THRESHOLD-BASED CHANNEL ESTIMATION FOR MSE OPTIMIZATION IN OFDM SYSTEMS

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ABSTRACT

In this paper, we describe a Threshold-Based Selection (TBS) approach for sparse channel estimation in an Orthogonal Frequency Division Multiplexing (OFDM) system. An optimal tap-tuned threshold is derived by minimizing the Mean Squares Error (MSE) per channel impulse response coefficient. Comparing the proposed TMSE approach to the Probabilistic Framework Estimator (PFE) and to former MSE optimization TBS algorithms using the criteria of Oliver, Kang and Rosati, shows its better performance in terms of true channel structure detection capacity and of channel response Normalized MSE (NMSE).

***Index Terms*—** Sparse channel estimation, threshold-based detection, structured estimation, MSE optimization.

1. INTRODUCTION

Orthogonal Frequency Division Multiplexing (OFDM) has received considerable interest as a high data rate wireless communication system thanks to its easy equalization over frequency selective channels implied by high data rate transmissions. However, the coherent equalizer good performance is conditioned upon the accuracy of the channel response estimation. Among the mostly used channel estimation methods are the Least Squares (LS) [5,6] and the Linear Minimum Mean Squares Error (LMMSE) [7] solutions. At low SNR, the LS performance is not acceptable because it does not use any information about the noise. Yet, the LMMSE algorithm, realizing higher performance, requires the channel statistics knowledge and a large amount of computations.

The enhanced versions of the LS, which have been introduced in [5,6], exploit the fact that the Cyclic Prefix (CP) in OFDM systems is longer than the Channel Impulse Response (CIR) in order to reduce the estimation noise.

Moreover, in practical wireless channels, only few of the multi-path components have significant energy, thus resulting in a sparse CIR structure. Indeed, several authors exploit the channel sparsity to further enhance its estimation performance. For instance, we distinguish the threshold-based algorithms [1-4,8,9] and a projection-based approach, named the Matching Pursuit method [10,11]. Also, a Generalized Akaike Information Criterion (GAIC) is proposed in [12].

The Threshold-Based Selection (TBS) approach consists in detecting the CIR structure by comparing the amplitude of a CIR raw estimate to a given threshold(s). TBS has been adopted differently in several methods, which can be divided into two categories. The first one considers a constant threshold value for all the channel taps, such as proposed in [2] by Kang *et. al*. In the same vein, Rosati *et. al*. [4] derive a sub-optimal threshold by minimizing the MSE of the global CIR. However, in the second category, an optimal threshold is tap-tuned such as in [1] and [3] where respectively the Probabilistic Framework Estimator (PFE) and the Threshold-based modified least squares (Tmls) method, optimizing the CIR structure detection performance, are presented.

This contribution proposes a novel TBS approach. Combining the criteria of Rosati *et. al*. [4] on MSE and the per-tap tuned threshold approach of PFE. The proposed TMSE optimizes for each tap a threshold to minimize the elementwise MSE. This elementwise MSE is a weighed sum of MSE corresponding to active and zero valued coefficients, respectively wheighed by the probability of the tap to be active or not. The local choice of threshold is expected to enhance the MSE performance compared to a global approach like that of Rosati [4], since it adapts for each CIR tap a threshold to minimize its specific MSE. These thresholds are then applied on the CIR LS estimate to detect its structure.

This paper is organized as follows. After introducing the system model in section 2, the TMSE algorithm is developed in section 3. Then, the performance evaluation, through numerical examples, is given in section 4 before the conclusion.

2. SYSTEM MODEL

We consider a cyclic prefixed OFDM system with N subcarriers operating over a multi-path channel. The subcarriers are modulated by symbols drawn from a normalized energy QAM or PAM constellation. We consider the case of a frequency selective multipath slow varying channel, whereby a block estimation can be envisaged. A pilot OFDM symbol is then regularly sent for channel estimation purpose.
A discrete version of the system is set, where the sampling rate is N times the subcarriers spacing, we assume that the Cyclic Prefix (CP), with length N_g samples, is longer than the channel memory L and that the synchronization is perfect. Then, at the receiver the concatenated N FFT outputs, after CP removal, are expressed as [2, 4]

