SIGNAL COMPRESSION USING THE DISCRETE LINEAR CHIRP TRANSFORM (DLCT)

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ABSTRACT

Signal compression aims to decrease transmission rate (increase storage capacity) by reducing the amount of data necessary to be transmitted. The discrete linear chirp transform (DLCT) is a joint frequency instantaneous-frequency transform that decomposes the signal in terms of linear chirps. The DLCT can be used to transform signals that are not sparse in either time or frequency, such as linear chirps, into sparse signals. In this paper, we propose a new algorithm for signal compression based on the direct and the dual DLCT, depending on the sparsity of the signal in either time or in frequency. Furthermore, we develop a data structure for the extracted coefficients of compressed signals. In the data structure, the extracted parameters are arranged in certain way that are predetermined for the compress and decompress processes. The ability of the proposed method in signal compression are demonstrated using test as well as actual signals. The results are compared with those obtained with compressive sensing (CS) method.

***Index Terms*—** discrete linear chirp transform, signal compression, compressive sensing, sparsity, duality

1. INTRODUCTION

The growth of communication systems and information technology, and their ability to serve voice, image, and video, requires more data to be transmitted or stored. Signal compression transforms a signal into an efficient compact form, for transmission or storage, that can be decompressed back to produce a close approximation of the original signal. The goal of signal compression is to minimize data rate to conserve bandwidth, while keeping the quality and intelligibility of the original signal. Unfortunately, the compression ratio is inversely proportional to the quality of the signal. Hence, there is always a tradeoff between compression ratio and quality [1, 2].

Compressive sensing (CS) [3] aims to take advantage of the signal's sparser representation dictated by the uncertainty principle. For instance, in [4] the signal to be compressed is represent in the sparser domain, using the discrete cosine transform (DCT). Taking random measurements from the new sparse signal so that the length of the measurement is smaller than the length of the original signal, the original signal can be reconstructed from the measurements using ℓ_1-optimization. Although CS provides very good results for signals that are sparse in either time or frequency, it does not for signals that are not significantly sparse in either time or frequency domains such as the case of chirp signals [5],[6]. Time-frequency analysis is needed to obtain an intermediate domain where the signal is sparser than in time or in frequency. The Fractional Fourier Transform [8, 9] or the polynomial time-frequency transforms [10] can be used, here we propose a joint frequency instantaneous-frequency and its dual joint time and instantaneous-frequency transform to obtain a sparse representation of a signal that is not sparse in time or frequency, or sparse in either of these domains.

In [7], the discrete linear chirp transform (DLCT) is introduced to provide a linear-chirp representation of signals. The DCLT is a joint frequency and instantaneous frequency transform. It is applicable in the analysis of non-stationary signals, it can be implemented with the fast Fourier transform, and it provides a dual transform jointly relating time and instantaneous-frequency. A clear application for the DLCT is compression of signals that are sparse in time or in frequency, or in neither time nor frequency. The DLCT parameters characterizing the signal would permit us to generate a compressed form useful for transmission or storage. This paper will provide the necessary information about the DLCT and illustrate its application to compression. Test signals will be used to compare our procedure with CS.

The rest of the paper is organized as follows. In section 2 we discuss the DLCT transform, so that in section 3 and 4 we are able to propose and demonstrate its application to data compression. Although our procedure is illustrated with test signals that are not necessarily sparse in either time or frequency, it is applicable to signals that are sparse in time or in frequency.

2. THE DISCRETE LINEAR CHIRP TRANSFORM (DLCT)

Sparseness or compressibility is fundamental in the transmission and storage of signals. Sparseness can be obtained, in some cases, using the uncertainty principle. For instance, a sinusoid of infinite time support is sparse in frequency, while an impulse is sparse in time but not in frequency. As proposed in [4], for a given signal one can select either the time or the frequency domain for which the signal is sparser, by forcing the sparseness through thresholding the transform of the signal. Using a random measurement matrix, the reconstruction of sparse signals is converted into a convex optimization. Determining the sparseness of signals requires considering joint time-frequency transformations.

