ADAPTIVE IIR NOTCH FILTERS FOR TRACKING OF QUASI-HARMONIC SIGNALS

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ABSTRACT

In this paper, a new procedure for the design of a cascade of IIR Adaptive Notch Filters (ANFs) is presented. The cascade is composed by second-order sections of modified Nehorai Constrained ANFs, with coefficients modeled by a single polynomial of arbitrary order. This approach allows for a design with less parameters to control notch positions than the number of cascaded sections. The resulting structure is particularly suitable for tracking components of quasi-harmonic signals. An example shows that the coefficient modeling can tackle the inherent inharmonicity of the notes emitted by a string instrument. Finally, a complete adaptation algorithm for the model parameters, derived from Nehorai's ANF recursive prediction error algorithm, is described.

***Index Terms*—** adaptive notch filter, inharmonicity, IIR

1. INTRODUCTION

Adaptive Notch Filters (ANFs) are commonly used for tracking partial components of harmonic and quasi-harmonic signals [1, 2], and the Nehorai structure [3], implemented as a cascade of second-order ANF sections, has been successfully applied to track partials of time-varying signals [4, 5]. Each Nehorai ANF section establishes a notch between 0 and π radians using, besides a parameter to control the notch width, a single coefficient to control the notch position.

We propose to model the set of notch coefficients of a cascade of second-order Nehorai ANF sections by a polynomial in m, which indexes the notch frequencies in ascending order. This approach allows the number of parameters that control notch positions to be inferior to the number of cascaded sections. The resulting structure is suitable for tracking the instantaneous frequencies of partials from quasi-harmonic signals, like that of most musical instruments.

This paper is organized as follows: in Section 2, we briefly describe the Nehorai ANF as a cascade of second-order notch filters and propose a model for their coefficients; in the same section, we show how to relate this model to the partials in string instrument notes; in Section 3 we derive a Recursive Prediction Error algorithm (RPE), adapted from Nehorai's ANF update algorithm [3] to adjust the model parameters; in the same section, we incorporate to the proposed model auxiliary techniques in order to improve tracking with IIR ANFs [4, 6]; in Section 4 we show computer simulations over synthetic signals; in Section 5, we present conclusions and plans for future work.

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2. FILTER STRUCTURE

In a cascade of Nehorai's second-order ANF sections, each section produces a single notch that requires only one coefficient a_m to control its central attenuation frequency.

The overall structure can be described by

$$H(q^{-1}) = \prod_{m=1}^{M} \frac{1 + a_m q^{-1} + q^{-2}}{1 + \rho a_m q^{-1} + \rho^2 q^{-2}}, \quad (1)$$

where ρ is the pole contraction factor (PCF) and q^{-1} is the unitary delay operator. The value of each coefficient a_m is related to its correspondent notch frequency ω_m by

$$a_m = -2\cos\omega_m. \quad (2)$$

The PCF ($0<\rho<1$) controls the notches' widths, which narrows as ρ gets closer to 1. The PCF can be made adaptive and unique to each section, enhancing the filter tracking capabilities [6].

2.1. Proposed Model

By modeling the a_m coefficients as an N^{th}-order polynomial in m, it is possible to obtain an ANF cascade with less parameters to control the notch positions than the number M of cascaded sections. Furthermore, the resulting filter maintains a structural relation between coefficients which favors the tracking of quasi-harmonic partials.

The model can be expressed as

$$a_m = \sum_{n=1}^{N} \alpha_n m^n - 2. \quad (3)$$

By combining Equation (2) and Equation (3) we obtain

$$\alpha_N m^N + \alpha_{N-1} m^{N-1} + \cdots + \alpha_1 m - 2 = -2\cos\omega_m. \quad (4)$$

A simple procedure for the algorithm initialization consists in solving a linear (determined or over-determined) equation system composed by (4) fed with a set of arbitrary m values.

As an example, Figure 1 displays the values of the notch filter coefficients a_m computed for a harmonic series, and its approximations obtained through a polynomial model of order $N=3$ solved for $m = \{1, 8, 16\}$. Such particular choices of m show that the model can yield a good approximation for a_m over the spectrum.

