GRADUALLY IMPROVING THE READABILITY OF THE TIME-FREQUENCY SPECTRA FOR NATURAL FREQUENCY IDENTIFICATION IN CANTILEVER BEAMS

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ABSTRACT

It is the purpose of the paper to stimulate a systematic investigation of the time-frequency methods used to extract the natural frequency of mechanical structures. In particular, we are interested in cantilever beams, which are simple structures in mechanics, and therefore most appropriate for detailed comparison. We will compare the refined time-frequency spectra for the short-time Fourier transform, for the Wigner transform and for the non-stationary Gabor transform and extract a set of natural frequencies from the accelerometer data placed at the free end of a cantilever beam.

Index Terms— short-time Fourier transform, non-stationary Gabor transform, Wigner–Ville, Lasso, re-assignment, natural frequency identification.

1. INTRODUCTION

Time-frequency methods are applied to mechanical structures analysis for investigating functional parameters that are evolving in time like structural damages, cracks or deformations ([6, 3, 4]). Their usefulness consists mainly in the fact that they are non-invasive techniques but they have the disadvantage of imprecision in frequency. It is the purpose of this paper to provide a method for improving the frequency identification by several post-processing operations like re-assignment ([1]) and LASSO ([8]). We will compare two classical time-frequency methods, namely the short-time Fourier transform (STFT) as linear transform and the Wigner-Ville transform (WV) as quadratic transform with a novel time-frequency representation namely non-stationary Gabor transform as adaptive transform (CQ) ([9]). The methodology of work constitutes the plan of the paper. We start by placing an accelerometer at the free end of a cantilever beam. Then the data is recorded using a Matlab interface to the sensor. We compute the spectra for the three time-frequency transforms STFT, WV and CQ. Then we post-process the time-frequency spectra using the LASSO and the re-assignment method. Finally, we perform the identification of the natural frequencies using a peak-picking technique and the comparison of the results is realized.

Due to its simplicity we will use a steel cantilever beam having the following geometrical characteristics: length $l = 1000mm$, wide $b = 50mm$, height $h = 5mm$ and consequently, for the undamaged state the cross-section $A = 250 \cdot 10^{-6}m^2$, moment of inertia $I = 520.833 \cdot 10^{-12}m^4$. The mechanical characteristics of the beam are mass density $\rho = 7850kg/m^3$, Young's modulus $E = 2.0 \cdot 10^{11}N/m^2$ Poissons ratio $\mu = 0.3$.

This beam is considered as a reference, for beams with other dimensions (l, b, h) or mechanical characteristics (ρ, E, μ), the problem can be solved by considering the scale influence. For recording the signal, we place an accelerometer on the free end of the unloaded beam.

We start by recalling the definition of the Fourier transform for functions on $\mathbb{R}^d$ using an integral transform. It is useful to assume that $f \in \boldsymbol{L}^1(\mathbb{R}^d)$, i.e. that f belongs to the space of Lebesgue integrable functions in the technical sense of being measurable and absolutely integrable. We prefer the normalization with the factor 2π in the exponent, because then the "pure frequency" $t \mapsto e^{10\pi it}$ has exactly $5 = 10/2$ full oscillations over

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any interval of length 1 while one would have to think in terms of multiples of a basic period of 2π in case of a normalization of pure frequencies of the form $t \mapsto e^{it}$).

$$\hat{f}(\omega) = \int_{\mathbb{R}^d} f(t) \cdot e^{-2\pi i \omega \cdot t}\, dt \tag{1}$$

The inverse Fourier transform then has the form

$$f(t) = \int_{\mathbb{R}^d} \hat{f}(\omega) \cdot e^{2\pi i t \cdot \omega}\, d\omega, \tag{2}$$

Strictly speaking this inversion formula only makes sense under the additional hypothesis that $\hat{f} \in \boldsymbol{L}^1(\mathbb{R}^d)$, which is not satifsied for arbitrary functions $f \in \boldsymbol{L}^1(\mathbb{R}^d)$. In the general case ($f \in \boldsymbol{L}^1(\mathbb{R}^d)$) one can obtain f from $\hat{f}$ using classical summability methods, convergent in the $\boldsymbol{L}^1$-norm.

One often speaks of Fourier analysis being the first step, telling us how much energy of f is concentrated at a given frequency ω (or more specifically $|\hat{f}(\omega)|^2$), and the Fourier inversion as a method to build f from the pure frequencies also call Fourier synthesis.

2. TIME-FREQUENCY SPECTROGRAMS

We introduce in this section the main tools for time-frequency analysis, namely the STFT, WV and CQ.