A chirp signal would be an example of a signal that is not sparse in time or in frequency, although it could be considered sparse in an intermediate domain. This points toward the application of the Fractional Fourier transform (FrFT) [8, 9] or of the polynomial time-frequency transforms [10]. What seems to be needed is a transformation that represents any signal in terms of chirps, having modulation and duality properties. Such a representation would determine if the signal is sparse, and if not how to transform it into an equivalent sparse signal. In [7] a local signal representation in terms of linear chirps is proposed. The modulation and duality properties of this transform permit us to change the given signal into a sparser domain, and to consider the sinusoidal and the impulse representations as special cases.

Given a discrete-time signal $x(n)$, $0 \leq n \leq N-1$, its discrete linear chirp transform (DLCT) is given as [7]

$$X(k,\beta) = \sum_{n=0}^{N-1} x(n) \exp\left(-j\frac{2\pi}{N}(\beta n^2 + kn)\right)$$
$$0 \leq n, k \leq N-1, \; -\Lambda \leq \beta < \Lambda. \tag{1}$$

The DLCT is obtained from a basis of linear chirps

$$\phi_{\beta,k}(n) = \exp\left(j\frac{2\pi}{N}(\beta n^2 + kn)\right)$$

characterized by a chirp rate β, a continuous variable connected with the instantaneous frequency of the chirp:

$$IF(n,k) = \frac{2\pi}{N}(2\beta n + k),$$

and by the discrete frequency $2\pi k/N$. Assuming a finite support for β, i.e., $-\Lambda \leq \beta < \Lambda$, it is possible to construct an orthonormal basis $\{\phi_{\beta,k}(n)\}$ with respect to k in the supports of β and n. To obtain a discrete transformation, we approximate the chirp rate as

$$\beta \approx \ell C, \text{ where } C = \frac{2\Lambda}{L} \text{ so that}$$
$$-\frac{L}{2} \leq \ell \leq \frac{L}{2} - 1 \text{ integer.}$$

The inverse discrete linear chirp transform is proposed in [7] as

$$x(n) = \sum_{\ell=-L/2}^{L/2-1} \sum_{k=0}^{N-1} \frac{X(k,\beta)}{LN} \exp\left(j\frac{2\pi}{N}(\beta n^2 + kn)\right)$$
$$0 \leq n, k \leq N-1, \; -\Lambda \leq \beta < \Lambda. \tag{2}$$

The DLCT is a joint instantaneous-frequency frequency transform that generalizes the discrete Fourier transform (DFT) as $X(k,0)$ is the DFT of $x(n)$.

A dual transformation is obtained by interchanging the time and frequency variables in (1) and 2 :

$$\hat{x}(n,\beta) = \sum_{k=0}^{N-1} \hat{X}(k) \exp\left(j\frac{2\pi}{N}(\beta k^2 + nk)\right)$$
$$\hat{X}(k) = \sum_{\ell=-L/2}^{L/2-1} \sum_{n=0}^{N-1} \frac{\hat{x}(n,\beta)}{LN} \exp\left(-j\frac{2\pi}{N}(\beta k^2 + nk)\right)$$
$$0 \leq n, k \leq N-1, \; -\Lambda \leq \beta < \Lambda. \tag{3}$$

It can be shown that $\hat{X}(k)$ is the DFT of $x(n)$ or $X(k,0)$. Thus, the DLCT can be used to represent signals that are combinations of sinusoids or chirps of small chirp rates, while the dual DLCT is more appropriate for signals that are combinations of impulses or chirps of large chirp rates.