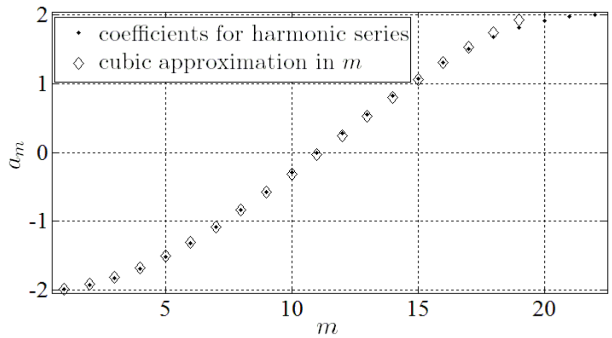

Fig. 1. Coefficient values a_m correspondent to notches on the harmonic series of 1 kHz (sampling frequency f_s=44.1 kHz) and an example of cubic approximation.

With this model, coefficient adaptation can be done over the N parameters of the polynomial, instead of over the M coefficients of the ANF cascade. The same N parameters can be used, for example, to design a filter that attenuates a sequence of partials $p = \{1, 2, ..., P\}$ of a quasi-harmonic string instrument signal, using $N < P$.

The Fletcher inharmonicity model for notes emitted by a string instrument [7] can be summarized as

$$\omega_p = p\,\omega_0\sqrt{1 + Bp^2}, \qquad p = \{1, 2, 3, ...\}, \tag{5}$$

where ω_0 is the nominal fundamental frequency (i.e. the signal fundamental frequency if the inharmonicity coefficient B is zero). Referring to Equation (4), the desired value of a_m such that the p^{th} partial is attenuated ($m = p$) is given by

$$a_p = -2\cos(p\,\omega_0\sqrt{1 + Bp^2}). \tag{6}$$

Hence, the same initialization procedure described before can be followed for tracking string instrument notes.

3. ADAPTATION ALGORITHM

The following algorithm is a modified version of the RPE-type algorithm developed by Nehorai [3]. This modification was devised considering the outputs of each filter section, as shown in Figure 2. Each section output is given, at time index $k \in \mathbf{Z}$, by $\varepsilon[m, k] = \varepsilon[m-1, k]H_m(q^{-1})$, where $\varepsilon[0, k]$ is the input signal. This approach, commonly used for adapting cascades of IIR ANFs [1, 4, 8], assumes that, using narrow notches, each section can be updated independently. The proposed algorithm adapts the N polynomial parameters, implicitly updating the M filter coefficients.

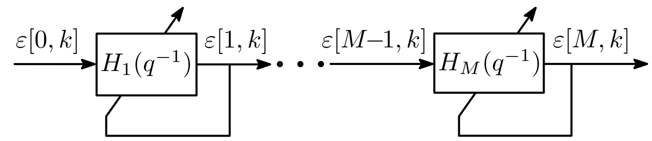

Fig. 2. Cascaded second-order ANF sections.

3.1. Recursive Adaptation of the Model Parameters

The output of the m^{th} section in the ANF cascade can be described as a function of $A_m = 1 + a_m q^{-1} + q^{-2}$ and the pole contraction factor ρ.

$$\varepsilon[m, k] = \varepsilon[m-1, k]\frac{A_m(q^{-1})}{A_m(\rho q^{-1})}. \tag{7}$$

Rearranging (7) and taking its derivative to α_n leads to

$$\frac{\partial(A_m(\rho q^{-1})\varepsilon[m, k])}{\partial\alpha_n} = \frac{\partial(A_m(q^{-1})\varepsilon[m-1, k])}{\partial\alpha_n}. \tag{8}$$

The chain rule can be applied to both sides of Equation (8).

$$\frac{\partial A_m(\rho q^{-1})}{\partial\alpha_n}\varepsilon[m, k] + \frac{\partial\varepsilon[m, k]}{\partial\alpha_n}A_m(\rho q^{-1}) = \frac{\partial A_m(q^{-1})}{\partial\alpha_n}\varepsilon[m-1, k] + \frac{\partial\varepsilon[m-1, k]}{\partial\alpha_n}A_m(q^{-1}). \tag{9}$$