The Short-Time Fourier Transform (STFT) of a function $f \in \boldsymbol{L}^2(\mathbb{R}^d)$ with respect to a window $g \in \boldsymbol{L}^2(\mathbb{R}^d)$ is defined as

$$V_g f(x,\omega) = \int_{\mathbb{R}^d} f(t)\overline{g(t-x)}e^{-2\pi i t\cdot\omega} dt$$

for $z = (x,\omega) \in \mathbb{R}^{2d}$.

The Wigner transform is by definition:

$$Wf(x,\omega) = \int_{\mathbb{R}^d} e^{-2\pi i\omega\cdot t} f(x+\tfrac{1}{2}t)\overline{f(x-\tfrac{1}{2}t)}dt. \tag{3}$$

The definition of the cross-Wigner transform of a pair of square-integrable functions is similar to that of the Wigner transform (3)

$$W(f,g)(x,\omega) = \int_{\mathbb{R}^d} e^{-2\pi i\omega\cdot t} f(x+\tfrac{1}{2}t)\overline{g(x-\tfrac{1}{2}t)}dt$$

and reduces to the latter when $f = g$.

We observe, however, that the cross-Wigner transform $W(f,g)$ and the STFT are related by the simple formulas

$$W(f,g)(z) = 2^d e^{4\pi i\omega\cdot x} V_{g^\vee} f(2z) \tag{4}$$

where $g^\vee(x) = g(-x)$ and

$$V_g f(z) = 2^{-d} e^{-i4\pi\omega\cdot x} W(f, g^\vee)(\tfrac{1}{2}z). \tag{5}$$

For the practical implementation, these transforms will be developed in a finite discrete setting using the Hilbert space $\mathbb{C}^L$. We consider a signal f of length L, with time-shifted window functions $\{\varphi_k\}_k$ and frequency shift parameters $\{b_k\}_k$. Using these considerations, the direct discrete versions of the STFT and of the Wigner transform can be immediately obtained. We will introduce now, a variation of the STFT called the non-stationary Gabor transform.

The finite, discrete non-stationary Gabor transform (CQ) is given by:

$$c_{n,k} = \sum_{j=0}^{L-1} f[j]\overline{\varphi_k[j]}e^{\frac{2\pi i n b_k j}{L}}. \tag{6}$$

This transform allows for windows to vary over time compared to the fixed resolution of the STFT. Therefore, it generates dynamic time-adaptive transforms that are still Fast Fourier transform (FFT)-based for fast implementation.

The (frequency-side) finite, discrete non-stationary Gabor transform (CQ) is given by:

$$c_{n,k} = \sum_{j=0}^{L-1} \hat{f}[j]\overline{\widehat{\varphi_k}[j]}e^{\frac{2\pi i n a_k j}{L}} = \sum_{l=0}^{L-1} f[l]\overline{\varphi_k[l-na_k]}. \tag{7}$$

In this case the windows vary over the frequency domain and generate frequency-adaptive transform design like wavelet transforms or nonuniform LTI filterbanks. The fast implementation requires a full FFT of f.

The comparison of our methods will include the linear frequency spacing over a fixed time-frequency resolution versus geometric frequency spacing. In figure (1) we present the spectrograms for the three time-frequency representations.

3. REFINEMENT OF THE SPECTROGRAMS

The Wigner-Ville distribution possesses a high resolution in the time-frequency plane, and satisfies a large number of desirable properties, but its use in practical application is limited by the presence of non-negligible cross-terms, resulting from interaction between signal components.

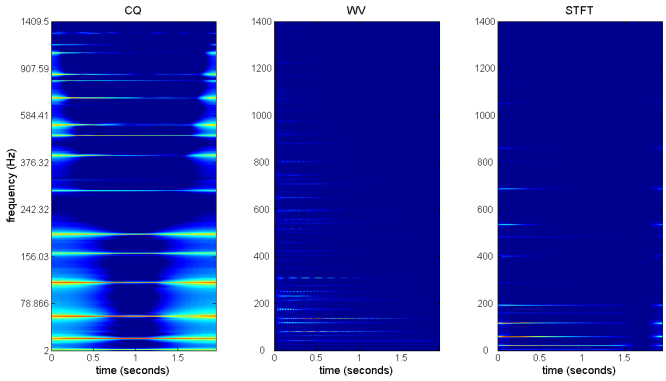

Fig. 1. STFT, WV, CQ.

We present in the following the re-assignment method that we have used to improve the time-frequency readability in the classical setting of the WV spectrum.

