A UNIFIED APPROACH TO STRUCTURED COVARIANCES: POLYNOMIAL VANDERMONDE BEZOUTIAN REPRESENTATIONS

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ABSTRACT

This paper shows how the theory of recurrence related polynomials is connected to the construction of covariance displacement operators and their diagonalization properties. It is demonstrated that covariance Bezoutians admit a broader class of polynomial Vandermonde based representations, and are not limited to factor circulants, commonly seen in the literature. We show that there is sufficient freedom in the choice of operators, such that more general eigenvector factorizations other than the DFT-based are possible. These become key to achieving efficient matrix-vector multiplications required in signal processing and communications, as the ones arising in modern multicarrier and frequency-domain equalization methods.

1. INTRODUCTION

Displacement theory is an attractive way to efficiently exploit structure when realizing mathematical formulas in signal processing and communications. In [1], this concept is approached in view of arbitrarily structured $N \times M$ data matrices $\mathcal{H}_{M,N}$ possessing a fixed relation between two successive rows $\{u_{M,k}\}$ of $\mathcal{H}_{M,N}$, i.e.,

$$\breve{u}_{M,N+1} = u_{M,N}\Psi_M. \tag{1}$$

where Ψ_M is a structured matrix induced by the input network. In this scenario, it is shown how the generating vectors of an $M \times M$ inverse covariance matrix $P_{M,N,L} = (\Pi^{-1} + \mathcal{H}^*_{M,N} W \mathcal{H}_{M,N})^{-1}$ are sequentially computed through an *Extended Generalized Fast Transversal Filter* (EGSWFTF), in a *causal* manner, where Π is a positive definite matrix and W provides a diagonal weighting as a sliding window with a single breakpoint after L past samples (although this can be extended to an arbitrary number of breakpoints) and forgetting factor λ, as illustrated below:

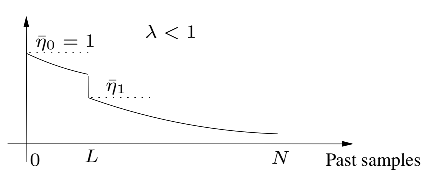

Fig. 1. *Generalized window.*

The EGSWFTF is truly fast, as long as matrix-vector multiplications of the form $\Psi_M^{-1} \cdot x$ which arise in the algorithm for some vector x, are efficiently as well. That is, although the recursions of [1], and consequently the displacement rank of $P_{M,N,L}$ are obtained regardless of Ψ_M, the EGSWFTF requires efficient multiplications with Ψ_M^{-1}, and requires that Ψ_M itself possesses a particular structure. The same holds for $\breve{P}_{M,N,L}$, with respect to $\breve{u}_{M,N}$.

The way the operator is generated is crucial not only for complexity reasons, but also for numerical stability when representing these covariances. Applications include minimum-mean-square-error (MMSE), least-squares (LS), or pure matrix inverse solutions encountered in channel estimation and equalization scenarios, where the data matrix is entirely known or estimated a priori [3]. In these cases, the estimate assumes the form of a matrix-vector multiplication $P_{M,N,L} \cdot x$, or simply $H^{-1}_{M,N} \cdot y$. An efficient, or compact representation of $P_{M,N,L}$ is thus highly desirable, and as we shall show in this work, it is obtained by solving its displacement equation with respect to suitable operators $\{\Phi_{M,\theta}, \Phi_{M,\varsigma}\}$.

More specifically, I have recently shown in [1] that any data structure such that (1) holds, induces displacement operators of the general form $\Phi_{M,\theta} = \Psi_M Z_\theta^{-1}$ [or $\Phi_{M,\theta} = Z_\theta^{-1}\Psi_M$ with respect to $\breve{u}_{M-1,N}$], where Z_θ^{-1} is a companion matrix, i.e.,

$$Z_\theta = \begin{bmatrix} 0 & 0 & \cdots & 0 & \theta_0 \\ 1 & 0 & \cdots & 0 & \theta_1 \\ \vdots & \ddots & & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & \theta_{M-1} \end{bmatrix}. \tag{2}$$

For example, tap-delay-line models allow us to choose the displacement operators as companion matrices $\Phi_{M,\theta} = Z_\theta^{-1}$ and $\Phi_{M,\varsigma} = Z_\varsigma^{-1}$, giving rise to the so-called *superfast* $\mathcal{O}(M \log^p M)$ representations [2], where $d \leq 2$. Efficiency in this case is a result of the Vandermonde structure of their eigenvectors, so when its defining nodes are equispaced on circles of radius, say, $|\phi_0|$ and $|\varrho_0|$, these matrices simply collapse to DFT filterbanks.

One way to achieving efficient realizations lies in the construction of operators induced by *recurrence related polynomials*, since in this case, inversion or eigenvector factorization can be quickly performed via fast or superfast realizations. Which recursion to be employed (*2-term, 3-term*, etc.) will depend on the objective at hand, usually targeting efficient realizations, compactness in modeling, and numerical robustness. For example, the choice of orthonormal IIR basis leads to a shorter basis representation, while the degree of freedom in the construction of the displacement operator further allows for a specific eigenvector factorization of the inverse covariance. I have shown that regardless of the input basis, the resulting displacement operator can always be designed such that its eigenvalues are equispaced on a circle, which does not necessarily imply that the corresponding *Bezoutian* (an inverse covariance) yields an efficient eigenvector decomposition. That is, efficiency and numerical

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robustness may require the construction of operators with eigenvalues placed at arbitrary locations, as it may be desirable to use transformations other than DFT-based [1]. Considering the aforementioned, in this paper we address the following fundamental questions on structured inverse decompositions:

1) Is it possible to obtain inverse or inverse covariance representations based on *arbitrary* displacement operators?