By defining $\psi_n[m, k] = -\dfrac{\partial\varepsilon[m, k]}{\partial\alpha_n}$ and using Equation (3), Equation (9) becomes

$$\frac{\partial\left(1+\rho(\alpha_N m^N+\cdots+\alpha_1 m-2)q^{-1}+\rho^2 q^{-2}\right)}{\partial\alpha_n}\varepsilon[m,k]-\psi_n[m,k]A_m(\rho q^{-1})= \frac{\partial\left(1+(\alpha_N m^N+\cdots+\alpha_1 m-2)q^{-1}+q^{-2}\right)}{\partial\alpha_n}\varepsilon[m-1,k]-\psi_n[m-1,k]A_m(q^{-1}),$$

which leads to

$$\rho m^n\varepsilon[m,k-1]-\psi_n[m,k]A_m(\rho q^{-1}) = m^n\varepsilon[m-1,k-1]-\psi_n[m-1,k]A_m(q^{-1}). \tag{10}$$

Equation (10) can be solved for

$$\psi_n[m,k]=\frac{m^n(-\varepsilon[m-1,k-1]+\rho\varepsilon[m,k-1])+\psi_n[m-1,k]A_m(q^{-1})}{A_m(\rho q^{-1})}. \tag{11}$$

By defining $\varphi[m, k] = -\varepsilon[m-1, k-1] + \rho\varepsilon[m, k-1]$, Equation (11) can be written as

$$\psi_n[m, k] = \frac{m^n\varphi[m, k] + \psi_n[m-1, k]A_m(q^{-1})}{A_m(\rho q^{-1})}. \tag{12}$$

Notice that $\psi_n[0,k] = -\partial\varepsilon[0,k]/\partial\alpha_n = 0$. Then, for $m=1$,

$$\psi_n[1,k] = \frac{\varphi[1,k]}{A_1(\rho q^{-1})}. \tag{13}$$

Equations (12) and (13) may be put in recursive form by substitution of $A_m(q^{-1})$ and $A_m(\rho q^{-1})$ and application of the delay operators.

Parameter update is achieved using an RLS equation error algorithm [9] modified to adapt the N polynomial parameters using M error functions. For each pair $\{m,k\}$, the RLS algorithm has complexity $\mathcal{O}(N^2)$.

A. For time index k, calculate for each m

$$\begin{aligned}\varepsilon[m,k] = {} & \varepsilon[m-1,k] + a_m\varepsilon[m-1,k-1] + \varepsilon[m-1,k-2] \\ & - \rho a_m\varepsilon[m,k-1] - \rho^2\varepsilon[m,k-2],\end{aligned} \tag{14}$$

$$\mathbf{P}[m,k]=\left(\mathbf{P}[m,k-1]-\frac{\mathbf{P}[m,k-1]\boldsymbol{\psi}[m,k]\boldsymbol{\psi}^T[m,k]\mathbf{P}[m,k-1]}{\lambda[k] + \boldsymbol{\psi}^T[m,k]\mathbf{P}[m,k-1]\boldsymbol{\psi}[m,k]}\right)\frac{1}{\lambda[k]}, \tag{15}$$

where $\mathbf{P}[m,k]$ is the pseudo-inverse of the $N \times N$ Hessian matrix for the m^{th} section, $\boldsymbol{\psi}[m,k]=[\psi_1[m,k] \cdots \psi_N[m,k]]^T$ and $\lambda[k]$ is the forgetting factor, $(0 \ll \lambda[k] < 1)$.

B. Update the parameter vector: $\boldsymbol{\theta}[k]=[\alpha_1[k]\ \alpha_2[k] \cdots \alpha_N[k]]^T$,

$$\boldsymbol{\theta}[k] = \boldsymbol{\theta}[k-1] + \frac{1}{M}\sum_{m=1}^{M}\mathbf{P}[m,k]\boldsymbol{\psi}[m,k]\varepsilon[m,k]. \tag{16}$$

C. Using $\boldsymbol{\theta}[k]$, update a_m as in Equation (3).

3.2. Recursive Adaptation of PCFs

To enhance the tracking capabilities of IIR ANFs, Dragošević and Stanković [6] adopted a different PCF $\rho_m[k]$ for each filter section and derived an RPE algorithm to update their values, with gradient $\psi_\rho[m,k]$ given by

$$\psi_\rho[m,k] = \frac{(a_m q^{-1} + 2\rho_m[k]q^{-2})\varepsilon[m,k]}{A_m(\rho_m[k]q^{-1})}. \tag{17}$$