It is important to remark that in a discrete chirp, obtained by sampling a continuous chirp satisfying the Nyquist criteria, the chirp rate β cannot be an integer. Indeed, if a finite support continuous chirp

$$x(t) = e^{j(\alpha t^2 + \Omega_0 t)} \qquad \alpha = \frac{\Delta\Omega}{2\Delta t}, \;\; 0 \leq t \leq T$$

is sampled using $\Omega_s = 2\pi/T_s = M\Omega_{max}$, $M \geq 2$, the discrete signal is

$$\begin{aligned} x(n) &= x(t)|_{t=nT_s} = e^{j([\alpha T_s^2]n^2 + [\Omega_0 T_s]n)} \\ &= e^{j(\hat{\beta} n^2 + \omega_o n)} \qquad 0 \leq n \leq T/Ts = N-1 \end{aligned}$$

where we let $\hat{\beta} = \alpha T_s^2$ be the chirp rate and $\omega_0 = \Omega_0 T_s$ be the discrete frequency. Then the modulated chirp

$$x(n)e^{-j\omega_0 n} = e^{j\hat{\beta}n^2} \text{ where}$$
$$\hat{\beta} = \alpha T_s^2 = \left[\frac{\Delta\Omega}{2\Delta t}\right] T_s^2 = \frac{\Omega_s T_s/M}{2N} = \frac{\pi/M}{N}$$

therefore,

$$\beta = \frac{N\hat{\beta}}{2\pi} = \frac{1}{2M}$$

is not an integer for $M \geq 2$. The discrete chirp-Fourier transform proposed in [12], assumes the chirp rate β is an integer — indicating that if the discrete chirp is obtained by sampling a continuous chirp it is aliased. For not aliased chirps, we need $|\beta| \leq 0.25$.

For each value of β it can be shown that

$$x_\beta(n) = \sum_{k=0}^{N-1} \frac{X(k,\beta)}{N} \exp\left(j\frac{2\pi}{N}(\beta n^2 + kn)\right)$$

equals $x(n)$ so that the inverse DLCT is the average over all values of β.

3. SIGNAL COMPRESSION USING THE DLCT

The main goal of signal compression is to reduce the amount of data that we want to transmit or store. The direct and the dual DLCT are used to represent signals that can be better represented by one of them locally. Considering that a sinusoid has a chirp rate $\beta = 0$, while an impulse has as chirp rate $\beta \to \infty$, we separate signals into two groups: one having $0 \leq |\beta| \leq 0.5$, corresponding to a linear chirp with a slope with an angle in $[-45^o\ 45^o]$, and the other for $0.5 < |\beta| < \infty$ corresponding to a linear chirp with a slope with an angle in $[45^o,\ 90^o]$ or $[-45^o,\ -90^o]$. The value of $\beta = 0.5$ is not arbitrarily chosen since it relates to the slope of the instantaneous frequency such that

$$\text{Slope} = \tan(\theta) = 2\beta$$

If $\beta = 0.5$, then $\theta = \pi/4$ which is the angle that separates the time-frequency space into two symmetric halves.

The performance of the proposed algorithm is measured by signal to noise ratio (SNR) and the compression ratio (Cr),

$$\text{SNR} = 10\log\left(\frac{\sigma_x^2}{\sigma_e^2}\right)$$

$$\text{Cr} = \frac{\text{length of original signal}}{\text{length of compressed signal}}$$

where σ_x^2 is the mean square of the original signal and σ_e^2 is the mean square of the error signal or the difference between the original and the reconstructed signals. Another factor that plays an important role in compression is a threshold. After calculating the DLCT of a signal, many of the coefficients of the resulted signal are close to or equal to zero. Thus, we can modify those coefficients to produce more zeros by zeroing out them using certain threshold.

3.1. The proposed compression algorithm

In this section, we present a new algorithm for signal compression using DLCT. Figure 1 shows the block diagram of the proposed method.