2) How do we choose the input basis and the displacement operator, in order to achieve a filterbank type, and more importantly, efficient decompositions for MMSE or matrix inverse solutions?

3) Given that DFT based formulas can be interpreted as simply as a DFT filterbank realization, it is only natural to ask whether exact decompositions based on other transforms are possible. For example, can we obtain exact forms based on real transformations such as discrete cosine and sine matrices, unlike the only existing DFT-based?

The answer to these questions is positive, and the core message here, is that one is not required to search for a specific operator that will lead to a low displacement-rank, and consequently to an efficient representation of a certain inverse or inverse covariance, usually desired in signal processing and communications applications. On the contrary, in these contexts, it is rather the operator that acts on the data, redefining its structure; in this sense, a low displacement rank will hold regardless of the operator, and the relevant question here is how one should pick a suitable basis that will induce an alternative representation useful for a certain purpose.

2. DIAGONALIZATION OF CONFEDERATE MATRICES

Consider a *transversal* system realization illustrated in Fig. 2.

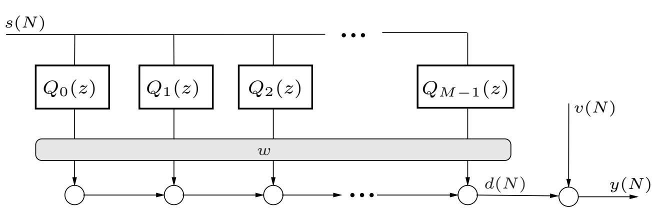

Fig. 2. *Transversal realization based on general basis functions*.

There are numerous ways in which we can construct the polynomial basis $\{Q_m(z)\}$. They can be realized via two-terms, three-terms, or generally, via M*-term* recurrence relations, each particular one serving to a different purpose, including better numerical conditioning, reduced computational complexity, compact representation of models, and so on. In this section, we shall see how recurrence relations lead to the solution of open problems and connections.

Thus, consider the realization of Fig. 3, which is obtained from the (shifted) M-term recurrence relation

$$Q_0(z) = 1\ ,\ \ Q_1(z) = \delta_0 z^{-1} Q_0(z) \quad (3)$$

$$Q_m(z) = (\bar{a}_{m-2,m-1} + \delta_{m-1} z^{-1}) Q_{m-1}(z) + \bar{a}_{m-3,m-1} Q_{m-2}(z) + \ldots + \bar{a}_{0,m-1} Q_1(z) \quad (4)$$

$$\breve{\Upsilon}_M(z) = \psi_0 Q_0(z) z^{-1} + \cdots + \psi_{M-1} Q_{M-1}(z) z^{-1} \quad (5)$$

for $m = 0, 1, \ldots, M-1$. The (unshifted) polynomial $\Upsilon_M(z)$ has been referred to as the *master polynomial* associated to $\{Q_m(z)\}$ [4].

This results in the following upper triangular matrix Ψ_M of (1):

$$\Psi_M = \begin{bmatrix} \delta_0 & \bar{a}_{01}\delta_0 & (\bar{a}_{12}\bar{a}_{01} + \bar{a}_{02})\delta_0 & \cdots & \psi_0 \\ 0 & \delta_1 & \bar{a}_{12}\delta_1 & \cdots & \psi_1 \\ 0 & 0 & \delta_2 & \cdots & \psi_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ddots & \psi_{M-2} \\ 0 & 0 & 0 & & \psi_{M-1} \end{bmatrix} \quad (6)$$

2.1. *Relation between Ψ_M and the Confederate structure*

For the M-term recurrence of (4), it can be verified that the matrix $\Phi_{M,\theta}^{-1} = \Psi_M^{-1} Z_\theta$ has a *Hessenberg* form, and has been referred to as a *confederate matrix* associated to the system of polynomials $\{Q_m(z)\}$ (see e.g., [5] and its references). The confederate matrix has several useful properties, and in particular, its eigenvalues are directly related to $\breve{\Upsilon}_M(z)$, which is a free polynomial (see next section). It is defined, e.g., in [5] under the notation "H_Q", where the role of the companion form Z_θ is fixed, and associated to the coefficients of the system transfer function $G(z)$ (therein their definition differs from ours with respect to its last column). Moreover, in [5], this association is such that the highest polynomial order on the right hand side of (4) is in terms of $Q_{m-1}(z)$, since in these references only FIR basis functions are considered. Here, we further allow for a first-order *rational* transfer function relating two successive basis functions, so that an additional m-th order term may arise on the r.h.s. of (4).

In [1], I have shown that whenever the displacement operator $\Phi_{M,\theta}$ has equispaced eigenvalues on a circle, there exists a general eigenvector representation for the covariance matrix $P_{M,N,L}$ which follows from the solution of its displacement equation with respect to the form $\Phi_{M,\theta} = Z_\theta^{-1}\Psi_M$. One natural question is whether it is possible to enforce this condition in general, and more importantly, realizing it in the context of the recurrence related polynomials. The answer is positive, and further suggests that the free structures of $\{Z_\theta, Z_\varsigma\}$ can be suitably chosen to construct operators $\{\Phi_{M,\theta}, \Phi_{M,\varsigma}\}$ with eigenvalues placed at any desired location.