The forgetting factors λ and λ_ρ, used in the RPE algorithms to adapt $\boldsymbol{\theta}$ and ψ_ρ, can also be different for each section. Results in [4] show that good stability and tracking can be achieved using $\lambda[m,k+1] = (1-\delta)\lambda[m,k]+\delta\rho_m[k+1]$, $0 < \delta \ll 1$. Also, λ_ρ should be higher than λ, thus making ρ adapt slower than the notch frequency. For the filter proposed in this paper, better results were obtained updating $\lambda[m,k]$ and $\lambda_\rho[m,k]$ as

$$\begin{aligned}&\lambda[m,k+1] = \min\{(1-\delta)\lambda[m,k]+\delta\rho_m[k+1]\kappa, 1\} \quad \text{and} \\ &\lambda_\rho[m,k+1]=\min\{(1-\delta)\lambda_\rho[m,k]+\delta\rho_m[k+1]\kappa_\rho,1\},\end{aligned} \tag{18}$$

where κ and κ_ρ are real values slightly higher than 1. Also, to improve stability and to help keep each filter section tracking only one partial, notch bandwidths $BW[m,k]$ are limited by maximum and minimum quality factors Q_h and Q_l. Each quality factor is computed as $Q[m,k] = \hat{\omega}_m[k]/BW[m,k]$, where $BW[m,k]$ can be approximated [3] by $\pi(1-\rho[m,k])$ and $\hat{\omega}_m[k]$ is the estimate of the tracked partial frequency at instant k.

The adaptation algorithm is given below. For initialization values please refer to the appendix in Section 6.

Adaptation Algorithm

for $k = 2,3,...$
 for $m = 1,2,...,M$
 • compute $\varepsilon[m,k]$ using Equation (14);
 • compute $\boldsymbol{\psi}[m,k]$ using Equation (12);
 • compute $\mathbf{P}[m,k]$ using Equation (15);
 • compute $\psi_\rho[m,k]$ using Equation (17);
 • $P_\rho[m,k]=\left(P_\rho[m,k-1]-\dfrac{P_\rho^2[m,k-1]\ \psi_\rho^2[m,k]}{\lambda_\rho[m,k]+P_\rho[m,k-1]\psi_\rho^2[m,k]}\right)\dfrac{1}{\lambda_\rho[m,k]}$;
 • $\rho_h[m,k]=1-\hat{\omega}_m[k]/(Q_h\pi)$; $\rho_l[m,k]=1-\hat{\omega}_m[k]/(Q_l\pi)$;
 • $\rho_m[k+1]=$
 $\max\{\min\{\rho_m[k]+P_\rho[m,k]\psi_\rho[m,k]\varepsilon[m,k],\rho_h[m,k]\},\rho_l[m,k]\}$;
 • compute $\lambda[m,k+1]$ and $\lambda_\rho[m,k+1]$ using (18);
 end for
 compute $\boldsymbol{\theta}[k]$ using Equation (16);
 for $m = 1,2,...,M$
 • update a_m and $\hat{\omega}_m[k+1]=\arccos(-a_m/2)$;
 end for
end for

3.3. Implementation Details

ANF tracking is sensitive to the initialization of notch coefficients. An attempt to mitigate this problem is to initialize each a_m such that its corresponding notch frequency is in the vicinity of the searched frequency. As this procedure requires previous knowledge about the signal, which may not be available, we present a different method in three steps.

In step 1, we use a single ANF section to locate and track any frequency partial. In step 2, we cascade another section to the ANF to locate and track a second partial. In step 3, we use the proposed model to track M signal partials. The outline of the algorithm is given ahead.

In step 1, a single ANF section is initialized with a notch centered around an arbitrary frequency $\hat{\omega}_1$. During adaptation $\rho[1,k]$ tends to increase as the notch gets closer to a partial [6]. This increase is accompanied by the value of $Q[1,k]$, the ANF section quality factor. That way, a smoothed version of $Q[1,k]$, $\overline{Q}[1,k] = (1-\delta_Q)\overline{Q}[1,k-1]+\delta_Q Q[1,k]$, can be used to indicate if a partial has been locked. When $\overline{Q}[1,k]$ reaches a threshold Q_1, step 2 begins.

