CAUSAL RECONSTRUCTION KERNELS FOR CONSISTENT SIGNAL RECOVERY

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ABSTRACT

This paper derives causal reconstruction kernels which allow for a consistent signal recovery of the past signal component from the past signal samples only. Our approach is based on classical Hilbert space methods of signal sampling and recovery. The causal reconstruction kernels are obtained as the causal dual frame for a given sequence of sampling functions. The proposed methodology is illustrated by a numerical example.

Index Terms— Causality, interpolation, sampling, signal reconstruction, stationary sequences

1. INTRODUCTION

One cornerstone of modern signal processing and information theory is the sampling theorem attributed to E. T. Whittaker, H. Nyquist, V. A. Kotelnikov, and C. E. Shannon [1]. It essentially states that a band-limited analog signal can be perfectly reconstructed from its uniform samples taken at a rate which is at least twice of the signal bandwidth. In later years, this result was extended in many different direction (see, e.g., [2, 3, 4] and references therein). It is remarkable that all of these sampling theorems are essentially non-causal, which means that for a perfect reconstruction of the original signal $x(t)$ at a certain (time) point t all signal samples from the infinite past to the infinite future are necessary. For many applications, e.g. in image processing, this non-causality is not a drawback since usually all signal samples are available at a certain moment and the signal (the pixels) may be processed in any desired spatial direction. However, in applications involving essentially infinite signal streams over time, like audio-, speech- or communication signals, the causality of the signal processing algorithms is a necessary property. Not only are the future samples unaccessible but also should they intuitively have no influence on the present and the past signal component. Therefore, they should not be necessary for signal recovery anyway.

Against this background, it is somewhat surprising that there seems to be no systematic attempt to obtain a consistent theory of sampling and reconstruction under a causality constraint, similar to the well developed non-causal theory. Recently the causality problem in signal reconstruction was addressed especially in the context of spline interpolation [5] [6], [7], whereas in [8] techniques from estimation theory of stationary sequences were used to obtain a causal reconstruction filter. Nevertheless, the reconstruction kernel in this work was non-causal. Signal reconstruction with a causality constraint was investigated in the framework of mean-square optimization and from a system-theoretic viewpoint in [9].

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The present paper will derive causal reconstruction kernels, based on the classical approach of consistent signal recovery in Hilbert spaces (see, e.g., [3]), and will characterize the subspace for which a perfect signal recovery is possible.

2. SIGNAL MODEL AND PRELIMINARIES

Our subordinate signal space is $L^2(\mathbb{R})$, that is the Hilbert space of square-integrable functions on the real axis $\mathbb{R}$ with the inner product $\langle x, y\rangle_{L^2(\mathbb{R})} = \int_{\mathbb{R}} x(t)\, \overline{y(t)}\, \mathrm{d}t$. The Fourier transform of any function $x \in L^2(\mathbb{R})$ is defined as

$$\widehat{x}(\omega) = \int_{\mathbb{R}} x(t)\, \mathrm{e}^{-\mathrm{i}\omega t}\, \mathrm{d}t\,, \quad \omega \in \mathbb{R}\,.$$

For any arbitrary positive constant $a \in \mathbb{R}$, the *translation operator* $\mathrm{S}_a : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is defined by $(\mathrm{S}_a x)(t) = x(t-a)$ and it is clear that $(\mathrm{S}_a^n\, x) = x(t - n\, a)$ for every $n \in \mathbb{Z}$. If $\mathcal{S}$ is a closed subset of $L^2(\mathbb{R})$ then $\mathrm{P}_{\mathcal{S}}$ denotes the orthogonal projection from $L^2(\mathbb{R})$ onto $\mathcal{S}$. $L^2(\mathbb{R})$ may be decomposed into two closed orthogonal subspaces:

$$L^2_+(\mathbb{R}) := \{x \in L^2(\mathbb{R}) \,:\, x(t) = 0 \text{ for all } t < 0\}$$
$$L^2_-(\mathbb{R}) := \{x \in L^2(\mathbb{R}) \,:\, x(t) = 0 \text{ for all } t \geq 0\}\,,$$

such that $L^2(\mathbb{R}) = L^2_+(\mathbb{R}) \oplus L^2_-(\mathbb{R})$. The orthogonal projection from $L^2(\mathbb{R})$ onto $L^2_-(\mathbb{R})$ will be denoted by P_-.
For $1 \leq p \leq \infty$ the common Lebesgue spaces on the unit circle $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$ are denoted by $L^p(\mathbb{T})$. In particular, $L^2(\mathbb{T})$ is a Hilbert space with the inner product

$$\langle x, y\rangle_{L^2(\mathbb{T})} = \frac{1}{2\pi}\int_{-\pi}^{\pi} x(\mathrm{e}^{\mathrm{i}\theta})\, \overline{y(\mathrm{e}^{\mathrm{i}\theta})}\, \mathrm{d}\theta\,.$$