2.2. *Diagonalization of Polynomial Vandermonde Matrices and Signal Flow Graph Connections*

Consider the vector multiplication in (1). Since $u_{M,N-1}$ is updated to $\breve{u}_{M,N}$ with only M operations through Ψ_M, it is obvious that applying the same structure to an arbitrary vector implies a one sample filtering step through the same network that originated Ψ_M in first place. Moreover, the inverse operation Ψ_M^{-1} can be simply realized by invoking *dual* realizations and its defining *Horner-like* polynomials — see, e.g., [5]. This is a consequence of the well known theory of realization of digital filters, which has been reemerged more recently in the context of fast structured matrix operations and factorizations. The procedure for obtaining the Horner-like polynomials is realized by reversing the signal flow graph directions of the original network, and identifying the Horner-like polynomials $\tilde{\mathcal{R}} = \{\tilde{R}_k(z^{-1})\}$ as the partial transfer functions seen from the input to the tap-delays inputs in the dual system. These dual polynomials easily realize inverse, transposition and matrix factorizations efficiently, and are paramount to the connections presented in this work. Just like the original basis, the associated Horner-like polynomials are computed through recurrences involving any number of terms as well. In a more general case of the M-term recursion of (3),

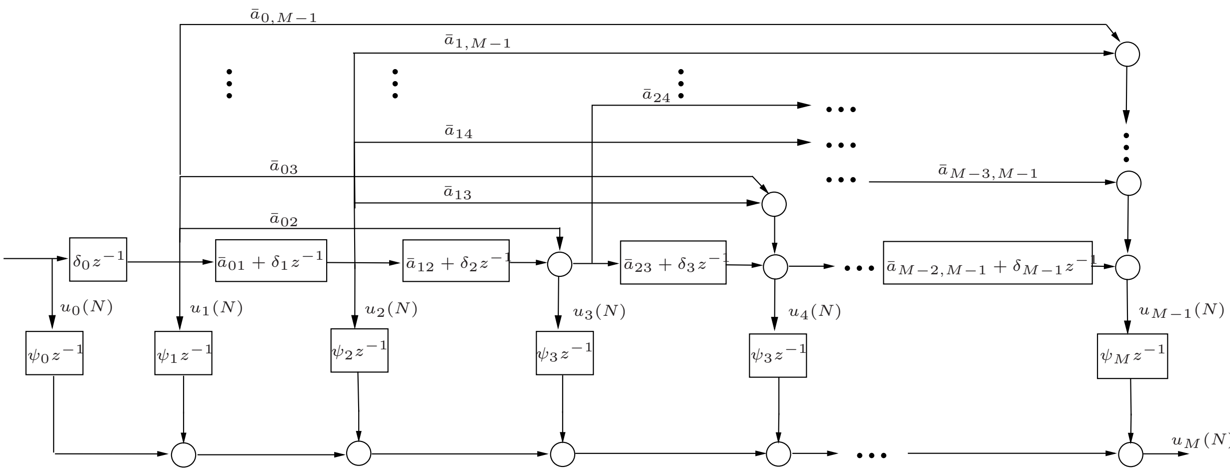

Fig. 3. *M-term realization.*

these satisfy the M-term relations

$$\tilde{R}_0(z) \;=\; 1\,, \quad R_1(z) = \tilde{\delta}_0\phi_M z^{-1} R_0(z) \tag{7}$$

$$\tilde{R}_k(z) \;=\; (\tilde{a}_{k-2,k-1} + \tilde{\delta}_{k-1}z^{-1})\tilde{R}_{k-1}(z) \;+\; \tilde{a}_{k-3,k-1}\tilde{R}_{k-2}(z) \;+\; \ldots \;+\; \tilde{a}_{0,k-1}\tilde{R}_1(z) + \phi_{M-k} \tag{8}$$

where $\{\phi_{M-k}\}$ are the coefficients of the master polynomial of order $M+1$, i.e.,

$$\tilde{\Upsilon}_{M+1}(z) = \phi_0 Q_0(z) \;+\; \phi_1 Q_1(z) \;+\; \cdots \;+\; \phi_M Q_M(z) \tag{9}$$

with $\tilde{\delta}_k = \delta_{M-k}$ and $\tilde{a}_{k,j} = \frac{\delta_{M-j}}{\delta_{M-k}}\bar{a}_{M-j,M-k}$. In the monomial case, these polynomials, denoted by $\tilde{\mathcal{P}} = \{\tilde{h}_k(z)\}$ are simply called *Horner polynomials*, and satisfy the recursion

$$\tilde{h}_k(z) = z^{-1}\tilde{h}_{k-1}(z) + \phi_{M-k}\;, \quad \tilde{h}_0(z) = \phi_M. \tag{10}$$

We now establish the connection between these polynomials and the associated displacement operators constructed along with the companion structures $\{Z_\theta, Z_\varsigma\}$.