$$\mathbf{Z} = \sqrt{N}\mathbf{AFh} + \tilde{\mathbf{N}}. \tag{1}$$

where $\mathbf{A}$ is a diagonal matrix whose entries are the pilot symbols denoted by $\{a_i\}$, $i = 1, \ldots, N$. $\mathbf{F}$ is the $N \times N$ unitary Discrete Fourier Transform (DFT) matrix with entries $\mathbf{F}_{l,m} = \frac{1}{\sqrt{N}} e^{\frac{-j2\pi(l-1)(m-1)}{N}}$ where $l, m = 1, \ldots, N$ and $\mathbf{h}$ is the sampled CIR. The complex noise $\tilde{\mathbf{N}}$ is Gaussian distributed as $\tilde{\mathbf{N}} \sim \mathcal{N}\left(0, \sigma^2 \mathbf{I}_N\right)$, where σ^2 is the noise variance.

3. OPTIMAL THRESHOLD FOR MSE MINIMIZATION PER CIR COEFFICIENT

In this section, first a survey on the unstructured and structured LS estimators is given. Then, the proposed TMSE scheme is detailed.

3.1. Overview of the unstructured and structured LS estimation

The CIR LS estimator [5,6] is expressed as

$$\hat{\mathbf{h}}_{ls} = \mathbf{h} + \frac{1}{\sqrt{N}} \mathbf{F}^H \mathbf{A}^{-1} \tilde{\mathbf{N}}. \tag{2}$$

The k^{th} tap LS CIR estimate is then

$$\hat{\mathbf{h}}_k = \mathbf{h}_k + \tilde{\mathbf{n}}_k^h, \tag{3}$$

where $\tilde{\mathbf{n}}_k^h \sim \mathcal{N}\left(0, 2\sigma_n^2\right)$, $\sigma_n^2 = \frac{\sigma^2}{2N^2} \sum_{i=1}^{N} \frac{1}{|a_i|^2}$ is the CIR LS estimation noise variance per dimension.
Since the CIR is sparse, only K out of L taps are effective, with non zero valued coefficients. Let $S = \{s_1, s_2, \ldots, s_K\}$ denote the active taps positions within the CIR and let $\mathbf{F}_s$ define the $N \times K$ sub-matrix of $\mathbf{F}$ obtained by selecting the columns corresponding to the active taps positions. The Channel Frequency Response (CFR) can then be expressed as

$$\mathbf{H} = \mathbf{Fh} = \mathbf{F}_s \mathbf{h}_s, \tag{4}$$

where the vector $\mathbf{h}_s$ of size K concatenates the non zero-valued entries of $\mathbf{h}$. In our study, the active taps positions are detected by a thresholding procedure and form the set $\hat{S} = \{\hat{s}_1, \hat{s}_2, \ldots, \hat{s}_{\hat{K}}\}$, where $\hat{K}$ is the number of taps detected as active.
The structured LS CIR estimate is obtained by combining (1) and (4) as

$$\hat{\mathbf{h}}_s = \frac{1}{\sqrt{N}} \hat{\mathbf{F}}_s^{\dagger} \mathbf{A^{-1} Z} = \hat{\mathbf{F}}_s^H \mathbf{F_s h_s} + \frac{1}{\sqrt{N}} \hat{\mathbf{F}}_s^H \mathbf{A}^{-1} \tilde{\mathbf{N}}, \tag{5}$$

where $\hat{\mathbf{F}}_s$ is a $N \times \hat{K}$ sub-matrix of $\mathbf{F}$ obtained by selecting the columns corresponding to the detected taps positions (elements of $\hat{S}$).
We hereafter characterize the statistical properties of the LS estimator. Due to the channel sparse structure, the amplitude of the CIR LS estimate given by (2) verifies

$$|\hat{\mathbf{h}}_k| = |\mathbf{h}_k + \tilde{\mathbf{n}}_k^h| \text{ for } k \in S, \tag{6}$$

$$|\hat{\mathbf{h}}_k| = |\tilde{\mathbf{n}}_k^h| \text{ for } k \notin S. \tag{7}$$

We assume that the CIR coefficients $\mathbf{h}_k$ are zero mean normal complex random variables with variance σ_{ck}^2 per dimention. Also, the coefficients $\tilde{\mathbf{n}}_k^h$ are modeled by a zero mean distribution with equal real and imaginary parts variance σ_n^2. So, the independence between $\mathbf{h}_k$ and $\tilde{\mathbf{n}}_k^h$ allows to deduce that $\hat{\mathbf{h}}_k = \mathbf{h}_k + \tilde{\mathbf{n}}_k^h$ of (3) is also zero mean normally distributed with variance σ_{sk}^2 per dimension such as

$$\sigma_{sk}^2 = \sigma_{ck}^2 + \sigma_n^2 \text{ for } k \in S, \tag{8}$$

$$\sigma_{sk}^2 = \sigma_n^2 \text{ for } k \notin S. \tag{9}$$

Consequently, the estimated CIR amplitude $|\hat{\mathbf{h}}_k|$ has a Rayleigh distribution with mean $\sqrt{\frac{\pi}{2}}\sigma_{sk}$ and variance $\frac{4-\pi}{2}\sigma_{sk}^2$.