Consider the local representation of a signal $x(n)$, $0 \leq n \leq N-1$, as a superposition of P linear chirps

$$\begin{aligned} x(n) &= \sum_{i=0}^{P-1} a_i \exp\left(j\frac{2\pi}{N}(\beta_i n^2 + k_i n) + j\varphi_i\right) \\ &= x_{\{|\beta_i| \leq 0.5\}}(n) + x_{\{|\beta_i| > 0.5\}}(n) \end{aligned}$$

where $\{a_i,\ \varphi_i,\ k_i,\ \beta_i\}$ are the amplitude, phase, frequency, and chirp rate of the i^{th} linear chirp. The algorithm has two paths for the signal, the upper which is the dual path and the lower which is the direct path. Depending on the minimum value of the extracted βs for certain segment of the signal, we can do the compression either by the dual path or by the direct path. The coefficients $\{a_i,\ \varphi_i,\ k_i,\ \beta_i\}$ are extracted and from theses coefficients we can reconstruct an approximation for the signal $x(n)$— where the arrangement of these coefficients is done according to the proposed data structure as will be shown in Fig. 2.

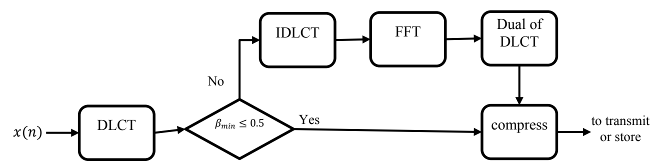

Fig. 1. Compression algorithm.

3.2. The Developed Data Structure

The proposed data structure for sending or storing the extracted parameters is shown in Fig. 2, we choose P chirp rates that correspond to the peaks of chirps which forms the signal and P is the order of the chirp model. Then, from each vector which corresponds to the chosen chirp rates from the chirp transform $X(k,\beta)$ or $\hat{x}(n,\beta)$ matrix, we select M_j amplitudes, phases, and frequencies or samples that have more power of the signal concentrated upon them. .

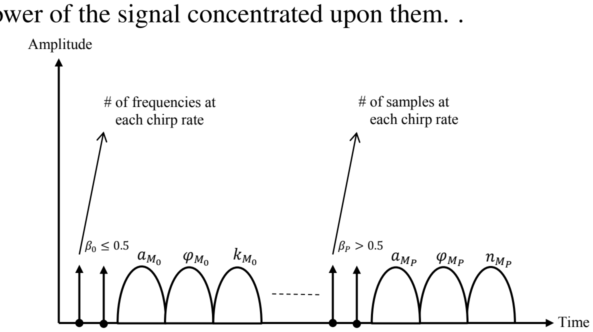

Fig. 2. Data structure.

4. SIMULATION RESULTS

In this section, we present three experiment to illustrate the performance of the proposed method, compare the results with the compressive sensing method, and explore the relation between the chirp rate and the proposed algorithm.

Compressive sensing (CS) is a compression technique that uses a fixed set of linear measurements providing that the signal is sparse. The signal can be reconstructed by a convex optimization process. Consider the real and finite length signal $x[n]$ represented by its coefficient vector $\mathbf{x} \in \mathcal{R}^N$. Let us assume that the basis $\psi = [\psi_1 ... \psi_N]$ where ψ is an $N \times N$ matrix, the signal can be expressed in terms of the basis as [3]

$$\mathbf{x} = \sum_{i=1}^{N} s_i \psi_i \quad \text{or} \quad \mathbf{x} = \psi\, \mathbf{s}$$

where $\mathbf{s}$ is a vector of size $N \times 1$. The basis can be any function that transforms $\mathbf{x}$ into a sparse signal. For instance, we can use sinusoidal basis such as discrete cosine transform. The signal $\mathbf{s}$ is a sparse signal in the new space and has K nonzero coefficients. Assume that the K nonzero coefficients are not extracted directly, but we project the vector $\mathbf{s}$ onto a matrix ϕ of size $M \times N$ where $M < N$. The matrix ϕ is called the measurement matrix and it satisfies the condition that the columns of the sparsity basis ψ cannot sparsely represent the rows of the measurement matrix ϕ. The measurements $\mathbf{y}$ can be obtained as follows

$$\mathbf{y} = \phi\, \mathbf{x} = \phi\, \psi\, \mathbf{s} = \theta \mathbf{s}$$

where $\mathbf{y}$ is a vector of size $M \times 1$. The reconstruction of the signal can be done via ℓ_1-optimization

$$\hat{\mathbf{s}} = \quad \text{argmin} \quad \|\mathbf{s}\|_1 \quad \text{subject to} \quad \mathbf{y} = \theta \mathbf{s}$$

as shown in [4].