Definition 1 (Polynomial Vandermonde Matrix and Inverse) *Let $\{z_0, z_1, \ldots, z_{M-1}\}$ be M distinct numbers. Given the set of recurrence related bases $\mathcal{Q} = \{Q_0(z), Q_1(z), \ldots, \breve{\Upsilon}_M(z)\}$, the corresponding polynomial Vandermonde matrix is given by*

$$V_{\mathcal{Q}}(\bar{z}) = \begin{bmatrix} Q_0(z_0) & Q_1(z_0) & \cdots & Q_{M-1}(z_0) \\ Q_0(z_1) & Q_1(z_1) & \cdots & Q_{M-1}(z_1) \\ \vdots & \vdots & \ddots & \vdots \\ Q_0(z_{M-1}) & Q_1(z_{M-1}) & \cdots & Q_{M-1}(z_{M-1}) \end{bmatrix}. \tag{11}$$

Also, the so-called Horner-like polynomials $\tilde{\mathcal{Q}} = \{\tilde{R}_k(z)\}$ define its inverse which is itself a polynomial matrix given by

$$V_{\mathcal{Q}}^{-1}(\bar{z}) = \frac{1}{\phi_M} R(\bar{z})\mathcal{D}_{\mathcal{Q}}(\bar{z}),$$

where

$$R_{\mathcal{Q}}(\bar{z}) = \begin{bmatrix} \tilde{R}_{M-1}(z_0) & \tilde{R}_{M-1}(z_1) & \cdots & \tilde{R}_{M-1}(z_{M-1}) \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{R}_1(z_0) & \tilde{R}_1(z_1) & \cdots & \tilde{R}_1(z_{M-1}) \\ \tilde{R}_0(z_0) & \tilde{R}_0(z_1) & \cdots & \tilde{R}_0(z_{M-1}) \end{bmatrix} \tag{12}$$

$$\mathcal{D}(z) = diag(d_0^{-1}, \ldots, d_{M-1}^{-1}), \quad with \quad d_i = \prod_{\substack{k=0 \\ k\neq i}}^{M-1}(z_k^{-1} - z_i^{-1}). \tag{13}$$

The recurrence polynomial formulas described above have been used to extend the Parker-Forney-Traub algorithm and the Björck-Pereyra algorithm for computing inverse Vandermonde, to polynomial Vandermonde inverses in several cases in $\mathcal{O}(M^2)$ operations, and more importantly, with excellent numerical precision, even for extremely high condition numbers— see [6] and the references therein for a summary of these works.

Lemma 1 (Rank-one displacement) *Let $Q_m(z)$ satisfy the recurrence relation (3)-(5), and define $\Theta'(z) \triangleq \sum_{m=0}^{M-1}\theta'_m Q_m(z)$ where $\{\theta'_m\}$ are associated with the companion matrix $Z_{\theta'}$. Assume that $z_0, \ldots, z_{M-1}$ are M distinct numbers and define the m-th pinning vector $\mathbb{e}_m \triangleq [\,0\ \cdots\ 0\ 1\ 0\ \cdots\ 0\,]^{\mathsf{T}}$. Then,*

$$V_{\mathcal{Q}}(\bar{z})\Psi_M \;-\; \Lambda_z V_{\mathcal{Q}}(\bar{z}) \;=\; \Lambda_{\bar{z}}\ell(\bar{z})\mathbb{e}_0 \tag{14}$$

where $V_{\mathcal{Q}}(\bar{z})$ is the polynomial Vandermonde defined in (11), and $\ell(\bar{z}) = \frac{1}{\theta'_0}\left[\breve{\Upsilon}_M(z_0) - \Theta'(z_0)\ \cdots\ \breve{\Upsilon}_M(z_{M-1}) - \Theta'(z_{M-1})\right]^T$.

Proof: The recurrence equations in (3)-(5) can be written as

$$\begin{aligned} &\left[\; Q_0(z) \quad Q_1(z) \quad \cdots \quad Q_{M-1}(z) \;\right]\Psi_M = \\ &\qquad z\left[\; Q_1(z) \quad Q_2(z) \quad \cdots \quad \breve{\Upsilon}_M(z) \;\right] \end{aligned} \tag{15}$$

$$\begin{aligned} &= \; z\left(\left[\; Q_0(z) \quad Q_1(z) \quad \cdots \quad Q_{M-1}(z) \;\right] Z_{\theta'} \right.\\ &\qquad \left. + \left[\breve{\Upsilon}_M(z) - \Theta'(z)\right]\mathbb{e}^*_{M-1}\right) \\ &= \; \left[\; Q_1(z) \quad Q_2(z) \quad \cdots \quad \Theta'(z) \;\right] Z_{\theta'}^{-1}\Psi_M \\ &= \; \left(\left[\; Q_1(z) \quad Q_2(z) \quad \cdots \quad \breve{\Upsilon}_M(z) \;\right]\right. \\ &\qquad \left. - \left[\breve{\Upsilon}_M(z) - \Theta'(z)\right]\mathbb{e}^*_{M-1}\right) Z_{\theta'}^{-1}\Psi_M \end{aligned} \tag{16}$$

so that for M distinct numbers $z_0, \ldots, z_{M-1}$, we obtain (14).

Corollary (Diagonalization of operators from recurrence polynomials) *Let $\{\bar{z}_i\}$ be a zero of $\Upsilon_M(z_i) - \Theta'(z_i)$, $m = 0, \ldots, M-1$, and pick $\theta'_i = \theta_i$ in (16), so that $Z_{\theta'} = Z_\theta$ and $\Theta(z) = \Theta'(z)$.*

Then, the operators $\Phi_{M,\theta}$ and $\breve{\Phi}_{M,\theta}$ admit the Jordan factorizations

$$\Phi_{M,\theta} \triangleq \Psi_M Z_\theta^{-1} = V_{\mathcal{Q}}^{-1}(\bar{z})\Lambda_{\bar{z}} V_{\mathcal{Q}}(\bar{z}) \tag{17}$$

$$\breve{\Phi}_{M,\theta} \triangleq Z_\theta^{-1}\Psi_M = \breve{V}_{\mathcal{Q}}^{-1}(\bar{z})\Lambda_{\bar{z}} \breve{V}_{\mathcal{Q}}(\bar{z}) \tag{18}$$

where

$$\breve{V}_{\mathcal{Q}}(\bar{z}) = \begin{bmatrix} Q_1(z_0) & Q_2(z_0) & \cdots & \breve{\Upsilon}_M(z_0) \\ Q_1(z_1) & Q_2(z_1) & \cdots & \breve{\Upsilon}_M(z_1) \\ \vdots & \vdots & \ddots & \vdots \\ Q_1(z_{M-1}) & Q_2(z_{M-1}) & \cdots & \breve{\Upsilon}_M(z_{M-1}) \end{bmatrix}, \text{ and} \tag{19}$$

