ASYMMETRIC AVERAGE CONSENSUS UNDER SINR-BASED INTERFERENCE

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ABSTRACT

Consensus algorithms are a family of distributed processes that are based on exchanging local information in order to obtain some particular global information. An example of these algorithms is the average consensus, in which the value to be obtained is the average of some initial data. Most of the existing consensus techniques assume unrealistic models of communications that require complex control mechanisms in practice. In contrast, we consider the average consensus algorithm under a realistic asynchronous and asymmetric scheme of communications, where the interferences constrain the information exchanged among the nodes. To ensure a correct operation in this scheme, we propose a link scheduling protocol that satisfies certain convergence conditions and maximizes the number of simultaneous links in each iteration of the consensus algorithm. This increase in the number of communications per iteration improves the performance of the consensus algorithm. Simulation results are presented to verify and clearly show the efficiency of our approach.

***Index Terms*—** Link Scheduling, Average Consensus, SINR-based interference, Wireless Sensor Networks

1. INTRODUCTION

The average consensus algorithm is a well-known case of study in Wireless Sensor Networks (WSNs). This is a distributed process to obtain the global average that avoids the the need of performing all the computations at one or more sink nodes, thus, reducing congestion around these nodes and incrementing the robustness of the network. However, consensus algorithms have been generally studied from a level of abstraction that ignores the details of communications that occur at lower levels. This leads to assume certain conditions that are not easily obtained in practice. A clear example is the consensus algorithm presented in [1], which assumes undirected graphs and fixed and perfect communications. In general, these assumptions are not readily available in a wireless environment because the communications are performed in the presence of environmental factors such as interferences and background noise. In particular, the interferences constrain the number of simultaneous communications and require the use of complex control mechanisms to ensure the convergence conditions. This problem is slightly mitigated by the so-called gossip algorithm [2]. In this process, each node randomly picks up a neighbor and iteratively computes a symmetric average pairwise. This variation still requires symmetric communications, except notable exceptions as the work in [6]. Since the number of simultaneous communications and the pairwise symmetry are still constrained by the interferences, an alternative scenario is to apply the consensus algorithm over an asymmetric and asynchronous scheme of communications. In this scheme, the instantaneous connectivity patterns are asymmetric and random, which require new conditions to ensure convergence [3] and a probabilistic analysis [3][4]. The work in [7] takes into account the interferences that occur in wireless communications. However, although this approach guarantees the convergence conditions proposed in [3], it presents a poor performance in terms of simultaneous links. The number of simultaneous communications determines the instantaneous random topologies that are generated during consensus. This parameter is directly related with the convergence rate of the average consensus algorithm and its associated MSE, as shown in [4].

In this paper, the consensus process is performed under an asymmetric and asynchronous scheme of communications in which the convergence is ensured by the simultaneous execution of a new link scheduling protocol. Moreover, we focus on maximizing the number of simultaneous links. Finally, we show experimentally how this effect increases the convergence rate of the average consensus and reduces the MSE.

The remainder of this paper is structured as follows: The problem formulation is given in Section II. The interference model and our link scheduling approach are presented in Section III. We then present, in Section IV, some numerical results to show the efficiency of our approach. Finally, the conclusions are summarized in Section V.

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2. PROBLEM FORMULATION

A WSN where the communications between the nodes are constrained by the wireless interferences and other environmental factors can be modeled as a time-varying graph $\mathbf{G}(k) = (\mathbf{V}, \mathbf{E}(k))$, consisting of a set $\mathbf{V}$ of N nodes and a set $\mathbf{E}(k) \subset \mathbf{E}$ of edges. In particular, we denote a directed edge from node j to node i as e_{ij}, where the presence of an edge e_{ij} between two nodes indicates that there exists a directed information flow between them. Given a time-varying graph $\mathbf{G}(k)$, we can assign an $N \times N$ adjacency matrix $\mathbf{A}(k)$ where an entry is equal to 1 if $e_{ij} \in \mathbf{E}(k)$ and 0 otherwise. The set of neighbors of a node i is defined as $\mathbf{\Omega}_i(k) = \{j \in \mathbf{V} : e_{ij} \in \mathbf{E}(k)\}$ and the in-degree matrix $\mathbf{D}(k)$ is a diagonal matrix whose entries are given by $d_i(k) = |\mathbf{\Omega}_i(k)|$. Then, the Laplacian of a graph $\mathbf{G}(k)$ is a matrix defined as $\mathbf{L}(k) = \mathbf{D}(k) - \mathbf{A}(k)$ whose smallest eigenvalue can be shown to be equal to zero. The matrix $\mathbf{A}(k)$ is non symmetric random and its expected value is $\bar{\mathbf{A}} = \mathbf{P}$, where P_{ij} denotes the probability of establishing a link from node j to node i.