In the first experiment, we use a segment of speech (1024 samples, sampling rate $f_s = 8kHz$) as shown in Fig. 3(a). Figure 3(b) and (c) give the magnitude of the DLCT and the Wigner distribution for this segment of speech. The compression ratio versus the SNR plot is shown in Fig. 3 (d). Since our goal is to obtain high SNR with high compression ratio, the proposed method gives more compression ratio than compressive sensing method, for an acceptable SNR. This segment of speech has very small chirp rates at high frequency components, with low concentrated energy, and sinusoids at low frequency components with high concentrated energy. Since the minimum value β is less than 0.5, the compression is obtained by the direct path. Even though, this segment of speech can be considered a sparse signal in the frequency domain, the proposed algorithm outperforms the compressive sensing method.

In the second experiment, a bird song signal (2048 samples and sampling rate $f_s = 7,350Hz$) with $\beta = 0.88$ is considered; see Fig. 4 (a). This signal is sparser in the time domain than in the frequency domain. Its dual DLCT and its Wigner distribution are shown in Figs. 4(b) and (c). Figure 4(d) displays SNR versus compression ratio. In this experiment, the minimum value of β is greater than 0.5. Thus, the dual path is used for the compression. The proposed method performs better than CS method.

In the third experiment, we consider the case of a bird signal (number of samples$=$ 2048 and $f_s = 7,350Hz$) with

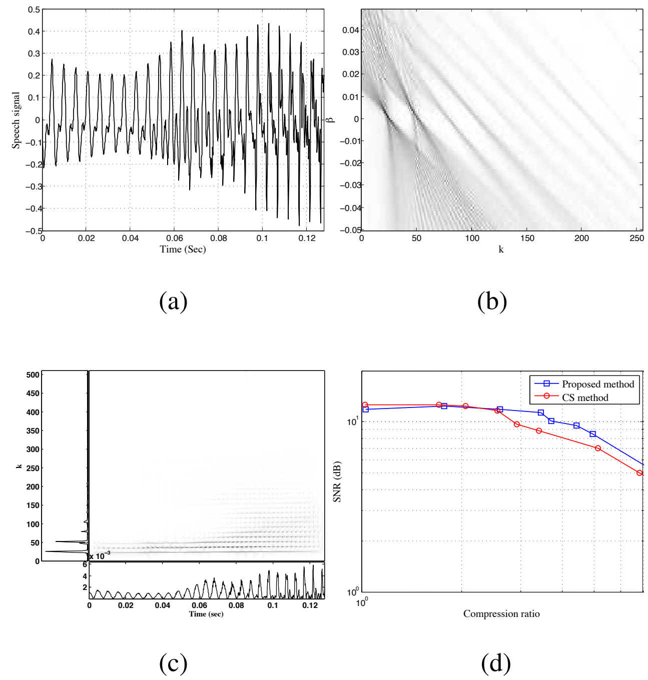

Fig. 3. Experiment 1: (a) Segment of speech; (b) $|X(k,\beta)|$ in two-dimensions; (c) The Wigner distribution of the signal showing time and frequency marginals; (d) Compression ratio vs SNR for different methods.

$\beta = 0.08$ and $\beta = 0.88$. Figures 5 (a), (b), and (c) show the signal, the magnitude of the DLCT, and the Wigner distribution of the signal. Since, this signal is a chirp based, the improvement of compression ratio for certain SNR is very large and it clearly shows the effectiveness of the proposed method over compressing sensing method as shown in Fig. 5 (d). The minimum value in this experiment is $\beta = 0.08$, so the compression is obtained by the direct path.