In step 2, the new cascaded ANF section is initialized with a notch centered around a frequency slightly higher than the current $\hat{\omega}_1[k]$. Since the partial component being tracked by the first ANF is attenuated at its output, the second ANF tends to track a different partial. As in step 1, when the smoothed version of the second ANF quality factor $Q[2,k]$ reaches a threshold Q_2, step 3 begins. During this step, the quality factor of the first notch is kept constant by updating its PCF as $\rho_1[k]{=}1-\hat{\omega}_1[k]/(Q_l\pi)$.

For each k, the ratio $r[k]$ between the lowest and the highest tracked partials is measured and smoothed, respectively, as $r[k] = \min\{\hat{\omega}_1[k], \hat{\omega}_2[k]\}/\max\{\hat{\omega}_1[k], \hat{\omega}_2[k]\}$ and $\bar{r}[k] = (1-\delta_r)\bar{r}[k-1]+\delta_r r[k], 0<\delta_r\ll 1$.

When step 3 is initialized, $\bar{r}[k]$ is compared with a set r_h of ratios between low integers η and τ, with $\eta < \tau$, such as $\{1/2, 1/3, 2/3, 3/4, 3/5, 4/5, 5/6, 5/7, 6/7\}$. The value in r_h closer to $\bar{r}[k]$ is likely to be the same as the ratio between the lowest and the highest tracked partial numbers. This way, the polynomial parameters can be initialized by solving a linear system composed by Equation (4) so that, for N arbitrary m values, notches are centered over frequencies $(m/\eta)\min\{\hat{\omega}_1[k], \hat{\omega}_2[k]\}$. Alternatively, polynomial parameters can be initialized so that, for N arbitrary m values, notches are centered over frequencies $\hat{\omega}_m[k] = m\,\hat{\omega}_0[k](1+\hat{B}[k]m^2)^{1/2}$. Estimates for $\omega_0[k]$ and $B[k]$ can be obtained by substituting the tracked partials frequencies in a 2 equations system composed by (5) for $p=\{\eta,\tau\}$. The ratios in the proposed set r_h were chosen assuming that none or just one partial falls in between $\hat{\omega}_1[k]$ and $\hat{\omega}_2[k]$. This is a reasonable assumption since, in step 2, $\hat{\omega}_2[k]$ is initialized close to $\hat{\omega}_1[k]$, making it unlikely that the the second ANF section misses 2 or more subsequent partials before locking.

In cases where the partials are in octave relation, a third ANF section is used to determine if there is another partial between them. This section is initialized with its notch frequency $\hat{\omega}_3[k]$ between $\hat{\omega}_1[k]$ and $\hat{\omega}_2[k]$. When the smoothed version of its quality factor $Q[3,k]$ reaches a threshold Q_3, the ratio $r_{oct} = \|\hat{\omega}_3[k]-\hat{\omega}_2[k]\|/\|\hat{\omega}_2[k]-\hat{\omega}_1[k]\|$ is computed. This ratio will be approximately equal to 0.5 if this section is tracking a partial at half the distance between $\hat{\omega}_1[k]$ and $\hat{\omega}_2[k]$. During this step, the quality factors of the first and second notches are kept constant by updating its PCFs as $\rho_m[k]{=}1-\hat{\omega}_m[k]/(Q_l\pi)$.

During steps 1 and 2, we use $\boldsymbol{\theta}[k] = a_1$ and $M{=}N{=}1$ for each ANF section and adapt them independently. During step 3, we use the proposed model.

4. TESTS AND EXAMPLES

Tests were performed using synthetized signals $\varepsilon[0,k]$, composed by chirps of quasi-harmonic sinusoids with random initial phases.

$$\varepsilon[0,k] = \sum_{p=1}^{P} A_p \cos(\varphi_p[k] + \phi_p) + e[k]. \qquad (19)$$

For partial p, A_p is its amplitude, $d(\varphi_p[k])/dk$ is its instantaneous frequency, ϕ_p is a random phase and $e[k]$ is white Gaussian noise. Instantaneous frequency $d(\varphi_p[k])/dk = \omega_p[k]$ is given by Equation (5). The signal parameters used in the tests were $P=8$, $A_p = 1/8$, SNR $=30$ dB, sampling frequency $f_s = 44.1$ kHz and $B = 0.003$. The ANF model parameters were $N{=}3$ and $M{=}7$.