Every node $i \in \mathbf{V}$ contains a scalar value $\mathbf{x}_i(k)$, defined as the state of node i at time k. The state is initialized at $k = 0$ and evolves in time by means of using only local data exchange, namely:

$$\mathbf{x}_i(k+1) = \mathbf{W}_{ii}(k)\mathbf{x}_i(k) + \sum_{j \in \mathbf{\Omega}_i(k)} \mathbf{W}_{ij}(k)\mathbf{x}_j(k) \quad (1)$$

where the weight matrix can be expressed as:

$$\mathbf{W}(k) = \mathbf{I} - \alpha \mathbf{L}(k) \quad (2)$$

where α is a constant independent of time and $\mathbf{I}$ denotes the identity matrix.

In our scenario, a matrix $\mathbf{A}(k)$ is determined by the set of randomly activated links in iteration k, among which collisions must be avoided. Every matrix $\mathbf{A}(k)$ is random and independent of each other. Therefore, the weight matrices $\mathbf{W}(k)$ at the various iteration steps are random, independent of each other, non-symmetric and row-stochastic with its largest eigenvalue equal to one. We model the initial set of values as real valued Gaussian random variables $\mathbf{x}(0) = [\mathbf{x}_1(0), \mathbf{x}_2(0), \cdots, \mathbf{x}_N(0)]^T$, with mean μ and variance σ_0^2, thus the average of the initial state $\mathbf{x}(0)$ is $\mathbf{x}_{\text{avg}} = \frac{1}{N}\mathbf{1}^T\mathbb{E}[\mathbf{x}(0)]\mathbf{1} = \mu\mathbf{1}$, where $\mathbb{E}[\cdot]$ denotes the expected value. Due to the randomness of both the activation of links in $\mathbf{A}(k)$ and the initial set of values $\mathbf{x}(0)$, the convergence of $\mathbf{x}(k)$ in (1) must be studied in probabilistic terms.

Under this probabilistic scenario, our problem is to design a new link scheduling protocol being capable to avoid collisions and ensure both the convergence in expectation of the vector $\mathbf{x}(k)$ and its convergence in the MSE sense with certain degree of accuracy. These forms of convergence are introduced in [3][4] and are recalled here for convenience.

Convergence in expectation: We say that the vector $\mathbf{x}(k)$ converges to the vector $\mathbf{x}_{\text{avg}}$ in expectation if:

$$\lim_{k\to\infty} \mathbb{E}[\mathbf{x}(k)] = \mathbf{x}_{\text{avg}} \quad (3)$$

Convergence of the MSE of the state: We also relate the convergence of the vector $\mathbf{x}(k)$ to the convergence of the MSE of the state, defined as:

$$\text{MSE}(\mathbf{x}(k)) = \frac{1}{N}\mathbb{E}\left[||\mathbf{x}(k) - \mathbf{x}_{\text{avg}}||_2^2\right] \quad (4)$$

The evolution of the vector $\mathbf{x}(k)$ in (1) can be rewritten as:

$$\mathbf{x}(k) = \prod_{l=1}^{k} \mathbf{W}(k-l)\mathbf{x}(0)$$

where the vector $\mathbf{x}(k)$ converges in expectation to the average if $\lim_{k\to\infty} \mathbb{E}[\mathbf{x}(k)] = \mathbf{x}_{\text{avg}}$, which is equivalent to:

$$\lim_{k\to\infty} \mathbb{E}\left[\prod_{l=1}^{k} \mathbf{W}(k-l)\mathbf{x}(0)\right] = \mathbf{x}_{\text{avg}}$$