Any $x \in L^2(\mathbb{T})$ can be written as a *Fourier series*

$$x(\mathrm{e}^{\mathrm{i}\theta}) = \sum_{n\in\mathbb{Z}} \widehat{x}(n)\, \mathrm{e}^{\mathrm{i}n\theta} \;\text{ with }\; \widehat{x}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} x(\mathrm{e}^{\mathrm{i}\theta})\, \mathrm{e}^{-\mathrm{i}n\theta}\, \mathrm{d}\theta\,,$$

with the *Fourier coefficients* $\widehat{x}(n)$. The closed subspace of all functions for which the Fourier coefficients with negative index are zero is denoted by $L^2_+(\mathbb{T}) := \{x \in L^2(\mathbb{T}) : \widehat{x}(n) = 0, \forall n < 0\}$. Every $x \in L^2_+(\mathbb{T})$ can be identified with a function x from the *Hardy space* H^2 which has the form $x(z) = \sum_{n=0}^{\infty} \widehat{x}(n) z^n$ and which is analytic for all $z \in \mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ [10]. The natural projection $L^2(\mathbb{T}) \to H^2$ will be denoted by

$$\mathrm{P}_+ : \textstyle\sum_{n=-\infty}^{\infty} \widehat{x}(n)\, \mathrm{e}^{\mathrm{i}n\theta} \mapsto \sum_{n=0}^{\infty} \widehat{x}(n)\, z^n\,.$$

The *Laurent* (or multiplication) operator $\mathrm{M}_\Phi : L^2(\mathbb{T}) \to L^2(\mathbb{T})$ is defined by $(\mathrm{M}_\Phi\, x)(\mathrm{e}^{\mathrm{i}\theta}) = \Phi(\mathrm{e}^{\mathrm{i}\theta})\, x(\mathrm{e}^{\mathrm{i}\theta})$, where $\Phi \in L^\infty(\mathbb{T})$ is called the *symbol* of M_Φ. The concatenation of P_+ and M_Φ, i.e. the mapping $\mathrm{T}_\Phi = \mathrm{P}_+\, \mathrm{M}_\Phi : H^2 \to H^2$ is called the *Toeplitz operator* with symbol Φ.

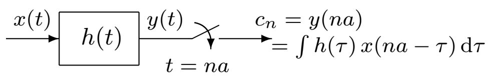

Fig. 1. Linear filtering followed by an ideal sampling with period a.

3. SAMPLING SCHEME

A simple model for a practical sampling device comprises a pre-filter with impulse response h followed by an ideal sampler [11], as depicted in Fig. 1. The resultant signal samples can be written as $c_n := \langle x, s_n \rangle_{L^2(\mathbb{R})}$, where

$$s_n(t) = (\mathrm{S}_a^n s)(t) = s(t - na) \quad \text{with} \quad s(t) := \overline{h(-t)}\ . \qquad (1)$$

It is always assumed that $h \in L^2(\mathbb{R})$, but in a real world system the linear filter will be causal, so that $h \in L^2_+(\mathbb{R})$ and consequently $s \in L^2_-(\mathbb{R})$. The functions s_n are called *sampling functions*, and the space spanned by these functions

$$\mathcal{S} := \overline{\operatorname{span}}\{s_n = \mathrm{S}_a^n s \ :\ n \in \mathbb{Z}\}$$

is called the *sampling space*. To simplify the presentation, it is always assumed that $\{s_n\}_{n\in\mathbb{Z}}$ forms a Riesz basis for $\mathcal{S}$.

4. STATIONARY SEQUENCES

The sampling space $\mathcal{S}$ is spanned by the sequence $\boldsymbol{s} := \{s_n\}_{n\in\mathbb{Z}}$ in $L^2(\mathbb{R})$. Since $\boldsymbol{s}$ is generated by the unitary operator S_a as in (1), it is easy to verify that $\boldsymbol{s}$ is a *stationary sequence* in $L^2(\mathbb{R})$ [8], which means that $\langle s_{n+k}, s_{m+k}\rangle = \langle s_n, s_m\rangle$ for all $n, m, k \in \mathbb{Z}$. As a consequence, the correlation function $r_{\boldsymbol{s}}(n) := \langle s_n, s_0\rangle$ has the following spectral representation

$$r_{\boldsymbol{s}}(n) = \langle s_n, s_0\rangle_{L^2(\mathbb{R})} = \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{e}^{-\mathrm{i}n\theta}\, \Phi_{\boldsymbol{s}}(\mathrm{e}^{\mathrm{i}\theta})\, \mathrm{d}\theta\ , \qquad (2)$$

where $\Phi_{\boldsymbol{s}} \in L^1(\mathbb{T})$ is the *spectral density* of $\boldsymbol{s}$, given by [8]

$$\Phi_{\boldsymbol{s}}(\mathrm{e}^{\mathrm{i}\theta}) = \tfrac{1}{a} \sum_{n\in\mathbb{Z}} \left|\widehat{s}\left(\tfrac{\theta+2\pi n}{a}\right)\right|^2$$

in terms of the Fourier transform $\widehat{s}$ of the generator s. Our assumption that $\boldsymbol{s}$ forms a Riesz basis for $\mathcal{S}$ can be expressed conveniently in terms of its spectral density [8, 12, 13]:

LEMMA 1: *Let $\boldsymbol{s} = \{s_n\}_{n\in\mathbb{Z}}$ be a stationary sequence in $L^2(\mathbb{R})$ with spectral density $\Phi_{\boldsymbol{s}}$. Then $\boldsymbol{s}$ forms a Riesz basis for $\mathcal{S}$ if and only if there exists two positive constants A, B such that*

$$A \le \Phi_{\boldsymbol{s}}(\mathrm{e}^{\mathrm{i}\theta}) \le B \quad \textit{for almost all } \theta \in [-\pi, \pi)\ . \qquad (3)$$