3.2. Proposed TBS approach

For each tap k, located in the CP interval, $k = 1, \ldots, N_g$, a threshold t_k is applied on the amplitude of the LS raw CIR estimate. The threshold t_k is tuned to optimize the CIR elementwise MSE.
In the following, we first derive the MSE per CIR coefficient, then analytically compute the optimal threshold expression. Some practical implementation issues of the proposed scheme are finally discussed.

3.2.1. MSE per CIR coefficient

The tap elementwise MSE over the CIR length, is given by

$$MSE_k = E\left(|\Delta \mathbf{h}_k|^2\right), \tag{10}$$

with $\Delta \mathbf{h}_k = \hat{\mathbf{h}}_k - \mathbf{h}_k$. For tap k detection, four events can occur:

⋆ **Correct Rejection** : a sample corresponding to a zero valued tap such as $|\hat{\mathbf{h}}_k| = |\tilde{\mathbf{n}}_k| < t_k$ is correctly rejected. This case does not contribute to MSE_k.
⋆ **False Alarm** : a sample that does not contain any channel energy is reckoned as active, since its energy is greater than the threshold. In this case, MSE_k is expressed as

$$MSE_k = E\left(|\tilde{\mathbf{n}}_k^h|^2\right) = 2\sigma_n^2.$$

⋆ **Missed Detection** : a sample corresponding to an active tap is not detected, since $|\hat{\mathbf{h}}_k| = |\mathbf{h}_k + \tilde{\mathbf{n}}_k| < t_k$. This leads to

$$MSE_k = E\left(|\mathbf{h}_k|^2\right) = 2\sigma_{ck}^2.$$

⋆ **Correct Detection** : an active tap is correctly detected, $|\mathbf{h}_k| = |\mathbf{h}_k + \tilde{\mathbf{n}}_k| > t_k$ and then

$$MSE_k = E\left(|\tilde{\mathbf{n}}_k^h|\right) = 2\sigma_n^2.$$

According to these four possibilities, the k^{th} tap MSE is

$$\begin{aligned} MSE_k &= P_a\left(P_m 2\sigma_{ck}^2 + (1-P_m)\,2\sigma_n^2\right) \\ &\quad + (1-P_a)\,2\sigma_n^2 P_{fa} \qquad (11) \\ &= 2P_a\left(\sigma_{ck}^2 - \sigma_n^2\right)P_m + 2P_a\sigma_n^2 \\ &\quad + 2(1-P_a)\,\sigma_n^2 P_{fa}. \qquad (12) \end{aligned}$$

where P_a is the probability of tap k to be active, P_{fa} and P_m are respectively the probabilities of false alarm and of missing given by

$$P_{fa}(t_k) = P\left(|\tilde{\mathbf{n}}_k^h| > t_k\right) = e^{-\frac{t_k^2}{2\sigma_n^2}}, \text{ and} \qquad (13)$$

$$P_m(t_k) = P\left(|\mathbf{h}_k + \tilde{\mathbf{n}}_k^h| < t_k\right) = 1 - e^{-\frac{t_k^2}{2\left(\sigma_{ck}^2+\sigma_n^2\right)}}. \qquad (14)$$

In this way,

$$\begin{aligned} MSE_k(t_k) &= 2P_a\left(\sigma_{ck}^2 - \sigma_n^2\right)\left(1 - e^{-\frac{t_k^2}{2(\sigma_n^2+\sigma_{ck}^2)}}\right) \\ &\quad + 2P_a\sigma_n^2 + 2(1-P_a)\,\sigma_n^2 e^{-\frac{t_k^2}{2\sigma_n^2}}. \qquad (15) \end{aligned}$$