$\{\bar{z}_i\}$ are the eigenvalues of $\Phi_{M,\theta}$ satisfying the (scaled) characteristic polynomial $\Omega_{M,\theta}(z) \triangleq \breve{\Upsilon}_M(z) - \Theta(z)$. Moreover, $\Omega(\bar{z}_i) = 0$ can be solved for ψ in terms of θ or vice-versa via (17) as

$$\psi = V_Q^{-1}(\bar{z})\Lambda_{\bar{z}}\Theta(\bar{z}) \quad or \quad \theta = V_Q^{-1}(\bar{z})\Lambda_{\bar{z}}^{-1}\Upsilon_M(\bar{z}). \tag{20}$$

or according to (18) as

$$\psi = Z_\theta \breve{V}_Q^{-1}(\bar{z})\Lambda_{\bar{z}}\Upsilon_M(\bar{z}) \quad or \quad \theta = \Psi_M \breve{V}_Q^{-1}(\bar{z})\Lambda_{\bar{z}}^{-1}\breve{\Upsilon}_M(\bar{z}). \tag{21}$$

Proof: From (16), it is easy to verify that $\Psi_M Z_\theta^{-1} = \Phi_{M,\theta} = V_Q^{-1}(\bar{z})\Lambda_{z,\theta} V_{\mathcal{Q}}(\bar{z}) Z_{\theta'} Z_\theta^{-1}$ so that choosing $\theta' = \theta$ yields (17), and (20) follows. The same argument applies when $\Phi_{M,\theta}$ is defined as $Z_\theta^{-1}\Psi_M$ instead, leading to (18) and consequently to (21).

Equations (20) and (21) guarantee that the Bezoutian representation in Theorem 1 of [1] is indeed feasible, since it allows for the construction of operators $\Phi_{M,\theta}$ having eigenvalues equispaced on a circle. The eigenvector matrices can be associated to a Vandermonde polyphase matrix of filterbank architecture, and therefore designed according to different criteria. Different nodes will lead to different designs for these matrices; As a consequence, one can expect that analogous representations based on other efficient filterbank designs are possible. This is to be further elaborated next.

3. VANDERMONDE REPRESENTATION FROM HESSENBERG BASED DISPLACEMENTS

In the previous sections, we showed that the construction of polynomial Vandermonde based representations is possible via proper choices for the "free" polynomials $\Upsilon_M(z)$ and $\Theta(z)$, which in turn define the operator $\Phi_{M,\theta}$. While efficient superfast $\mathcal{O}(M \log M)$ DFT filterbanks are easily obtained from a Vandermonde based representation, with nodes equispaced on circles, it is natural to ask whether other bases yield superfast decompositions as well. For instance, it is widely known that proper choice of the nodes in polynomial Vandermonde matrices lead to efficient DST and DST realizations, which provide perfect numerical conditioning in signal processing applications (see, e.g., [7]). Similar characteristics are obtained through Szego-Vandermonde operators, which are unitary (or almost unitary) and also highly desirable for practical purposes. These are missing connections which have been open for a while and which constitute the central result of this paper.

In [1], I have obtained the displacement equation for the Ricatti variable $\breve{P}_{M,N,L}$ in connection with its defining fast Kalman recursion variables as

$$\begin{aligned}
&\nabla_{\{\breve{\Phi}_{M,\theta},\breve{\Phi}^*_{M,\varsigma}\}}(\breve{P}_{M,N,L}) \triangleq \breve{P}_{M,N,L} - \lambda^{-1}\breve{\Phi}_{M,\theta}\breve{P}_{M,N,L}\breve{\Phi}^*_{M,\varsigma} \\
&= \lambda^{N-2}\breve{\Phi}_{M,\theta}\bar{\breve{k}}^{d_o}_{M,N}\bar{\breve{k}}^{d_o*}_{M,N}\breve{\Phi}^*_{M,\varsigma} + \bar{w}^b_{M-1,N,L}\bar{w}^{b*}_{M-1,N,L} \\
&- \lambda^{-1}Z_\theta^{-1}\bar{w}^f_{M-1,N-1,L}\bar{w}^{f*}_{M-1,N-1,L}Z_\varsigma^{-*} \\
&- \lambda^{-1}\tilde{k}^d_{M-1,N-1,L-1}\tilde{k}^{d*}_{M-1,N-1,L-1} - \tilde{k}_{M-1,N,L}\tilde{k}^*_{M-1,N,L}
\end{aligned} \tag{22}$$

with $\bar{w}^b_{M-1,N,L}, \bar{w}^f_{M-1,N-1,L}, \bar{\breve{k}}^{d_o}_{M,N}, \tilde{k}_{M-1,N,L}, \tilde{k}^d_{M-1,N-1,L-1}$ corresponding to normalized backward and forward prediction vectors, and Kalman gains associated to the 3 breakpoints in data, i.e., at 0, L_0, and L_1 (see Fig. 1). By solving (22), we arrive at the following main result (as an abuse of notation, we shall denote by $1/\bar{z}$ the entrywise inversion of the vector $\bar{z}$):