Assuming matrices $\mathbf{W}(k)$ independent on each other and $\prod_{l=1}^{k} \mathbf{W}(k-l)$ independent of $\mathbf{x}(0)$:

$$\lim_{k\to\infty} \mathbb{E}\left[\prod_{l=1}^{k} \mathbf{W}(k-l)\right]\mathbb{E}[\mathbf{x}(0)] = \mathbf{x}_{\text{avg}}$$

It has been shown in [4], that $\mathbb{E}\left[\prod_{l=1}^{k} \mathbf{W}(k-l)\right] = \bar{\mathbf{W}}^k$, which allows us to express the convergence as:

$$\lim_{k\to\infty} \bar{\mathbf{W}}^k\mathbb{E}[\mathbf{x}(0)] = \mathbf{x}_{\text{avg}}$$

where $\bar{\mathbf{W}} = \mathbf{I} - \alpha\bar{\mathbf{L}}$. In order to ensure convergence, the matrix $\bar{\mathbf{W}}$ should satisfy the following well-known conditions:

$$\bar{\mathbf{W}}\mathbf{1} = \mathbf{1}; \quad \mathbf{1}^T\bar{\mathbf{W}} = \mathbf{1}^T; \quad \rho\left(\bar{\mathbf{W}} - \frac{\mathbf{1}\mathbf{1}^T}{N}\right) < 1 \quad (5)$$

Our contribution is to ensure average consensus convergence in a new asymmetric, asynchronous and collision-free scheme of communications. This is possible by the simultaneous execution of a new link scheduling protocol that satisfies all the convergence conditions in (5). The first and the last conditions in (5) are ensured by the structure of the matrix $\bar{\mathbf{W}} = \mathbf{I} - \alpha\bar{\mathbf{L}}$ and a value of α sufficiently small [4]. However, the second condition is explicitly fulfilled by our link scheduling protocol, providing a sequence of connectivity patterns that lead to a matrix $\bar{\mathbf{A}} = \mathbf{P}$ symmetric. This guarantees convergence in expectation (3) and reduces the error in (4) as we show experimentally in Section IV.

3. LINK SCHEDULING PROTOCOL

In this section, we propose a variation of the link scheduling protocol presented in [7]. Our new protocol considerably increases the number of simultaneous links, by correctly adapting the communication areas, which directly affects the performance of the average consensus algorithm.

3.1. Interference model

We consider a WSN where the communications between the nodes are affected by environmental factors such as background noise and interferences. We define a transmit power, denoted by P_t, that is common to every node in the network. Moreover, the channel gain between a transmitter node j and a receiving node i is $\frac{1}{r_{ij}^{\gamma}}$, where $\gamma \geq 2$ is the path loss exponent.

We assume the SINR interference model according to which the successful reception of a packet sent by node j and destined to node i depends on the SINR at node i, that is, a packet between j and i is correctly received only if

$$\frac{\frac{P_t}{r_{ij}^{\gamma}}}{\varsigma + \sum\limits_{v \in \mathbf{V}, v \neq j} \frac{P_t}{r_{iv}^{\gamma}}} \geq \beta \tag{6}$$

where ς is the background noise and β is a given constant threshold, that depends on the hardware component being used. In this model, all the simultaneous transmissions are considered when evaluating whether a single transmission is valid. Therefore, every link can affect each other even if they do not share the intended receiver, which makes the problem to be NP-hard [5]. This implies that a heuristic link scheduling protocol should be used.