5. CONCLUSIONS

In this paper, we present a new algorithm for signal compression based on the discrete linear-chirp transform (DLCT) and its dual. The extracted coefficients can be arranged in the developed data structure. The simulation results show the effectiveness of the proposed method over the compressive sensing (CS) method. The improvement in the compression ratio depends on the nature of the signal. The effect of chirp rate on the performance of the direct and dual paths is also investigated. It turns out the compression ratio depends on the minimum chirp rate of the linear chirps that forms the signal. The value of $\beta = 0.5$ is the decision maker. If $\beta_{mim} \leq 0.5$ we use direct path, otherwise dual path is used.

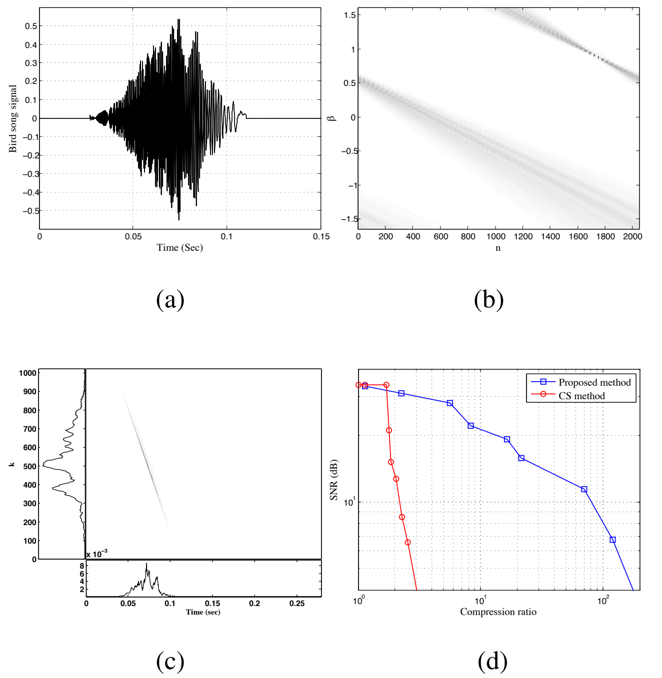

Fig. 4. Experiment 2: (a) Bird chirping; (b) $|X(k, \beta)|$ in two-dimensions; (c) The Wigner distribution of the signal showing time and frequency marginals; (d) Compression ratio vs SNR for different methods.

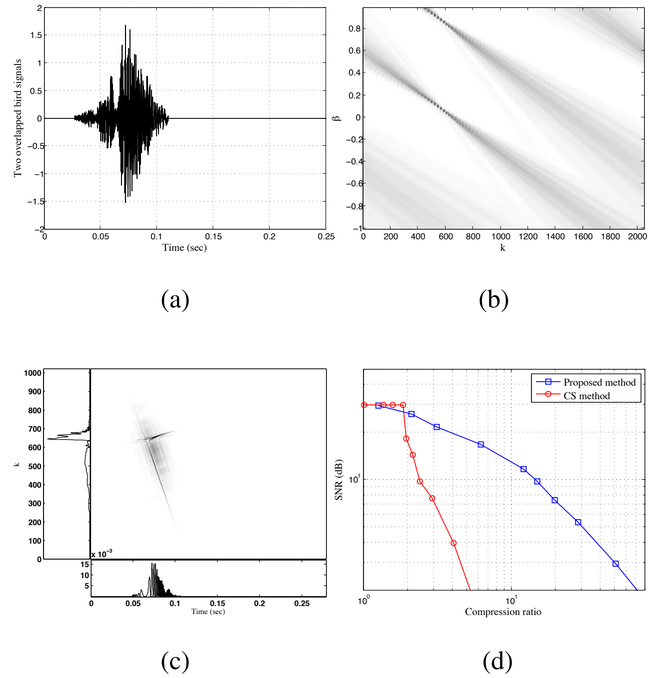

Fig. 5. Experiment 3: (a) Bird chirping; (b) $|X(k, \beta)|$ in two-dimensions; (c) The Wigner distribution of the signal showing time and frequency marginals; (d) Compression ratio vs SNR for different methods.

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