Theorem 1 (Polynomial Vandermonde Representation of Covariance Bezoutians) *Let $\breve{P}_{M,N,L}$ be a covariance matrix arising in a generalized window LS formulation for an arbitrary recurrence related polynomial basis $\{Q_k(z)\}$. Let $\bar{z}_1 = [z_1(0) \ \cdots \ z_1(M-1)]^T$ be the distinct eigenvalues of $\breve{\Phi}_{M,\theta} \triangleq Z_\theta^{-1}\Psi_M$ satisfying the characteristic polynomial $\Omega_{M,\theta}(z)$. Also, given $\Upsilon_M(\bar{z}_1)$ and the nodes $\bar{z}_1$, define the following corresponding* matrix-valued *polynomial:*

$$\bar{\Omega}_\theta(\bar{\Phi}_{M,\theta}) = \phi_0 I + \sum_{m=1}^{M} \lambda^{1/2}\bar{\phi}_m\bar{\Phi}^{-m}_{M,\theta} \tag{23}$$

$$with \quad \theta = \Psi_M V_{\mathcal{Q}}^{-1}(\bar{z}_1)\Lambda_{\bar{z}_1}^{-1}Q_{M-1}(\bar{z}_1), \tag{24}$$

as well as its slightly changed version,

$$\bar{\Omega}'_\theta(\bar{\Phi}_{M,\theta}) = \varrho_0 I + \sum_{m=1}^{M} \lambda^{1/2}\bar{\phi}_m\bar{\Phi}^{-m}_{M,\theta} \tag{25}$$

obtained by replacing ϕ_0 with another arbitrary scale ϱ_0. Set $\bar{z}_2 = [z_2(0) \ z_2(1) \ \cdots \ z_2(M-1)]^T$, such that $\bar{z}_1^*(m)\bar{z}_2(m) \neq 1$, $\{\bar{z}_1, \bar{z}_2\} \in \mathbb{C}$, and let $\{\breve{V}_{\mathcal{Q}}(\bar{z}_1), \breve{V}_{\mathcal{Q}}(\bar{z}_2)\}$ be the eigenvector matrices corresponding to $\{\breve{\Phi}_{M,\theta}, \breve{\Phi}_{M,\varsigma}\}$ respectively. Then, if $\bar{\Omega}_\theta(\Lambda^{-*}_{\bar{z}_2}) \neq 0$, $\breve{P}_{M,N,L}$ can be decomposed as

$$\begin{aligned}
\breve{P}_{M,N,L} = \ & \breve{V}_{\mathcal{Q}}^{-1}(\bar{z}_1)\Big[\Lambda_{V_1,\tilde{k}}V_{\mathcal{P}}(\bar{z}_1)R_{\mathcal{P}}(1/\bar{z}_2^*)\Lambda^*_{V_2,\tilde{k}} \\
& - \Lambda_{V_1,\bar{\breve{k}}^{d_o}}V_{\mathcal{P}}(\bar{z}_1)R_{\mathcal{P}}(1/\bar{z}_2^*)\Lambda^*_{V_2,\bar{\breve{k}}^{d_o}} \\
& + \Lambda_{V_1,\bar{w}_f}V_{\mathcal{P}}(\bar{z}_1)R_{\mathcal{P}}(1/\bar{z}_2^*)\Lambda^*_{V_2,\bar{w}_f} \\
& - \Lambda_{V_1,\bar{w}_b}V_{\mathcal{P}}(\bar{z}_1)R_{\mathcal{P}}(1/\bar{z}_2^*)\Lambda^*_{V_2,\bar{w}_b} \\
& + \Lambda_{V_1,\tilde{k}^d}V_{\mathcal{P}}(\bar{z}_1)R_{\mathcal{P}}(1/\bar{z}_2^*)\Lambda^*_{V_2,\tilde{k}^d}\Big] \cdot \\
& \bar{\Omega}_\theta^{-1}(\Lambda^{-*}_{\bar{z}_2})\breve{V}_{\mathcal{Q}}^{-*}(\bar{z}_2)
\end{aligned} \tag{26}$$

where $R_{\mathcal{P}}(1/\bar{z}_2^*)$ is the dual polynomial matrix of (12) with respect to $\Omega_{M,\theta}(z)$ and $\Lambda_{V_1,\bar{\breve{k}}^{d_o}} = diag(q_{\bar{\breve{k}}^{d_o},\theta})$, $\Lambda_{V_2,\bar{\breve{k}}^{d_o}} = diag(q_{\bar{\breve{k}}^{d_o},\varsigma})$, $\Lambda_{V_1,\bar{w}_f} = diag(q_{f,\theta})$, $\Lambda_{V_2,\bar{w}_f} = diag(q_{f,\varsigma})$ $\Lambda_{V_1,\bar{w}_b} = diag(q_{b,\theta})$, $\Lambda_{V_2,\bar{w}_b} = diag(q_{b,\varsigma})$, $\Lambda_{V_1,\tilde{k}^d} = diag(q_{\tilde{k}^d,\theta})$ $\Lambda_{V_2,\tilde{k}^d} = diag(q_{\tilde{k}^d,\varsigma})$ $\Lambda_{V_1,\tilde{k}} = diag(q_{\tilde{k},\theta})$, $\Lambda_{V_2,\tilde{k}} = diag(q_{\tilde{k},\varsigma})$ and where we define the frequency-domain vectors