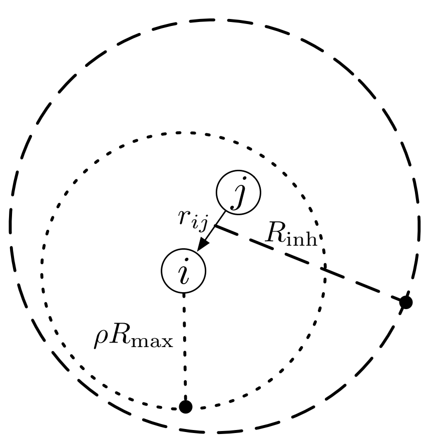

Fig. 1. Relation between the different areas and radii that are used by our link scheduling protocol. The center of the inhibition area is forced to be in the middle of a link in order to ensure a symmetric matrix $\mathbf{P}$ [7].

3.2. Link scheduling protocol

Our link scheduling protocol is based on defining the suitable distances between transmitters and receivers in order to ensure the correct operation of the average consensus algorithm by means of avoiding collisions. This operation increases the number of simultaneous links and saves energy due to the avoidance of unsuccessful communications.

Taking into account (6), the maximum transmission radius denoted $R_{\max}$, is defined as the maximum distance up to which a packet can be correctly received in absence of interference:

$$R_{\max} = \left(\frac{P_t}{\varsigma \beta} \right)^{\frac{1}{\gamma}} \tag{7}$$

Since a transmission from a node located at a distance equal to $R_{\max}$ implies that no other link can be scheduled simultaneously without collision, we assume that the length of the links to be scheduled in each iteration of the consensus process is lower or equal than the product of $R_{\max}$ and a constant factor $0 < \rho < 1$. Although the value of ρ must be large enough $\rho R_{\max} \geq \sqrt{2 \log N / N}$ to ensure connectivity with high probability [2], it is accomplished that, the smaller the value of ρ, the greater number of simultaneous links can be obtained in each step of the link scheduling protocol.

Our link scheduling protocol is based on inhibiting the links that are at certain distance of every randomly activated link. The inhibition radius R_{inh} is defined from (6), and it refers to the distance from which all the nodes j, that are at distance $r_{ij} \leq R_{\text{inh}}$ from the receiving node i, are inhibited. In a more formal way, we have the following expression:

$$R_{\text{inh}} \geq \left(\frac{\beta \eta P_t}{\frac{P_t}{(\rho R_{\max})^{\gamma}} - \varsigma \beta} \right)^{\frac{1}{\gamma}} \tag{8}$$

where η is an estimation of the maximum number of simultaneous transmissions in the current deployment. For obtaining expression (8), it is assumed that the transmitter and the η interferers are separated from the receiver by distances $\rho R_{\max}$ and R_{inh} respectively. Finally, substituting in (8) the expression of (7), it becomes:

$$R_{\text{inh}} \geq \left(\frac{\eta P_t}{\frac{\varsigma}{\rho^{\gamma}} - \varsigma} \right)^{\frac{1}{\gamma}} \tag{9}$$

The main purpose of our scheduling algorithm is to distributively avoid the occurrence of collisions during the execution of the consensus algorithm. This increases the number of simultaneous links and avoids the waste of energy due to unsuccessful communications. For this purpose, nodes randomly wake up, e.g. by using timers, and randomly choose one of its candidate links to be activated. Since all the links should have the same opportunities, nodes with a greater number of neighbors are activated more often. Then, the timers are proportional to the degree of the nodes. Once a link is activated, all other links in the same inhibition area are inhibited. To inhibit such links, the nodes having links at distance smaller than R_{inh} should be able to detect the energy variation produced by the activation of the new link, or alternatively, the current node should explicitly notify the new activation to the rest of the nodes within its inhibition area. An assumption required to avoid all the collisions is that two nodes in the same inhibition area do not wake up simultaneously. In practice, this is not possible without some coordination between the nodes. However, the occurrence of these collisions, which are dependent on the period of the timers, only implies a slightly reduction in the total number of simultaneous links per iteration.