For the following derivation, it is crucial to note (see, e.g., [14]) that the sampling space $\mathcal{S}$ is isometric isomorph to the Hilbert space $L^2(\mathbb{T}, \Phi_{\boldsymbol{s}})$ of functions on $\mathbb{T}$, equipped with the inner product

$$\langle f, g\rangle_{L^2(\mathbb{T},\Phi_{\boldsymbol{s}})} = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\mathrm{e}^{\mathrm{i}\theta})\, \overline{g(\mathrm{e}^{\mathrm{i}\theta})}\, \Phi_{\boldsymbol{s}}(\mathrm{e}^{\mathrm{i}\theta})\, \mathrm{d}\theta\ .$$

The Hilbert space isomorphism is established by the linear mapping $\Lambda_{\boldsymbol{s}} : \mathcal{S} \to L^2(\mathbb{T}, \Phi_{\boldsymbol{s}})$ defined by

$$\Lambda_{\boldsymbol{s}} : s_n \mapsto \varphi_{-n}(\mathrm{e}^{\mathrm{i}\theta}) := \mathrm{e}^{-\mathrm{i}n\theta} \quad \text{for all } n \in \mathbb{Z}\ . \qquad (4)$$

Since $\boldsymbol{s}$ is assumed to be a Riesz basis for $\mathcal{S}$, every $x \in \mathcal{S}$ has the form $x = \sum_{n\in\mathbb{Z}} a_n\, s_n$ and can be identified with the function

$$\xi(\mathrm{e}^{\mathrm{i}\theta}) = (\Lambda_{\boldsymbol{s}} x)(\mathrm{e}^{\mathrm{i}\theta}) = \textstyle\sum_{n\in\mathbb{Z}} a_n\, \varphi_{-n}(\mathrm{e}^{\mathrm{i}\theta}) = \sum_{n\in\mathbb{Z}} a_{-n}\, \mathrm{e}^{\mathrm{i}n\theta}$$

in $L^2(\mathbb{T}, \Phi_{\boldsymbol{s}})$. Then it follows from (2) that for any $x, y \in \mathcal{S}$ the relation $\langle x, y\rangle_{L^2(\mathbb{R})} = \langle \Lambda_{\boldsymbol{s}} x, \Lambda_{\boldsymbol{s}} y\rangle_{L^2(\mathbb{T},\Phi_{\boldsymbol{s}})}$ holds. In particular, $\{s_n\}_{n\in\mathbb{Z}}$ is a Riesz basis for $\mathcal{S}$ if and only if $\{\varphi_n\}_{n\in\mathbb{Z}}$ is a Riesz basis for $L^2(\mathbb{T}, \Phi_{\boldsymbol{s}})$. Note also that because of (3), $L^2(\mathbb{T}, \Phi_{\boldsymbol{s}})$ is isomorph to $L^2(\mathbb{T})$, i.e. $L^2(\mathbb{T}, \Phi_{\boldsymbol{s}})$ may be considered as $L^2(\mathbb{T})$ equipped with an equivalent norm which satisfies

$$\sqrt{A}\, \|x\|_{L^2(\mathbb{T})} \le \|x\|_{L^2(\mathbb{T},\Phi_{\boldsymbol{s}})} \le \sqrt{B}\, \|x\|_{L^2(\mathbb{T})}\ , \quad \forall x \in L^2(\mathbb{T})\ .$$

5. CONSISTENT SIGNAL RECOVERY

The first subsection briefly reviews the well known non-causal reconstruction method (see, e.g., [3]), because the derivation of the causal reconstruction kernels in Subsection 5.2 will follow a similar reasoning.

5.1. Non-causal reconstruction

Let $x \in L^2(\mathbb{R})$ be a signal and assume that we have acquired all signal samples $c_n = \langle x, s_n\rangle$ for $n \in \mathbb{Z}$. Since $\boldsymbol{s} = \{s_n\}_{n\in\mathbb{Z}}$ is assumed to be a Riesz basis for the sampling space $\mathcal{S}$, a signal reconstruction is given by

$$\widetilde{x}(t) = \textstyle\sum_{n\in\mathbb{Z}} \langle x, s_n\rangle\, \sigma_n(t)\ , \qquad (5)$$

wherein $\boldsymbol{\sigma} = \{\sigma_n\}_{n\in\mathbb{Z}}$ is the *dual Riesz basis* of $\boldsymbol{s} = \{s_n\}_{n\in\mathbb{Z}}$, which is known to be bi-orthogonal to $\boldsymbol{s}$, i.e. $\langle \sigma_n, s_n\rangle = 1$ for all $n \in \mathbb{Z}$ and $\langle \sigma_n, s_m\rangle = 0$ for $n \neq m$. Moreover, it is known that

$$\widetilde{x} = \arg\min_{s\in\mathcal{S}} \|x - s\| = \mathrm{P}_{\mathcal{S}}\, x\ ,$$

i.e. the reconstructed signal $\widetilde{x}$ is the best approximation of x in $\mathcal{S}$, and that one obtains a perfect reconstruction (i.e. $\widetilde{x} = x$) for all signals $x \in \mathcal{S}$. The above signal recovery was termed *consistent* [3, 11] since a re-sampling of the reconstructed signal $\widetilde{x}$ yields the same signal samples as before, i.e. $\langle \widetilde{x}, s_n\rangle = c_n = \langle x, s_n\rangle$, which follows from the bi-orthogonality of $\boldsymbol{\sigma}$ and $\boldsymbol{s}$.