$$q_{\bar{\breve{k}}^{d_o},\theta} = \lambda^{(N-2)/2}\breve{V}_{\mathcal{Q}}(\bar{z}_1)\bar{\breve{k}}^{d_o}_{M,N}, \tag{27}$$

$$q_{f,\theta} = \lambda^{-1/2}\Lambda_{\bar{z}_1}^{-1}\breve{V}_{\mathcal{Q}}(\bar{z}_1)Z_\theta^{-1}\bar{w}^f_{M-1,N-1,L}, \tag{28}$$

$$q_{b,\theta} = \Lambda_{\bar{z}_1}^{-1}\breve{V}_{\mathcal{Q}}(\bar{z}_1)\bar{w}^b_{M-1,N,L}, \tag{29}$$

$$q_{\tilde{k}^d,\theta} = \lambda^{-1/2}\Lambda_{\bar{z}_1}^{-1}\breve{V}_{\mathcal{Q}}(\bar{z}_1)\tilde{k}^d_{M-1,N,L-1}, \tag{30}$$

$$q_{\tilde{k},\theta} = \Lambda_{\bar{z}_1}^{-1}\breve{V}_{\mathcal{Q}}(\bar{z}_1)\tilde{k}_{M-1,N,L}, \tag{31}$$

with analogous definitions with respect to $\{\breve{V}_{\mathcal{Q}}(\bar{z}_2), \Lambda_{\bar{z}_2}, Z_\varsigma\}$.

The above result requires a long proof and we shall omit it here due to space limitations.

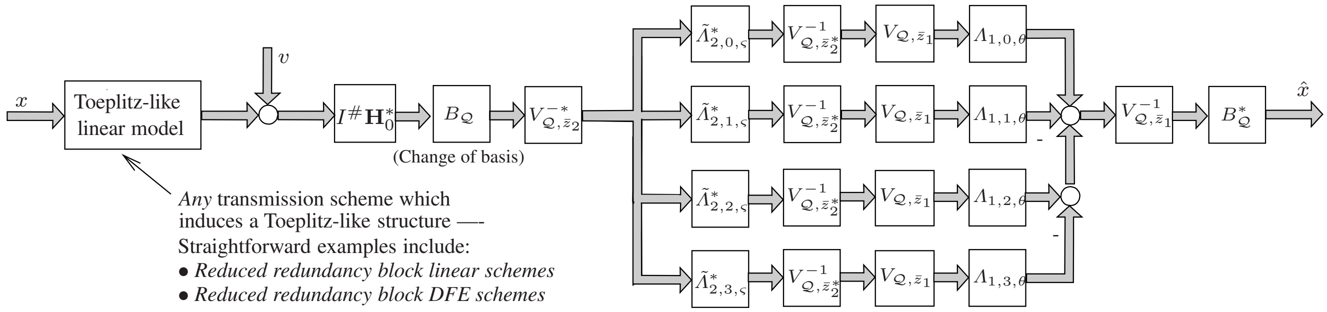

Fig. 4. *SC-FD-MMSE Decomposition*. For compactness of notation, we denote $\breve{V}_{\mathcal{Q},\bar{z}_1} \triangleq \breve{V}_{\mathcal{Q}}(1/\bar{z}_1)$, $\breve{V}_{\mathcal{Q},\bar{z}_2^*} \triangleq \breve{V}_{\mathcal{Q}}(1/\bar{z}_2^*)$, $\Lambda_{V_i,b_k,\theta} \triangleq \Lambda_{i,k,\theta}$, and similarly to $\tilde{\Lambda}_{V_i,b_k,\theta}$ and the same variables with dependency on ς.

4. APPLICATION TO BLOCK EQUALIZATION

Consider the linear model $\boldsymbol{y} = \mathbf{H}_0\mathbf{x} + \boldsymbol{v}_i$, with $\mathbf{H}_0$ a tall Toeplitz-like matrix, which can represent a hybrid zero-padding/discarding scheme [8]. The LS estimate of $\boldsymbol{s}$ is given by $\hat{\mathbf{x}} = \boldsymbol{K}_{\text{LS}}\boldsymbol{y}$, where

$$\boldsymbol{K}_{\text{LS}} = (\Pi^{-1} + \mathbf{H}_0^*\mathbf{H}_0)^{-1}\mathbf{H}_0^* = B_{\mathcal{Q}}^*(\epsilon\boldsymbol{I} + \mathbf{H}_B^*\mathbf{H}_B)^{-1}B_{\mathcal{Q}}\mathbf{H}_0^* \tag{32}$$

where $\mathbf{H}_B = \mathbf{H}_0 B_{\mathcal{Q}}^*$, with $B_{\mathcal{Q}}$ representing a change of basis matrix, and ϵ is a scalar inverse signal-to-noise ratio. Using (33) into (32), we obtain the general receiver realization illustrated in Fig. 4. Observe that multiplication with $\mathbf{H}_0$ is performed separately.