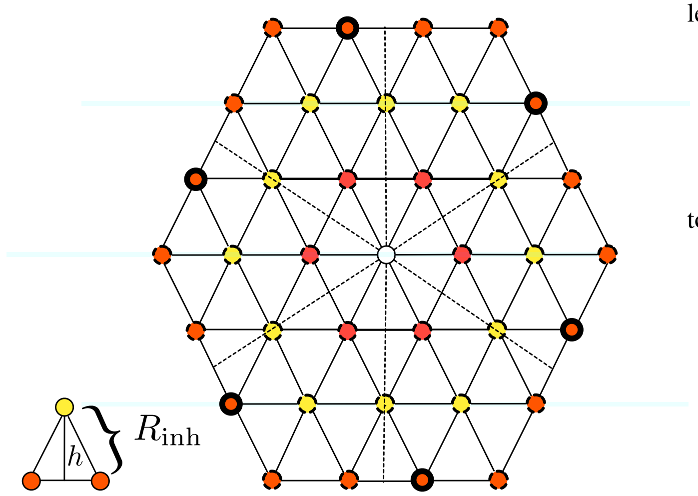

Fig. 2. The packing of the interfering links in the worst case. R_{inh} is the side of the triangle and h denotes its height.

3.3. Estimation of the parameter η

The densest packing of nodes with the R_{inh} distance requirement is the hexagon packing, see Fig. 2. Given this packing of interfering links in the worst case, we propose an iterative method for obtaining an approximation to the number of simultaneous links η and the corresponding inhibition radius R_{inh}. This is based on obtaining the maximum interference of nodes at distance R_{inh} from a given node. For this purpose, we argue in terms of right triangles within the hexagon packing. The calculation is divided in two parts (dotted and solid nodes) and also for odd and even levels (red and yellow nodes), see Fig. 2. Moreover, for a given value of R_{inh}, we define the maximum number of levels in the unit square area, denoted by l_{MAX}, as $l_{\text{MAX}} = \sqrt{2}/R_{\text{inh}}$.

Note that a node at distance $2R_{\text{inh}}$ produces $\frac{1}{2^{\gamma}}$ less interference than a node at R_{inh}, which is the distance used as a reference.

The interference produced by the dotted nodes that belong to the odd level is determined by the following expression:

$$\sum_{k=0}^{\lfloor \ell/2 \rfloor} \frac{6}{\left(\left(\frac{1}{2}+k\right)^2 + (\ell h)^2\right)^{\frac{\gamma}{2}}}$$

where k denotes the relative position of the current node in terms of triangles and ℓ denotes the level being considered. The equivalent expression for the even level is:

$$\sum_{k=0}^{\ell/2} \frac{6}{\left(k^2 + (\ell h)^2\right)^{\frac{\gamma}{2}}}$$

We can compact the equations for the odd and the even level as follows:

$$\sum_{k=0}^{\lfloor \ell/2 \rfloor} \frac{6}{\left(\left(\ell/2 - \lfloor \ell/2 \rfloor + k\right)^2 + (\ell h)^2\right)^{\frac{\gamma}{2}}} \tag{10}$$

We also have to consider the solid nodes. Then, we need to consider the following for the odd level:

$$\sum_{k=1}^{\lfloor \ell/2 \rfloor} \frac{6}{\left(\left(k-\frac{1}{2}\right)^2 + (\ell h)^2\right)^{\frac{\gamma}{2}}}$$

The equivalent expression for the even level is:

$$\sum_{k=2}^{\ell/2} \frac{6}{\left((k-1)^2 + (\ell h)^2\right)^{\frac{\gamma}{2}}}$$

Compacting again:

$$\sum_{k=2}^{\lfloor \ell/2 \rfloor} \frac{6}{\left(\left(\ell/2 - \lfloor \ell/2 \rfloor + k - 1\right)^2 + (\ell h)^2\right)^{\frac{\gamma}{2}}} + \frac{6}{((1/2)^2 + (\ell h)^2)^{\frac{\gamma}{2}}} \tag{11}$$

Combining (10) and (11) we obtain the final equation for the parameter η given by (12). For example, it is easy to verify that for a given $\gamma = 2$, the first level $\ell = 1$ produces the interference of six nodes, the second level $\ell = 2$ produces the interference of 3.5 nodes, etc.

