most of the SNR range. For the "CONFAC-3"method, we have used $S = 100$ randomly drawn derivative points. Note that the performance is sensitive to the choice of the points at which the derivatives are calculated. We believe that improved results can be obtained with a better choice of these points.

6. CONCLUSIONS

A fourth-order CONFAC tensor decomposition has been used to solve the problem of blind identification of underdetermined mixtures of complex sources. We have shown that third-order derivatives of the CGF of the observations can be stored in fourth-order tensors admitting similar CONFAC decompositions with known constrained structures. The proposed method combines four different CGF derivatives into a single fourth-order CONFAC tensor model with a known constrained structure, from which an estimation if the mixing matrix is obtained by means of the ALS algorithm. This work generalizes [17] to the case of third-order derivatives and, therefore, it allows to deal with higher underdeterminacy levels. A deeper study of the uniqueness conditions of the proposed CONFAC model is a perspective of this work. From an algorithmic viewpoint, the use of a CONFAC-based enhanced line search (CONFAC-ELS) method in conjunction wit the alternating least squares algorithm and a numerical complexity analysis are left for a future work.

[2] The intermediate steps leading to the construction of the unfoldings $\bar{\mathbf{X}}_i$, $i = 1, 2, 3, 4$, have been ommited due to lack of space and can be found in [16].

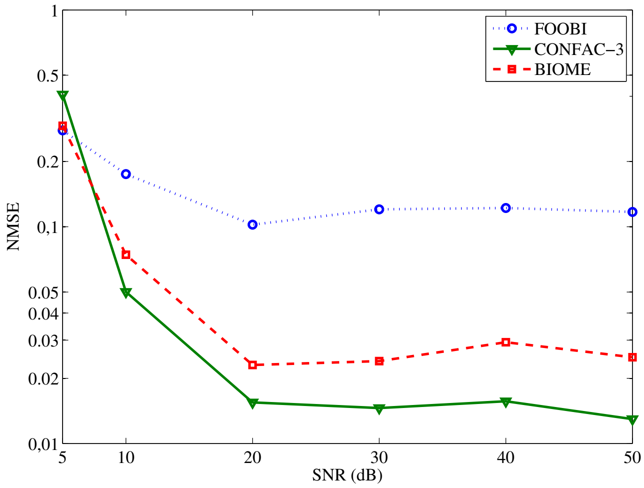

Fig. 1. NMSE vs. SNR ($K = 5$, $N = 3$, $M = 5000$, $S = 100$)

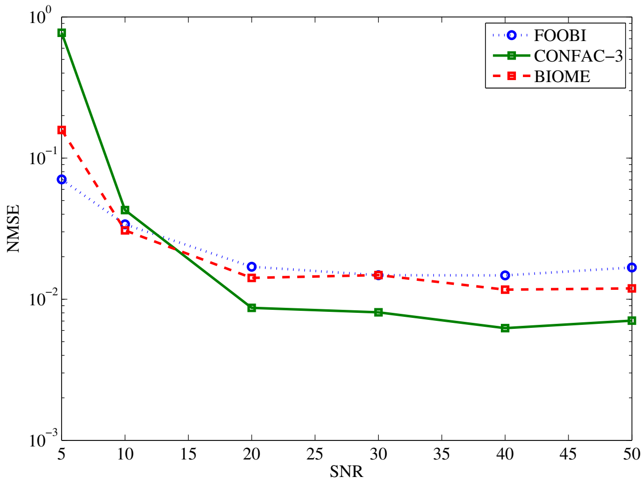

Fig. 2. NMSE vs. SNR ($K = 7$, $N = 4$, $M = 20000$, $S = 100$)

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