3.2.2. *Optimal Threshold*

Our aim is to find the optimal threshold t_{kopt} which minimizes MSE_k. In other words, we focus on the search of the global minimum of MSE (15) as

$$t_{kopt} = argmin_{t_k \geq 0} MSE_k(t_k). \qquad (16)$$

To this end, the derivative of $MSE_k(t_k)$ w.r.t. t_k is analyzed. From (15), this derivative is expressed as

$$\begin{aligned} \frac{\partial MSE}{\partial t_k} &= 2t_k(1-P_a)\,e^{-\frac{t_k^2}{2(\sigma_n^2+\sigma_{ck}^2)}} \\ &\quad \times \left[\frac{P_a}{1-P_a}\frac{\sigma_{ck}^2-\sigma_n^2}{\sigma_n^2+\sigma_{ck}^2} - e^{-\frac{t_k^2\sigma_{ck}^2}{2\sigma_n^2(\sigma_n^2+\sigma_{ck}^2)}}\right] \qquad (17) \end{aligned}$$

We set $\alpha = \frac{P_a}{1-P_a}\frac{\sigma_{ck}^2-\sigma_n^2}{\sigma_n^2+\sigma_{ck}^2}$ and $\beta = \alpha - e^{-\frac{t_k^2\sigma_{ck}^2}{2\sigma_n^2(\sigma_n^2+\sigma_{ck}^2)}}$. The last derivative (17) equals zero at two possible values of t_k, denoted t_{k1} and t_{k2}, which verify

$$t_{k1} = 0, \qquad (18)$$

$$\alpha - e^{-\frac{t_{k2}^2\sigma_{ck}^2}{2\sigma_n^2(\sigma_n^2+\sigma_{ck}^2)}} = 0. \qquad (19)$$

The second zero t_{k2} makes sense if and only if $0 \leq \alpha \leq 1$. In the following, we distinguish two cases.
• $\sigma_{ck}^2 \leq \sigma_n^2$: In this case, $\alpha \leq 0$ and $\beta \leq 0$ leading to $\frac{\partial MSE}{\partial t_k} \leq 0$ then, the threshold minimizing the MSE is $t_{kopt} = +\infty$, leading to $P_{fa} = 0$ and $P_m = 1$.
• $\sigma_{ck}^2 > \sigma_n^2$: In this case, $\alpha > 0$ and we distinguish two sub-cases.
-If $P_a \leq 0.5$: $\frac{P_a}{1-P_a} \leq 1$ and then, $\alpha < 1$, therefore, (19) admits a solution denoted t_{kopt} and given by

$$t_{kopt} = \sqrt{2\sigma_n^2\left(1 + \frac{\sigma_n^2}{\sigma_{ck}^2}\right)\ln\left(\frac{1-P_a}{P_a}\frac{\sigma_{ck}^2+\sigma_n^2}{\sigma_{ck}^2-\sigma_n^2}\right)}, \qquad (20)$$

which coincides with t_{k2} (19).
-If $P_a > 0.5$: we distinguish two-sub cases.
⋆ For $\sigma_{ck}^2 > \frac{\sigma_n^2}{2P_a-1}$: $\alpha > 1$ and the optimal threshold is equal to $t_{k1} = 0$ which leads to $P_{fa} = 1$ and $P_m = 0$.
⋆ For $\sigma_{ck}^2 \leq \frac{\sigma_n^2}{2P_a-1}$: $\alpha \leq 1$ and the optimal threshold is given by (20).

3.2.3. *TMSE implementation*

The equation (20) shows that for t_{kopt} evaluation, both of σ_n^2 and σ_{ck}^2 must be known. So, we propose to estimate them from the LS CIR noisy estimate $\hat{\mathbf{h}}$, as

$$\hat{\sigma}_n = \sqrt{\frac{2}{\pi}}\frac{1}{N-N_g}\sum_{i=N_g+1}^{N}|\hat{\mathbf{h}}_i|, \text{ and} \qquad (21)$$

$$\hat{\sigma}_{ck} = \sqrt{\max\left(\frac{2}{\pi}|\hat{\mathbf{h}}_k|^2 - \hat{\sigma}_n^2, 0\right)}, k = 1, \ldots, N_g, \qquad (22)$$

where (21) exploits the fact that $\hat{\mathbf{h}}_i$ for $i \geq N_g$ is a pure noise component. Also, the threshold computation requires the knowledge of the probability of active taps value P_a. If

no a priori knowledge about the Channel Degree of Sparsity (CDS) is available, we set $P_a = 0.5$. Otherwise, if K is known at the receiver, we set $P_a = \frac{K}{N_g}$.
It is worth noting at this level that even if the proposed scheme is formulated in the case of slow fading block type channel estimation, it can be easily extended to non stationary channels, using a raw comb channel estimation.