Algorithm 1 η and R_{inh} calculation

Ensure: maximum value of η obtained for a given packing
  $\eta = 1$;
  while the value of η is increased **do**
    obtain R_{inh} using expression (9);
    $l_{\text{MAX}} = \sqrt{2}/R_{\text{inh}}$;
    update η using expression (12);
  end while

$$\sum_{\ell=1}^{l_{\text{MAX}}} \left(\sum_{k=0}^{\lfloor \ell/2 \rfloor} 6 \left((\ell/2 - \lfloor \ell/2 \rfloor + k)^2 + (\ell h)^2 \right)^{-\frac{\gamma}{2}} + \sum_{k=2}^{\lfloor \ell/2 \rfloor} 6 \left((\ell/2 - \lfloor \ell/2 \rfloor + k - 1)^2 + (\ell h)^2 \right)^{-\frac{\gamma}{2}} + 6 \left((1/2)^2 + (\ell h)^2 \right)^{-\frac{\gamma}{2}} \right) \tag{12}$$

4. SIMULATION RESULTS

We model a WSN as a random geometric graph of $N = 100$ nodes inside a $2D$ unit square area, where each node measurement is modeled as an independent Gaussian random variable with mean $\mu = 20$ and variance $\sigma_0^2 = 8$. The information is mixed as described in (2) where the instantaneous topology determines which data is mixed. Channel gains are computed based on node positions, and on the radio propagation model. Radio signal propagation is assumed to follow log-normal shadowing, with path loss exponent $\gamma = 3$. Additionally, for a given background noise $G = 10^{-9} mW$ and a given value of $\beta = 1$, we choose a combination of the values P_t and ρ to ensure connectivity on average, that is, $\rho R_{\max} = \sqrt{2 \log N / N}$, which has been shown in [2] to ensure connectivity with high probability in a random geometric graph.

Fig. 3 shows the convergence behavior of the average consensus algorithm when this is simultaneously executed with our link scheduling and also with a CSMA based protocol, in which the transmitters decide whether to transmit or not based on a given constant threshold. The symmetric version of the protocol presented in [7] is also compared with our link scheduling protocol, assuming both 5% and 0% of unsuccessful transmissions produced by nodes waking up at a time in the same inhibition area. Fig. 3 also shows that the increase in the number of simultaneous links reduces both the convergence time and the MSE of the average consensus algorithm.

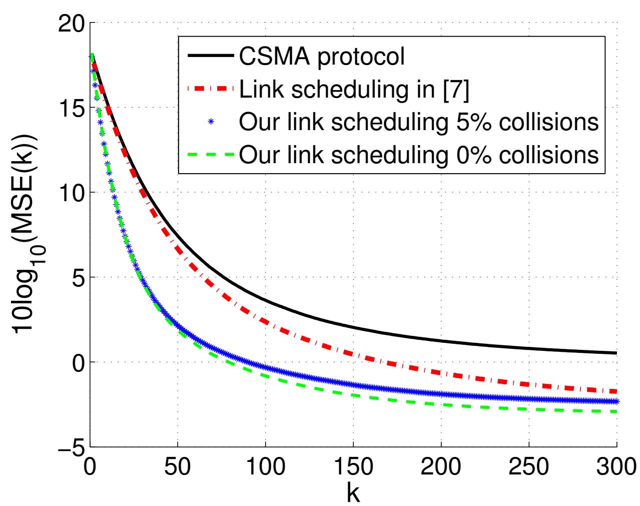

Fig. 3. Convergence comparison with N=100 nodes over 100 random geometric graphs.

5. CONCLUSIONS

In this paper, we have considered a realist communications scheme in which the average consensus algorithm is executed. For ensuring its convergence, we have guaranteed, by the simultaneous execution of a new link scheduling protocol, both the symmetry of the links in probabilistic terms and the maximization of the number of simultaneous links. The convergence time and the MSE are both reduced as compared with existing methods in the literature. Simulation results are presented to verify the efficiency of our approach.

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