The expression of the WV spectrum shows that the value of a time-frequency representation at any point (t, ω) of the time-frequency plane can be considered as the contributions of the weighted Wigner-Ville distribution values at the neighboring points $(t-u, \omega-\Omega)$. The transform average the signal energy in a domain centered on (t, ω) and delimited by the essential support of $g = \phi_{TF}(u, \Omega)$. This averaging leads to the attenuation of the oscillating cross-terms, but also to a signal component broadening. Therefore, one way to avoid this is to change the attribution point of this average, and to assign it to the center of gravity of these energy contributions, whose coordinates are:

$$\hat{t}(x;t,\omega) = t - \\ - \frac{\int\int u \cdot \phi_{TF}(u,\Omega) WV(x;t-u,\omega-\Omega) du \frac{d\Omega}{2\pi}}{\int\int \phi_{TF}(u,\Omega) WV(x;t-u,\omega-\Omega) du \frac{d\Omega}{2\pi}}$$

$$\hat{\omega}(x;t,\omega) = \omega - \\ - \frac{\int\int \Omega \cdot \phi_{TF}(u,\Omega) WV(x;t-u,\omega-\Omega) du \frac{d\Omega}{2\pi}}{\int\int \phi_{TF}(u,\Omega) WV(x;t-u,\omega-\Omega) du \frac{d\Omega}{2\pi}}$$

rather than to the point (t, ω) where it is computed. This reassignment leads to the construction of a modified version of the time frequency representation, whose values at any point (t', ω') is therefore the sum of all the representation values moved to the point:

$$MTFR(x;t',\omega') = \int\int TFR(x;t,\omega)\delta(t'-\hat{t}(x;t,\omega)) \cdot \\ \cdot \delta(\omega'-\hat{\omega}(x;t,\omega)) dt \frac{d\omega}{2\pi}$$

where $\delta(t)$ denotes the Dirac impulse.

In figure (2) we present the spectrograms for the three time-frequency representations after re-assignment.

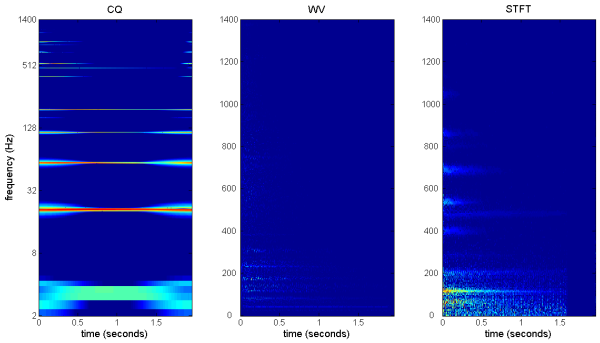

Fig. 2. STFT, WV, CQ after re-assignment

The second method we have used for comparison has its origins in the classical task of denoising. Efficient noise reduction methods can be obtained using thresholding strategies. Coefficient thresholding has introduced itself naturally into the context of variational formulations of denoising. The 'lasso' technique, or 'least absolute shrinkage and selection operator' was originally introduced by R. Tibshirani as a solution to the standard regression problem is applicable in the denoising of spectra ([8]). The lasso was defined in the following manner:

Suppose that we have the data $(x^i, y_i), i = 1, 2, ..., N$, where $x^i = (x_{i1}, ..., x_{ip})^T$ are the predictor variables and y_i are the responses. As in the usual regression set-up, we assume either that the observations are independent or that the y_is are conditionally independent given the x_{ij}s. We assume that the x_{ij} are standardized so that $\sum_i x_{ij}/N = 0, \sum_i x_{ij}^2/N = 1$. Letting $\hat{B} = (\hat{B}_1, ..., \hat{B}_p)^T$, the lasso estimate $\hat{\alpha}, \hat{B}$ is defined by

$$(\hat{\alpha}, \hat{B}) = argmin \left\{ \sum_{i=1}^{N} \left(y_i - \alpha - \sum_j B_j x_{ij} \right)^2 \right\}$$

subject to

$$\sum_j |B_j| \leq t. \tag{8}$$

Computation of the solution of this equation is a quadratic programming problem with linear inequality constraints.

In the context of spectra denoising, the lasso method leads to the following result:

Given an orthonormal basis ϕ_k, the variational denoising problem:

$$\min_{\alpha} \left[\frac{1}{2} \|x - \sum_k \alpha_k \phi_k\|_2^2 + \lambda \|\alpha\|_1 \right] \tag{9}$$

is solved by $\alpha_k = S_\lambda \langle x, \phi_k \rangle$. This is not true anymore when ϕ_k is a more general frame, but dedicated iterative algorithms (based upon soft thresholding) can yield the solution.