Signals with 8 different nominal fundamental frequencies where tested, all chirping during 2 s from f_0 to $2^{(4/12)}f_0$, an interval of 4 semitones. Ten tests were done for each initial f_0, each with a different set of ϕ_p.

All the tests initialized with f_0 equal to 500, 700, 900 and 1100 Hz were successful in all steps, closely tracking the first 7 partials. The same happened to 9 out of the 10 tests with f_0 initialized as 300 Hz. Figure 3 shows the results of a test with f_0 varying from 1100 Hz at $t=0$ s to 1388.4 Hz at $t=2$ s. Although the majority of tests were successful, most tests initialized with $f_0 = 200$ Hz and one initialized with $f_0{=}300$ Hz were not. This ocurred because the tracking results of steps 1 and 2 were noisy and could not provide good estimates of the ratios between the two first tracked partial frequencies.

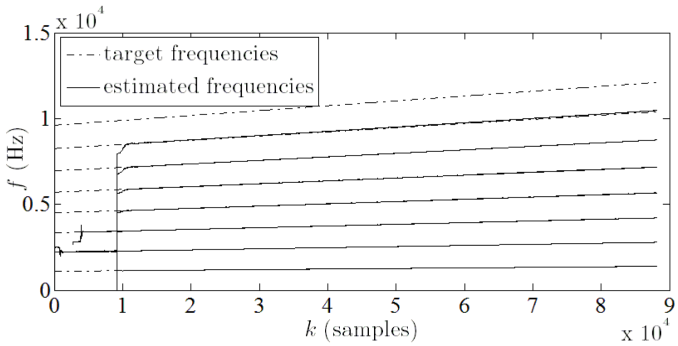

Fig. 3. Tracking of inharmonic chirp with $f_0 = 1100$ Hz at $t{=}0$ s (model order $N{=}3$).

It was possible to observe on the results of tests initialized with $f_0{=}1500$ Hz that the tracking of the seventh partial became less precise as its frequency increased. In Figure 4 it can be seen that the seventh notch deviates towards the eighth partial. This problem occurred because, for the increasing distance between partials (due to high f_0 combined with inharmonicity), the chosen model order $N=3$ is too low for closely tracking all 7 first partials. It can be easily solved by raising the value of N as shown in Figure 5.

5. CONCLUSIONS

In this paper we proposed an IIR ANF cascade design suitable for tracking partials of quasi-harmonic signals. Tracking tests, performed with a cascade of 7 ANF sections with

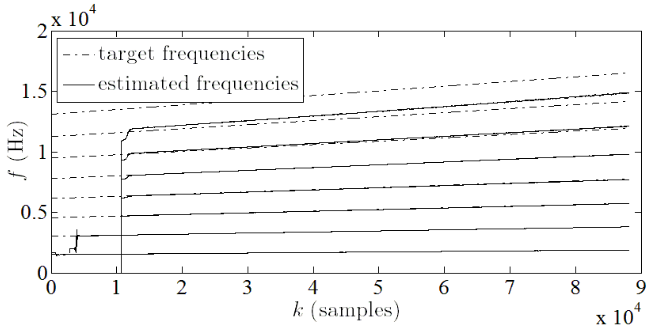

Fig. 4. Tracking of inharmonic chirp with $f_0 = 1500\,\text{Hz}$ at $t = 0\,\text{s}$ (model order $N = 3$).

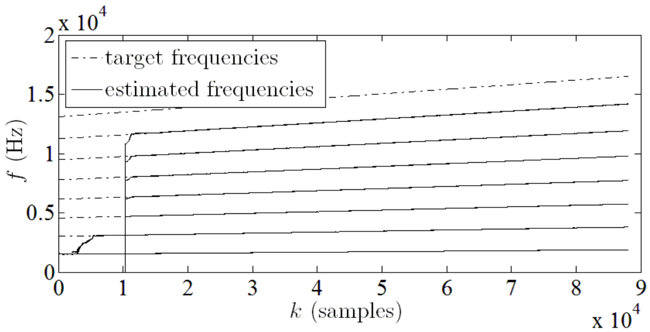

Fig. 5. Tracking of inharmonic chirp with $f_0 = 1500\,\text{Hz}$ at $t = 0\,\text{s}$ (model order $N = 4$).

notch coefficients modeled by an adaptive polynomial of order $N = 3$, showed that the proposed design and adaptation algorithm were able to successfully track the first 7 partials of several quasi-harmonic chirp signals. For high frequency regions, it was necessary to raise the polynomial order by 1 to allow close tracking of all first 7 partials. Limitations, inherent to IIR ANF, appeared in the low frequency range. Future work will focus on finding solutions for this low-frequency issue. In addition, further investigation is needed to establish a systematic way of model order selection.