4. PERFORMANCE EVALUATION

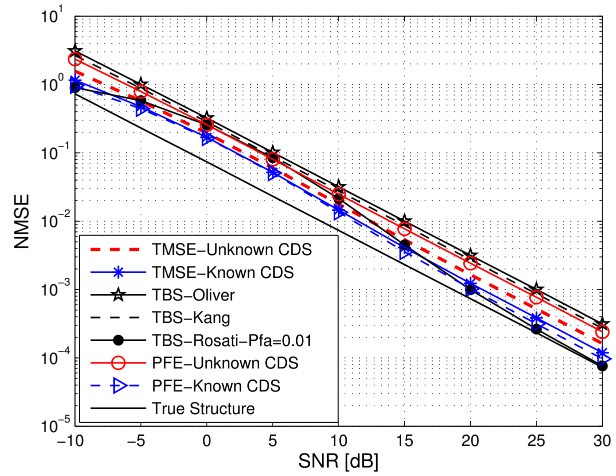
We here compare the proposed TMSE method performance to those of the following methods of channel estimation: the PFE [1], Kang *et. al.* [2], Oliver *et. al.* [3] and Rosati *et. al.* [4] where their sub-optimal version is considered for an overall probability of false alarm set to 10^{-2}.
The performance of the different schemes is assessed in terms of the Normalized Mean Squares Error (NMSE) on the CFR and of the capacity of true CIR structure detection, where the rates of tap missing, false alarm and true structure detection are presented versus the $SNR = \frac{E_s}{\sigma^2}$, ranging from -10dB to 30dB, where E_s is the transmitted energy per symbol. The NMSE is defined as NMSE $= \frac{E\left(\|\mathbf{H}-\hat{\mathbf{H}}\|_2^2\right)}{E\left(\|\mathbf{H}\|_2^2\right)}$, where $\hat{\mathbf{H}}$ denotes the CFR estimate recovered as $\hat{\mathbf{H}} = \hat{\mathbf{F}}_s \hat{\mathbf{h}}_s$.

4.1. Simulation parameters

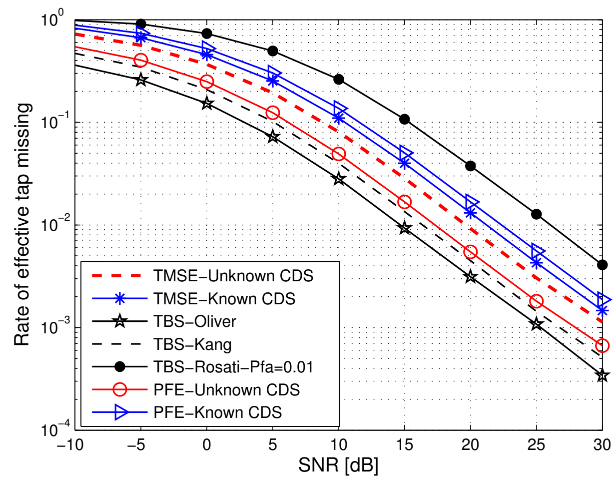
An OFDM system with $N = 64$ subcarriers and symbols drawn from a normalized energy 16 QAM is used over a multipath channel. The CP is of length $N_g = \frac{N}{4}$. An OFDM block thus consists of 80 samples, 16 of which are included in the CP. The channel coherence time is supposed to cover a number of OFDM blocks. One OFDM symbol is used as pilot to derive a LS coarse block estimate. The channel power-delay profile is exponential. It has a maximal memory length of $L = N_g = 16$. To evaluate the performance of the channel estimators, the results are averaged over 8.10^4 Monte Carlo trials for each SNR value. At each realization, the $K = 3$ taps are randomly positioned within the CP interval at positions s_1, s_2 and s_3 with coefficient $\mathbf{h}_{s_i}$ verifying $\mathbf{h}_{s_i} = |\mathbf{h}_{s_i}|e^{j\Phi_i}$ with $\mathbf{h}_{s_i} \sim \mathcal{N}\left(0, 2\sigma_{ck}^2 = e^{-\beta s_i}\right)$ and $\Phi_i \sim \mathcal{U}_{[0,2\pi]}$. The power-delay profile has a decreasing speed of $\beta = \frac{8}{N_g}$. Both cases of known and unknown CDS are envisaged.