In figure (3) we present the spectrograms for the three time-frequency representations after applying the LASSO method.

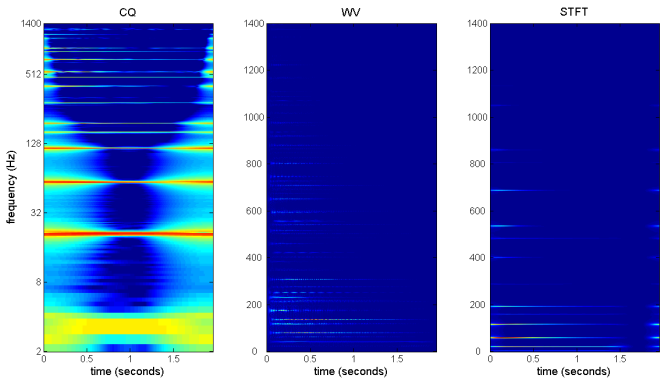

Fig. 3. STFT, WV, CQ after LASSO

4. NATURAL FREQUENCIES IDENTIFICATION METHOD

The following methods used for determining the natural frequencies of a signal makes use of the three time-frequency representation methods presented.

At first glance both the WV and the STFT offer the same advantage in frequency resolution. In principle, the CQ can have a similar resolution with the appropriate choice of bins per octave, but that is at the expense of some of its desirable properties such as emphasis on specific frequency ranges. The advantage of the WV over the other two methods is the fact that the signal is correlated with itself thus giving the most accurate localization of the natural frequencies. This is also a drawback since it causes many prominent cross-terms to appear. In order to determine the relevant frequencies from the WV, the method presented here uses prior knowledge from the STFT and CQ. This can be done in two different ways:

The first approach determines the frequencies that are most significant in the STFT and CQ and determines a range in which all the natural frequencies may lie. This range is comprised of the frequency bins with the largest contribution and their closest neighbors (within a distance that is variable depending on the bin value). By searching for high contributions in the WV only within this pre-determined range, we exclude the possibility of reading cross-terms of any intensity. This approach is heavily dependent on the choice of the neighborhoods around the key frequency bins determined in the STFT/CQ. Also, small correlation effects might appear in the STFT between the signal and the window function. Although unlikely, this might lead to an unwanted extension of the range in which the WV is investigated and to inconclusive results.

The second approach is a progressive refinement of the search range going through all the three representation methods, starting with the CQ. First an appropriate choice of window functions must be made so that only natural frequencies are detected, even with an imprecise resolution. The CQ is ideal for this because of the advantage of allocating a variable number of bins to each octave. Having determined this initial set of peak frequencies, the STFT is investigated in the areas of interest and a further refinement is made. This intermediate step is necessary in order to approach the natural frequencies and account for the change in window length. Equivalently, this could be done by taking an STFT with variable windows, but using both the CQ and the STFT ensures a smooth transition. The final step is identifying the peak frequencies in the WV that lie within the refined area of interest.

One should note that in the end, both approachesyield the same results in the optimal case, but the second algorithm minimizes the risk of having too wide of a search range in the WV. Both paths use the high resolution and accuracy of the WV representation to

Standard	Re-assigned	Lasso
5.12	*	4.27
23.55	*	21.32
62.47	66.7	59.65
136.19	143.36	137.34
232.45	235.53	257.34
352.26	322.56	337.43
485.39	481.28	485.01
650.25	645.13	646.25
826.4	824.33	811.28
1035.4	1034.1	1057.82

Table 1. Comparison of the first 10 natural frequencies, for the non-refined case, the re-assigned case and the LASSO. We marked with * the values that are not readable

determine peak frequencies within a certain range and they employ the CQ transform and the STFT in order to determine where the natural frequencies lie up to a certain degree of precision. This algorithm makes use of the advantages of the three spectral representation methods, while avoiding each of their drawbacks, so that in the end, it reaches better results than any of the three methods individually.

The set of natural frequencies analytically computed from the equation of the beam correponding to the first ten frequencies modes are $f_i = (4.076, 22.549, 60.539, 138.188, 241.741, 337.182, 485.510, 646.721, 816.832, 1042.824)$. In table 1, we gather the results of the comparison for the values obtained using the proposed method.

5. CONCLUSIONS

We provide in this paper a method to improve the natural frequencies readability using complementary information extracted from the spectra of the STFT, WV and CQ. The results show that we are able to improve the accuracy of natural frequencies identification using our method. The re-assignment although it gives a bad visual impression allows for good identification but the LASSO thresholding pre-processed with the proposed method is the most accurate. One can use this approach when the analytic equation is not available like in the case of damaged structures.

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