Due to space restrictions, important comparisons with other methods will also be portrayed in future work. Mainly, tests will be used to evaluate ANFs robustness while tracking quasi-harmonic partials subject to interferent partials. It is expected that, by adapting the ANF coefficients using the proposed model, deviations of the notches trajectories during frequency crossings can be more easily prevented then when adapting the ANF coefficients directly.

6. APPENDIX

Single ANF Initialization (for step 1)

$$k=\{0,1\},\ M=N=1,\ \hat{\omega}_1[k]=2\pi\, 900/f_s,\ \mathbf{P}[1,k]=10^{-3},$$
$$Q_l=Q[1,k]=10,\ Q_h=180,\ \overline{Q}[1,k]=0,\ P_\rho[1,k]=2\times10^{-2},$$
$$Q_1=30, \rho[1,k]=1-\hat{\omega}_1[k]/(Q[1,k]\pi),\ \delta_Q=1.5\times10^{-3},$$
$$\delta=10^{-4}, \lambda[1,k]=0.992, \lambda_\rho[1,k]=0.9992, \kappa=1, \kappa_\rho=1.002.$$

Single ANF Initialization (for step 2)

$$M=N=1,\ \mathbf{P}[2,k]=\hat{\omega}_1[k]2\times10^{-3},\ P_\rho[2,k]=2\times10^{-2},$$
$$\hat{\omega}_2[k]=1.2\,\hat{\omega}_1[k],\ \rho[2,k]=1-\hat{\omega}_2[k]/(Q[2,k]\pi),$$
$$Q_h=35,\ Q_2=22,\ \lambda[2,k]=0.992,\ \lambda_\rho[2,k]=0.9999,$$
$$\overline{Q}[2,k]=0,\ Q_l=Q[1,k]=Q[2,k]=6,\ \bar{r}[k]=0,\ \delta_r=10^{-2}.$$

Single ANF Initialization (octave resolution)

$$M=N=1,\ \hat{\omega}_3[k]=\max\{\hat{\omega}_1[k],\hat{\omega}_2[k]\}+0.5\|\hat{\omega}_2[k]-\hat{\omega}_1[k]\|,$$
$$\mathbf{P}[3,k]=2\times10^{-3},\ \rho[3,k]=1-\hat{\omega}_3[k]/(Q[3,k]\pi),\ \delta_Q=10^{-4},$$
$$Q_3=25,\ \overline{Q}[3,k]=0,\ Q_l=Q[3,k]=6, Q_h=35,\ Q[2,k]=6,$$
$$P_\rho[3,k]=2\times10^{-2},\ \lambda[3,k]=0.992,\ \lambda_\rho[3,k]=0.9999.$$

ANF Cascade Initialization (for step 3)

$$M=7, N=3,\ \kappa=1.01,\ \kappa_\rho=1.001,\ m=\{1,\ldots,M\},$$
$$\hat{\omega}_m[k]=(m/\eta)\min\{\hat{\omega}_1[k],\hat{\omega}_2[k]\},\ Q_l=6,\ Q_h=100,$$
$$\lambda[m,k]=\lambda_\rho[m,k]=\rho[m,k],\ Q[m,k]=40,\ \delta=0.002,$$
compute α_n for an N-equation system using (4),
$$\mathbf{P}[m,k]=2.5\times10^{-5}\hat{\omega}_1[k]\mathrm{diag}\{(5m)^{-3},(5m)^{-2},(5m)^{-1}\},$$
$$P_\rho[m,k]=P_\rho[1,k-1],\ \rho[m,k]=1-\hat{\omega}_m[k]/(Q[m,k]\pi).$$

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