Now, in view of (25), assume that we choose $\bar{z}_2 = 1/\bar{z}_1'$, in a way that the coefficients of the M highest powers of $\bar{\Omega}_{M,\theta}(z)$ and $\bar{\Omega}'_{M,\theta}(z)$ coincide. This gives $\bar{\Omega}_\theta(\bar{\Phi}_{M,\varsigma}^{-*}) = (\varrho_0^* - \phi_0)I$, and $R_{\mathcal{P}}(1/\bar{z}_2^*) = V_{\mathcal{P}}^{-1}(1/\bar{z}_2^*)\mathcal{D}^{-1}(1/\bar{z}_2^*)$. After some algebra, noting that $V_{\mathcal{Q}}(z)$ is obtained from $V_{\mathcal{P}}(z)$ through a change of basis, i.e., $V_{\mathcal{P}}(z) = V_{\mathcal{Q}}(z)B_{\mathcal{Q}}^{-*}$, and similarly, $\breve{V}_{\mathcal{P}}(z) = \breve{V}_{\mathcal{Q}}(z)\breve{B}_{\mathcal{Q}}^{-*}$, we can conclude that $V_{\mathcal{Q}}(\bar{z}_2) = \Lambda_{\bar{z}_2}^{-(M-1)}V_{\mathcal{Q}}(1/\bar{z}_2)B_{\mathcal{Q}}^{-*}I^{\#}B_{\mathcal{Q}}^*$, where $I^{\#}$ reverses the entries (columns) of a vector (matrix). Hence (26) results in

$$\breve{P}_{M,N,L} = \frac{\phi_M}{(\phi_0 - \varrho_0^*)}\sum_{k=1}^{4}\iota_k\breve{V}_{\mathcal{Q}}^{-1}(\bar{z}_1)\Lambda_{V_1,b_{k,\theta}}V_{\mathcal{Q}}(\bar{z}_1)V_{\mathcal{Q}}^{-1}(1/\bar{z}_2^*) \cdot \tilde{\Lambda}_{V_2,b_{k,\varsigma}}^*\breve{V}_{\mathcal{Q}}^{-*}(1/\bar{z}_2)B_{\mathcal{Q}}I^{\#}B_{\mathcal{Q}}^{-1} \tag{33}$$

where $\tilde{\Lambda}_{V_2,b_{k,\varsigma}}^* = \mathcal{D}^{-1}(1/\bar{z}_2^*)\Lambda_{V_2,b_{k,\varsigma}}^*\Lambda_{\bar{z}_2}^{(M-1)*}$, and $\phi_0 \neq \varrho_0^*$.

We see that the well known DFT-representation is just a special case of the above formula, considering the zeros of the master polynomials $\bar{\Omega}_\theta(z) = \phi_0 + z^{-M}$, and $\bar{\Omega}'_\theta(z) = \varrho_0 + z^{-M}$ calculated at $\{\bar{z}_1, 1/\bar{z}_2^*\}$. That is, in the latter, $\bar{z}_1(m) = \phi e^{j\frac{2\pi m}{M}}$, where $\phi = |\phi_0|^{-1/M}e^{j\frac{\angle -\phi_0^{-1}}{M}}$, and $\bar{z}_2(m) = \varrho e^{j\frac{2\pi m}{M}}$, where $\varrho = |\varrho_0|^{1/M}e^{j\frac{\angle -\varrho_0}{M}}$. This results in the DFT filterbanks $V_{\mathcal{P}}(\bar{z}_1) = \sqrt{M}FD_\phi$, $V_{\mathcal{P}}(\bar{z}_2) = \sqrt{M}FD_\varrho$, $V_{\mathcal{P}}(1/\bar{z}_2^*) = \sqrt{M}FD_{1/\varrho^*}$, and $\mathcal{D}^{-1}(1/\bar{z}_2^*) = M\varrho^{*(M-1)}diag\left(e^{j\frac{2\pi m}{M}}|_{m=0}^{M-1}\right)$, which leads to

$$\breve{P}_{M,N,L} = \frac{D_{1/\phi}F^*}{(\phi_0 - \varrho_0^*)}\sum_{k=1}^{4}\iota_k\Lambda_{V_1,b_{k,\theta}}FD_{\phi\varrho^*}F^*\tilde{\Lambda}_{V_2,b_{k,\varsigma}}^*FD_{1/\varrho^*}$$

where $\tilde{\Lambda}_{V_2,b_{k,\varsigma}} = \varrho^{(M-1)}\Lambda_{V_2,b_{k,\varsigma}}diag\left(e^{-j\frac{2\pi m}{M}}|_{m=0}^{M-1}\right)$. Alternatively, we can factor $\mathcal{D}^{-1}(1/\bar{z}_2^*)/M$ into the definition of $\Lambda_{V_2,b_{k,\varsigma}}$ itself, in which case the above expression collapses to the one obtained in (87) of [1], specialized for tap-delay-line models.

The generality of this decomposition provides a solution to our third initial question; that is, since it is well known that Chebyshev polynomials of the first and second kind yield the so-called *Chebyshev-Vandermonde* matrices, it can be verified that choosing the nodes $\{\bar{z}_i\}$ as the roots of $Q_M(z)$, along with border polynomial $\Upsilon_M(z) = Q_{M-1}(z)$, these become essentially different versions of the DCT or DST matrices [7], and some algebra will lead to such real representations. Due to the simple recurrence relation that yields the Chebyshev basis, *multiplication of $B_{\mathcal{Q}}$ or $B_{\mathcal{Q}}^*$ by a vector is as efficient as the tap-delay line network*, since there is actually no multiplication inherent to these operations (just adds and shifts).

5. CONCLUSION

Our results show how the choice of free companion structures along with recurrence related basis representations yields an exact polynomial Vandermonde based decomposition, from the solution of the corresponding displacement equation of inverse covariances.

The above framework can be easily applied in a multicarrier scheme, and holds similarly for pure block zero forcing equalizers requiring only 2 receive branches with optimized redundancy. Moreover, such general decomposition is straightforwardly applied to reduced redundancy block decision feedback schemes, which also yield *superfast* receivers. We remark that in a multicarrier context, a proper power loading scheme must be taken into consideration, since the change of basis may represent significant increase in the conditioning of the transmitted signals.

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