Thus, in order to reconstruct (or approximate) the analog signal x we have to determine the dual Riesz basis $\boldsymbol{\sigma}$ of $\boldsymbol{s}$. Even though this derivation is well known (see, e.g., [3, 13]), we will sketch it here based on our approach using the theory of stationary sequences.

The determination of the dual Riesz basis of $\{s_n\}_{n\in\mathbb{Z}}$ in $\mathcal{S}$ is equivalent to the determination of the dual Riesz basis of $\{\varphi_n\}_{n\in\mathbb{Z}}$ in $L^2(\mathbb{T}, \Phi_{\boldsymbol{s}})$. To this end we consider the Hilbert space $L^2(\mathbb{T})$ with the orthonormal basis $e_n(\mathrm{e}^{\mathrm{i}\theta}) = \mathrm{e}^{\mathrm{i}n\theta}$, $n \in \mathbb{Z}$, and the linear mapping $\mathrm{F} : L^2(\mathbb{T}) \to L^2(\mathbb{T}, \Phi_{\boldsymbol{s}})$ given by

$$(\mathrm{F}x)(\mathrm{e}^{\mathrm{i}\theta}) = x(\mathrm{e}^{\mathrm{i}\theta})\ . \qquad (6)$$

Since $\Phi_{\boldsymbol{s}}$ is assumed to satisfy (3), it is easy to see that F is bounded, one-to-one, and onto. Moreover, we obviously have $\mathrm{F}e_n = \varphi_n$ for all $n \in \mathbb{Z}$. Consequently, F is the so-called *pre-frame operator* associated with $\{\varphi_n\}_{n\in\mathbb{Z}}$ and its adjoint $\mathrm{F}^* : L^2(\mathbb{T}, \Phi_{\boldsymbol{s}}) \to L^2(\mathbb{T})$ is apparently

$$(\mathrm{F}^* x)(\mathrm{e}^{\mathrm{i}\theta}) = (\mathrm{M}_{\Phi_{\boldsymbol{s}}}\, x)(\mathrm{e}^{\mathrm{i}\theta}) = \Phi_{\boldsymbol{s}}(\mathrm{e}^{\mathrm{i}\theta})\, x(\mathrm{e}^{\mathrm{i}\theta})\ .$$

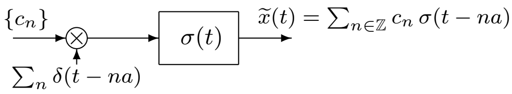

Fig. 2. Non-causal signal reconstruction with linear filter σ.

This F^* possesses a unique (left) inverse, i.e. there exists a mapping $\mathrm{G} : L^2(\mathbb{T}) \to L^2(\mathbb{T}, \Phi_{\boldsymbol{s}})$ which satisfies $\mathrm{G}\,\mathrm{F}^* x = x$ for every $x \in L^2(\mathbb{T}, \Phi_{\boldsymbol{s}})$, and it is easy to verify that G is given by

$$(\mathrm{G}\, x)(\mathrm{e}^{\mathrm{i}\theta}) = (\mathrm{M}_{1/\Phi_{\boldsymbol{s}}}\, x)(\mathrm{e}^{\mathrm{i}\theta}) = \frac{1}{\Phi_{\boldsymbol{s}}(\mathrm{e}^{\mathrm{i}\theta})}\, x(\mathrm{e}^{\mathrm{i}\theta}) \ .$$

Now it is well known [13] that the dual Riesz basis $\{\psi_n\}_{n\in\mathbb{Z}}$ of $\{\varphi_n\}_{n\in\mathbb{Z}}$ is given by

$$\psi_n(\mathrm{e}^{\mathrm{i}\theta}) = (\mathrm{G}\, e_n)(\mathrm{e}^{\mathrm{i}\theta}) = \frac{\mathrm{e}^{\mathrm{i}n\theta}}{\Phi_{\boldsymbol{s}}(\mathrm{e}^{\mathrm{i}\theta})} \ , \qquad n \in \mathbb{Z} \ . \tag{7}$$

By writing $1/\Phi_{\boldsymbol{s}}$ as a Fourier series, these dual functions can be written as

$$\psi_n(\mathrm{e}^{\mathrm{i}\theta}) = \sum_{k\in\mathbb{Z}} \alpha_k\, \varphi_{n+k}(\mathrm{e}^{\mathrm{i}\theta}) \ \text{ with } \ \alpha_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\mathrm{e}^{-\mathrm{i}k\theta}}{\Phi_{\boldsymbol{s}}(\mathrm{e}^{\mathrm{i}\theta})}\, \mathrm{d}\theta.$$

Using again the isomorphism between $L^2(\mathbb{T}, \Phi_{\boldsymbol{s}})$ and $\mathcal{S}$, one obtains the desired dual Riesz basis of $\{s_n\}_{n\in\mathbb{Z}}$ as

$$\sigma_n = \Lambda_{\boldsymbol{s}}^{-1} \psi_{-n} = \sum_{k\in\mathbb{Z}} \alpha_k\, s_{n-k} = \mathrm{S}_a^n \left(\sum_{k\in\mathbb{Z}} \alpha_k\, s_{-k} \right) = \mathrm{S}_a^n \sigma$$

for all $n \in \mathbb{Z}$ and where

$$\sigma(t) := \textstyle\sum_{k\in\mathbb{Z}} \alpha_k\, s_{-k}(t) = \sum_{k\in\mathbb{Z}} \alpha_k\, s(t + ka) \tag{8}$$

is the generator of this dual Riesz basis. Note that $\{\sigma_n\}$ has the same shift-invariant structure as the given sequence of sampling functions $\{s_n\}$. Therefore, the signal reconstruction has a very simple implementation as sketched in Fig. 2. The reconstructed signal $\widetilde{x}$ is obtained by sending a puls-stream modulated with the signal samples $\{c_n\}$ through a linear filter with impulse response σ. However, it follows from (8) that $\sigma \notin L^2_+(\mathbb{R})$ in general, i.e. σ is the impulse response of a non-causal filter even if h is assumed to be causal.