4.2. Numerical results

The NMSE performance of the TBS algorithms is depicted in figure (a), whereby a structured LS is performed based on the beforehand detected CIR structures, for all of the compared methods. It shows that for known CDS, the PFE and TMSE methods achieve comparable performance with a slight advantage of the PFE at high SNR.

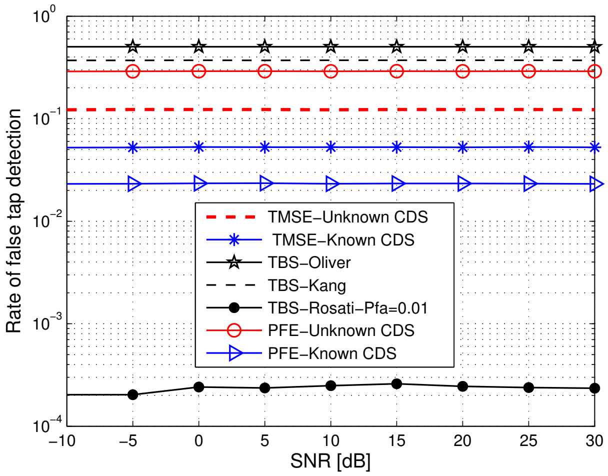


(a) CFR Normalized Mean Squares Error versus SNR.

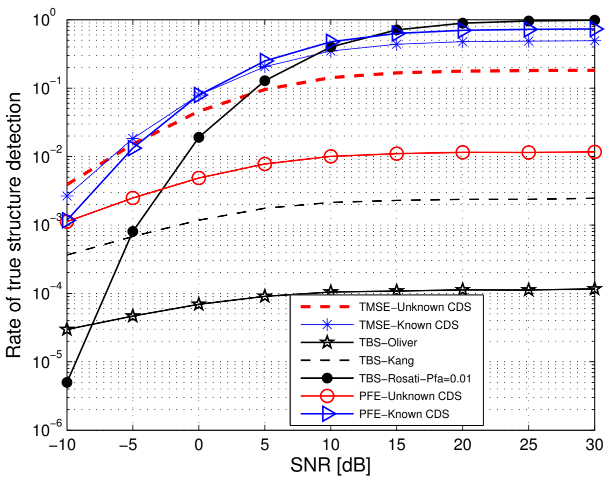


(b) Rate of active tap missing versus SNR.

For the more realistic scenario of unknown CDS, the TMSE achieves a gain of 2dB w.r.t. the PFE algorithm throughout all the SNR range thanks to its elementwise MSE optimization.
Even if the TBS-Rosati NMSE attains that obtained in the true CIR structure case at high SNR, it is outperformed by the proposed TMSE approach below 19dB because of its low rate of true structure detection. However above 15 dB, the NMSE achieved by Rosati is lower because its detection capacity of the structure is much better than that of TMSE.
Examining the active tap missing and the false tap detection rates obtained by the different schemes and displayed respectively in figures (b) and (c), shows an obvious compromise between these two detection performance measures. Indeed, for the active tap missing rate, Rosati algorithm has the poorest result, followed by PFE then TMSE in the known CDS case, then in the unknown CDS case, Kang and Oliver meth-



(c) Rate of false tap detection versus SNR.



(d) Rate of true CIR structure detection versus SNR.

ods lead to the lowest missing rates. This order is exactly reversed in the false alarm tap detection rates. It can be noticed in figure (c) that, for all methods, the false tap detection rates exhibit a flat behavior as a function of the SNR, in contrast to the missing rates, which are decreasing functions of the SNR.

Figure (d) displays the rate of true CIR structure detection and shows that discarding the known CDS case, which leads to the best performance, the higher true detection rate averaged over the SNR range is achieved by the TMSE approach.

5. CONCLUSION

In the frame of OFDM channel estimation, we proposed a novel threshold-based scheme, which we named TMSE. This scheme allows to detect the channel impulse response structure, which is further used to perform a structured channel response estimation in the least squares sense. The optimized threshold is tuned for each channel coefficient to minimize the corresponding mean squares error. Comparing the TMSE performance to those of existing threshold based selection solutions shows its better performance in terms of normalized MSE and of true structure detection performance.

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