5.2. Causal Reconstruction

In a causal setup, we assume that the sampling filter h in Fig.1 is causal, i.e. that $s \in L^2_-(\mathbb{R})$, and we perform signal recovery at time $t = 0$. At this moment, we have only received the signal component $x_- := \mathrm{P}_- x \in L^2_-(\mathbb{R})$ and sampled the signal up to $t = 0$. Therefore we only know the *past signal samples*

$$c_n = \langle x, s_n \rangle = \langle x_-, s_n \rangle \quad \text{ for } \quad n = 0, -1, -2, \dots \ . \tag{9}$$

The second equality follows from the assumption that $s \in L^2_-(\mathbb{R})$. Because then $s_n \in L^2_-(\mathbb{R})$ for all $n \leq 0$, and consequently $\langle x, s_n \rangle = \langle x, \mathrm{P}_- s_n \rangle = \langle \mathrm{P}_- x, s_n \rangle = \langle x_-, s_n \rangle$ for all $n \leq 0$ since P_- is self-adjoint. The subspace spanned by the past sampling functions

$$\mathcal{S}_0 := \overline{\mathrm{span}}\{s_n : n = 0, -1, -2, \dots\} \subset L^2_-(\mathbb{R})$$

will be called the *past sampling space*.

Our aim is now to reconstruct (or approximate) the past signal component x_- based on the past signal samples (9). The formal approach is the same as in the non-causal case. To this end, we notice first that under the assumption that $\{s_n\}_{n\in\mathbb{Z}}$ is a Riesz basis for $\mathcal{S}$, the past sampling functions $\{s_n\}_{n\leq 0}$ will form a Riesz basis for the past sampling space $\mathcal{S}_0$. The simple proof is omitted.

LEMMA 2: *Let $\boldsymbol{s} = \{s_n\}_{n\in\mathbb{Z}}$ be a stationary sequence in $L^2(\mathbb{R})$, and let $\mathcal{S}$ and $\mathcal{S}_0$ be the corresponding sampling space and past sampling space, respectively. If $\{s_n\}_{n=-\infty}^{\infty}$ is a Riesz basis for $\mathcal{S}$ then $\{s_n\}_{n=-\infty}^{0}$ is a Riesz basis for $\mathcal{S}_0$.*

Consequently, a consistent reconstruction of x_- based on the past signal samples is given by

$$\widetilde{x}_-(t) = \textstyle\sum_{n=-\infty}^{0} \langle x, s_n \rangle\, \varsigma_n(t) \tag{10}$$

wherein $\{\varsigma_n\}_{n\leq 0}$ is the dual Riesz basis of $\{s_n\}_{n\leq 0}$ for $\mathcal{S}_0$. Similar to the non-causal case, (10) represents the orthogonal projection of x onto $\mathcal{S}_0$, i.e.

$$\widetilde{x}_- = \arg\min_{s\in\mathcal{S}_0} \|x - s\| = \mathrm{P}_{\mathcal{S}_0} x \ .$$

This implies that $\widetilde{x}_-$ is the best approximation of x_- in $\mathcal{S}_0$ and that a perfect reconstruction of the past signal component x_- is obtained for all signals x for which $x_- = \mathrm{P}_- x \in \mathcal{S}_0$. It remains to determine the dual Riesz basis $\{\varsigma_n\}_{n\leq 0}$ of $\{s_n\}_{n\leq 0}$. This is done in the following theorem.

THEOREM 3: *Let $\boldsymbol{s} = \{s_n\}_{n\in\mathbb{Z}}$ be a stationary sequence in $L^2(\mathbb{R})$ which forms a Riesz basis for $\mathcal{S} = \overline{\mathrm{span}}\{s_n : n \in \mathbb{Z}\}$, and let $\Phi_{\boldsymbol{s}}$ be the spectral density of $\boldsymbol{s}$. Then $\boldsymbol{s}_0 = \{s_{-n}\}_{n=0}^{\infty}$ is a Riesz basis for $\mathcal{S}_0$ and the corresponding dual Riesz basis $\{\varsigma_{-n}\}_{n=0}^{\infty}$ is given by*

$$\varsigma_{-n} = \sum_{k=0}^{\infty} \widehat{\psi}_n(k)\, s_{-k} \ , \quad n = 0, 1, 2, \dots \tag{11}$$

with

$$\widehat{\psi}_n(k) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \psi_n(\mathrm{e}^{\mathrm{i}\theta})\, \mathrm{e}^{-\mathrm{i}k\theta}\, \mathrm{d}\theta \ , \quad n = 0, 1, 2, \dots \tag{12}$$

and where the functions $\psi_n \in H^2$ are given by

$$\psi_n(\mathrm{e}^{\mathrm{i}\theta}) = \frac{1}{\Phi^+_{\boldsymbol{s}}(\mathrm{e}^{\mathrm{i}\theta})}\, \mathrm{P}_+ \left[\frac{\mathrm{e}^{\mathrm{i}n\theta}}{\Phi^-_{\boldsymbol{s}}(\mathrm{e}^{\mathrm{i}\theta})} \right] \ , \quad n = 0, 1, 2, \dots \tag{13}$$

and wherein $\Phi^+_{\boldsymbol{s}}$ and $\Phi^-_{\boldsymbol{s}}$ are the spectral factors of $\Phi_{\boldsymbol{s}}$.

Proof: The proof follows the same ideas as in the non-causal case. By the isomorphism between $\mathcal{S}$ and $L^2(\mathbb{T}, \Phi_{\boldsymbol{s}})$, it follows that $\mathcal{S}_0$ is isomorph to $L^2_+(\mathbb{T}, \Phi_{\boldsymbol{s}}) = \overline{\mathrm{span}}\{\varphi_n : n = 0, 1, 2, \dots\}$, where the isomorphism is given by the mapping $\Lambda_{\boldsymbol{s}}$ in (4). Therefore, the determination of the dual Riesz basis of $\{s_{-n}\}_{n=0}^{\infty}$ in $\mathcal{S}_0$ is equivalent to the determination of the dual Riesz basis of $\{\varphi_n\}_{n=0}^{\infty}$ in $L^2_+(\mathbb{T}, \Phi_{\boldsymbol{s}})$, with $\varphi_n(\mathrm{e}^{\mathrm{i}\theta}) = \mathrm{e}^{\mathrm{i}n\theta}$.
To this end, we consider the mapping $\mathrm{F} : H^2 \to L^2_+(\mathbb{T}, \Phi_{\boldsymbol{s}})$, defined as in (6). Since $\boldsymbol{s}$ is a Riesz basis for $\mathcal{S}$, the spectral density $\Phi_{\boldsymbol{s}}$ satisfies (3). It follows that F is bounded, invertible, and that $\mathrm{F}\, e_n = \varphi_n$ for all $n = 0, 1, 2, \dots$, where $\{e_n(z) = z^n\}_{n=0}^{\infty}$ is the standard orthonormal basis of H^2. Therefore $\mathrm{F} : H^2 \to L^2_+(\mathbb{T}, \Phi_{\boldsymbol{s}})$ is the pre-frame operator associated with $\{\varphi_n\}_{n=0}^{\infty}$, and its adjoint $\mathrm{F}^* : L^2_+(\mathbb{T}, \Phi_{\boldsymbol{s}}) \to H^2$ is given by

$$\mathrm{F}^*\, x = \mathrm{P}_+\, \mathrm{M}_{\Phi_{\boldsymbol{s}}}\, x = \mathrm{T}_{\Phi_{\boldsymbol{s}}}\, x \ .$$

Indeed, for every $x \in H^2$ and every $y \in L^2_+(\mathbb{T}, \Phi_{\boldsymbol{s}})$ one has

$$\begin{aligned}\langle \mathrm{F}\, x\,,\, y\rangle_{L^2_+(\mathbb{T},\Phi_{\boldsymbol{s}})} &= \langle x\,,\, \mathrm{M}_{\Phi_{\boldsymbol{s}}}\, y\rangle_{L^2(\mathbb{T})} = \langle \mathrm{P}_+\, x\,,\, \mathrm{M}_{\Phi_{\boldsymbol{s}}}\, y\rangle_{L^2(\mathbb{T})} \\ &= \langle x\,,\, \mathrm{P}_+\, \mathrm{M}_{\Phi_{\boldsymbol{s}}}\, y\rangle_{H^2}\ .\end{aligned}$$

Thus F^* is the Toeplitz operator with symbol $\Phi_{\boldsymbol{s}}$, and we need to find a (left) inverse $\mathrm{G} : H^2 \to L^2_+(\mathbb{T}, \Phi_{\boldsymbol{s}})$ such that $\mathrm{G}\,\mathrm{F}^*\, x = x$ for all $x \in L^2_+(\mathbb{T}, \Phi_{\boldsymbol{s}})$. To this end, let

$$\Phi_{\boldsymbol{s}}(\mathrm{e}^{\mathrm{i}\theta}) = \Phi^+_{\boldsymbol{s}}(\mathrm{e}^{\mathrm{i}\theta})\, \Phi^-_{\boldsymbol{s}}(\mathrm{e}^{\mathrm{i}\theta}) \quad \text{for all } \theta \in [-\pi, \pi) \tag{14}$$

be the spectral factorization of $\Phi_{\boldsymbol{s}}$. Therein, $\Phi^+_{\boldsymbol{s}} \in H^2$ with $\Phi^+_{\boldsymbol{s}}(z) \neq 0$ for all $z \in \mathbb{D}$ and $\Phi^-_{\boldsymbol{s}}(z) = \overline{\Phi^+_{\boldsymbol{s}}(1/\overline{z})}$ are the spectral factors of $\Phi_{\boldsymbol{s}}$. Such a factorization exists since $\Phi_{\boldsymbol{s}}$ satisfies (3) (see, e.g., [15]). Moreover, the properties of $\Phi^+_{\boldsymbol{s}}$ and relation (3) imply that $\Phi^+_{\boldsymbol{s}}$ and $1/\Phi^+_{\boldsymbol{s}}$ are in $H^\infty = H^2 \cap L^\infty(\mathbb{T})$ and that $\Phi_{\boldsymbol{s}}(\mathrm{e}^{\mathrm{i}\theta}) = |\Phi^+_{\boldsymbol{s}}(\mathrm{e}^{\mathrm{i}\theta})|^2$ for all $\theta \in [-\pi, \pi)$. Based on these properties of $\Phi^+_{\boldsymbol{s}}$ and $\Phi^-_{\boldsymbol{s}}$, it can be shown (see, e.g., [16, Lemma 2.3.5]) that the Toeplitz operator $\mathrm{F}^* = \mathrm{T}_{\Phi_{\boldsymbol{s}}}$ is invertible, with the inverse

$$\mathrm{G} = \mathrm{T}^{-1}_{\Phi_{\boldsymbol{s}}} = \mathrm{T}_{1/\Phi^+_{\boldsymbol{s}}}\, \mathrm{T}_{1/\Phi^-_{\boldsymbol{s}}} = \mathrm{M}_{1/\Phi^+_{\boldsymbol{s}}}\, \mathrm{P}_+\, \mathrm{M}_{1/\Phi^-_{\boldsymbol{s}}}\ .$$

The dual Riesz basis $\{\psi_n\}_{n=0}^\infty$ of $\{\varphi_n\}_{n=0}^\infty$ is then be determined by $\psi_n = \mathrm{G}\, e_n$, which is equivalent to (13). Finally, every $\psi_n \in H^2$ can be written as a Fourier series

$$\psi_n(\mathrm{e}^{\mathrm{i}\theta}) = \textstyle\sum_{k=0}^\infty \widehat{\psi}_n(k)\, \mathrm{e}^{\mathrm{i}k\theta} = \sum_{k=0}^\infty \widehat{\psi}_n(k)\, \varphi_k(\mathrm{e}^{\mathrm{i}\theta})$$

with Fourier coefficients (12). By applying the isomorphism (4), one obtains finally

$$\varsigma_{-n} = \Lambda^{-1}_{\boldsymbol{s}}\, \psi_n = \textstyle\sum_{k=0}^\infty \widehat{\psi}_n(k)\, \Lambda^{-1}_{\boldsymbol{s}}\, \varphi_k = \sum_{k=0}^\infty \widehat{\psi}_n(k)\, s_{-k}\ ,$$

which is (11). ■

As (11) shows, every vector ς_{-n}, $n = 0, 1, 2, \dots$ of the dual basis is an element of $\mathcal{S}_0 \subset L^2_-(\mathbb{T})$, i.e. every $\varsigma_n(t)$ is zero for $t > 0$. Consequently, the reconstructed signal $\widetilde{x}_-(t) \in \mathcal{S}_0$ given by (10) is also only supported in the past. A property which we expect for a causal reconstruction method.

Although the formal derivation of the dual Riesz basis is very similar for the non-causal and the causal case, the actual solution has a very different structure. The non-causal dual Riesz basis $\{\sigma_n\}_{n\in\mathbb{Z}}$ is again a stationary sequence in $L^2(\mathbb{R})$. So every σ_n is a time translate of a fixed function $\sigma \in \mathcal{S}$: $\sigma_n(t) = (\mathrm{S}^n_a\, \sigma)(t) = \sigma(t - n\, a)$. Therefore, one only has to determine the generator σ in order to obtain the dual Riesz basis $\{\sigma_n\}$. The causal dual $\{\varsigma_n\}_{n\leq 0}$ does not have such a simple structure, but every ς_{-n}, $n = 0, 1, 2, \dots$ has to be determined separately (cf. Theorem 3), and there is generally no simple recursive method to obtain ς_n from its predecessor ς_{n-1}. However, it should be noted that for $n \to \infty$, the part which is cut-off by P_+ in (13) becomes negligible. Then a comparison of (7) and (13) shows that for a sufficiently large index n the causal dual basis element ς_{-n} is almost equal to the non-causal dual basis element σ_{-n} (cf. also the example in Sect. 6).

For the determination of the causal dual basis, a spectral factorization (14) is necessary, and we refer to [15] for a survey of various suitable algorithms. This operation has a fairly complicated behavior which makes it harder to investigate the analytic properties of the causal reconstruction, e.g. the decay behavior of the Fourier coefficient (12) which in turn influences the stability and robustness of this causal reconstruction scheme. However, a detailed investigation of the spectral factorization mapping may be found, e.g., in [17, 18]. It provides, for example, a relation between the decay of Fourier coefficients (12) and the smoothness of the spectral density $\Phi_{\boldsymbol{s}}$.

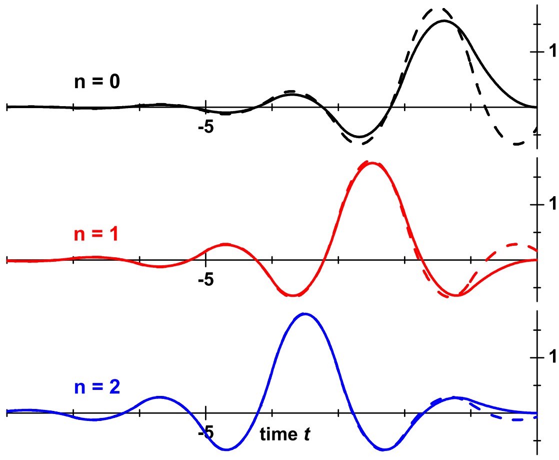

Fig. 3. Solid lines: causal reconstruction kernels $\varsigma_{-n}(t)$ according to Theorem 3. Dashed lines: truncated non-causal reconstruction kernels $\sigma_{-n}(t) = \sigma(t + n)$.

6. EXAMPLE: CAUSAL SPLINE RECONSTRUCTION

In the following we illustrate the above concepts for the impulse response h of the sampling filter in Fig. 1 being a B-spline and for a sampling period a normalized to 1. A B-spline of degree N is defined recursively (see, e.g., [19]) by

$$\beta^N(t) = (\beta^{N-1} * \beta^0)(t) = \int_{\mathbb{R}} \beta^{N-1}(\tau)\, \beta^0(t-\tau)\, \mathrm{d}\tau$$

$$\text{with} \qquad \beta^0(t) = \begin{cases} 1 & -1/2 \leq t \leq 1/2 \\ 0 & \text{otherwise} \end{cases}\ .$$

We set $s(t) = h(-t) := \beta^N(-t - (N+1)/2)$, where the shift by $(N+1)/2$ ensures that h is causal, and $s_n(t) := s(t-n)$. From the recursive definition of β^N, it follows that the spectral density $\Phi_{\boldsymbol{s}}$ of the sequence of sampling functions $\boldsymbol{s} = \{s_n\}_{n\in\mathbb{Z}}$ is a trigonometric polynomial of degree N given by

$$\Phi_{\boldsymbol{s}}(\mathrm{e}^{\mathrm{i}\theta}) = \textstyle\sum_{k=-N}^{N} \beta^{2N+1}(k)\, \mathrm{e}^{-\mathrm{i}k\theta}\ ,$$

and one can verify that $\Phi_{\boldsymbol{s}}(\mathrm{e}^{\mathrm{i}\theta}) > 0$ for all $\theta \in [-\pi, \pi)$ such that by the theorem of Fejér-Riesz [20] there exists a factorization (14) with $\Phi^+_{\boldsymbol{s}}(z) = \sum_{k=0}^N \alpha_k\, z^k$. Moreover, [19] readily provides closed form expressions for $\Phi^{-1}_{\boldsymbol{s}}(z)$ and its poles. So that the factorization (14) and the coefficients $\widehat{\psi}_n(k)$ in (12) are easily obtained.
In particular, we choose a B-spline of degree $N = 2$. For this case, Fig. 3 compares the non-causal dual functions $\sigma_{-n}(t) = \sigma(t+n)$, determined by (8), with the causal dual functions $\varsigma_{-n}(t)$ of Theorem 3, for $n = 0, 1, 2$. It shows that for small n, and for times t close to zero, there is a substantial difference between the non-causal and the causal dual functions. However, as n increases this difference becomes negligible, as expected from the discussion above.
Next we want to illustrate that the causal reconstruction (10) provides a perfect reconstruction from the past signal samples for signals in the past sampling space $\mathcal{S}_0$. To this end, we generated a signal $x \in \mathcal{S}_0$ which has the form

$$x(t) = \textstyle\sum_{n\leq 0} x_n\, s(t-n) = \sum_{n\leq 0} x_n\, \beta^2(-[t-n] - \frac{N+1}{2})\ .$$

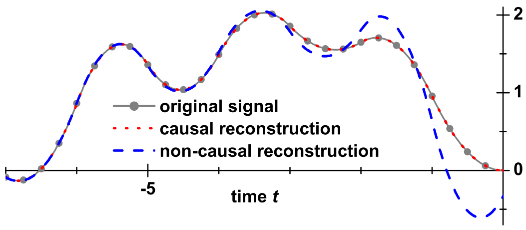

Fig. 4. Causal and non-causal signal reconstruction based on (10) and (15), respectively. Note that the graph for the causal recovery lies exactly over the graph of the original signal.

The coefficients $\{x_n\}$ were chosen at random and they were drawn independently from a normal distribution. Then the past signal samples (9) were determined. Afterwards, the past signal component x_- was reconstructed in two different ways. First, signal recovery was based on (10), using the causal reconstruction kernels ς_n. Then this result was compared with a reconstruction based on the truncated non-causal reconstruction formula (5), i.e. with

$$\widetilde{x}_{\mathrm{nc}}(t) = \textstyle\sum_{n=0}^{\infty} c_{-n}\, \sigma(t+n)\ , \quad t \leq 0\ . \tag{15}$$

Fig. 4 shows that (10) provides indeed a perfect reconstruction of $x \in \mathcal{S}_0$ whereas the reconstruction with (15) yields significant errors for times t close to zero. On the other hand, for time instances earlier than approximately 3 sampling periods, almost no difference in the signal reconstruction can be observed. Therefore (15), in conjunction with a sufficiently large decision lag, is often used in practical applications for reconstruction. Nevertheless, if no or only a small decision lag is acceptable, the causal reconstruction (10) provides much better results at a price of a slightly higher computational complexity.

7. SUMMARY AND OUTLOOK

This paper derived causal reconstruction kernels for signal recovery from generalized samples which satisfies the so-called the consistency condition. In future works, the proposed framework will be extended to more general U-invariant sampling schemes in atomic subspaces [8, 21] and to other recovery techniques [3, 4]. Moreover, the convergence behavior of the causal reconstruction series will be investigated for signal space on which the classical non-causal interpolation techniques run into some fundamental limits [22].

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